Domains of analyticity for response solutions in strongly dissipative forced systems

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Abstract

We study the ordinary differential equation $\varepsilon\ddot{x}+\dot{x}+\varepsilon g(x)=\varepsilon f(\boldsymbol{\omega}t)$, where g and f are real-analytic functions, with f quasi-periodic in t with frequency vector $\boldsymbol{\omega}$. If $c_0\in\mathbb{R}$ is such that $g(c_0)$ equals the average of f and $g'(c_0)\neq 0$, under very mild assumptions on $\boldsymbol{\omega}$ there exists a quasi-periodic solution close to c_0 with frequency vector $\boldsymbol{\omega}$. We show that such a solution depends analytically on ε in a domain of the complex plane tangent more than quadratically to the imaginary axis at the origin.

1 Introduction

Consider the ordinary differential equation in $\mathbb{R}$

$$\varepsilon\ddot{x}+\dot{x}+\varepsilon\, g(x)=\varepsilon\, f(\boldsymbol{\omega}t), \tag{1.1}$$

where $\varepsilon\in\mathbb{R}$ is small and $\boldsymbol{\omega}\in\mathbb{R}^d$, with $d\in\mathbb{N}$, is assumed (without loss of generality) to have rationally independent components, i.e. $\boldsymbol{\omega}\cdot\boldsymbol{\nu}\neq 0\ \forall\boldsymbol{\nu}\in\mathbb{Z}^d_*:=\mathbb{Z}^d\setminus\{\mathbf{0}\}$. For $\varepsilon>0$ the equation describes a one-dimensional system with mechanical force g, subject to a quasi-periodic forcing f with frequency vector $\boldsymbol{\omega}$ and in the presence of strong dissipation. We refer to [3] for some physical background. A quasi-periodic solution to (1.1) with the same frequency vector $\boldsymbol{\omega}$ as the forcing will be called a *response solution*.

Hypothesis 1. *The functions* $g\colon\mathbb{R}\to\mathbb{R}$ *and* $f\colon\mathbb{T}^d\to\mathbb{R}$ *are real-analytic. There is* $c_0\in\mathbb{R}$ *such that* $g(c_0)=f_{\mathbf{0}}$*, where* $f_{\mathbf{0}}$ *is the average of* f *on* $\mathbb{T}^d$*, and* $a:=g'(c_0)\neq 0$.

In other words we assume that c_0 is a simple zero of the function $g(x)-f_{\mathbf{0}}$. Denote by $\Sigma_\xi:=\{\boldsymbol{\psi}=(\psi_1,\ldots,\psi_d)\in(\mathbb{C}/2\pi\mathbb{Z})^d:|\mathrm{Im}\,\psi_k|\leq\xi \text{ for } k=1,\ldots,d\}$, with $\xi>0$, the strip where f is analytic. By the analyticity assumptions one can write

$$f(\boldsymbol{\psi})=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}f_{\boldsymbol{\nu}},\qquad g(x)=\sum_{p=0}^{\infty}a_p\,(x-c_0)^p,$$

where

$$|f_{\boldsymbol{\nu}}|\leq\Phi\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|},\qquad a_p:=\frac{1}{p!}\frac{\mathrm{d}^pg}{\mathrm{d}x^p}(c_0),\qquad |a_p|\leq\Gamma\,\rho^p,$$

for suitable constants Φ, Γ and ρ. Set $N(f) = N$ if f is a trigonometric polynomial of degree N and $N(f) = \infty$ otherwise, and define

$$\beta_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n, |\boldsymbol{\nu}| \le N(f)\right\}, \qquad \varepsilon_n(\boldsymbol{\omega}) := \frac{1}{2^n}\log\frac{1}{\beta_n(\boldsymbol{\omega})},$$

$$\alpha_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n\right\}, \qquad \mathfrak{B}(\boldsymbol{\omega}) := \sum_{n=0}^{\infty}\frac{1}{2^n}\log\frac{1}{\alpha_n(\boldsymbol{\omega})}.$$

Hypothesis 2. $\lim_{n\to\infty}\varepsilon_n(\boldsymbol{\omega}) = 0$.

In particular no assumption at all is required on $\boldsymbol{\omega}$ if f is a trigonometric polynomial, since $\beta_n(\boldsymbol{\omega})$ is definitively constant in that case.

Before stating our results we need some more notations. We define the sets $C_R := \{\varepsilon \in \mathbb{C} : |\mathrm{Re}\,\varepsilon^{-1}| > (2R)^{-1}\}$ and $\Omega_{R,B} := \{\varepsilon \in \mathbb{C} : |\mathrm{Re}\,\varepsilon| \ge B\,(\mathrm{Im}\,\varepsilon)^2$ and $0 < |\varepsilon| < 2R\}$. C_R consists of two disks with radius R and centers $(R, 0)$ and $(-R, 0)$, while $\Omega_{R,B}$ is the intersection of the disk of center $(0, 0)$ and radius $2R$ with two parabolas with vertex at the origin: all such sets are tangent at the origin to the imaginary axis. Note that the smaller B, the more flattened are the parabolas. If $2RB < 1$ one has $C_R \subset \Omega_{R,B}$.

The following result has been proved in [1].

Theorem 1.1. *Assume Hypotheses 1 and 2 for the system (1.1) and denote by Σ_ξ the strip of analyticity of f. Then there exist $\varepsilon_0 > 0$ and $B_0 > 0$ such that for all $B > B_0$ there is a response solution $x(t) = c_0 + u(\boldsymbol{\omega}t, \varepsilon)$ to (1.1), with $u(\boldsymbol{\psi}, \varepsilon) = O(\varepsilon)$ analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$.*

In the theorem above ε_0 has to be small, while B_0 must be large enough. However, for B as close as wished to B_0 one can take $\bar\varepsilon < \varepsilon_0$ small enough for the condition $\bar\varepsilon B < 1$ to be satisfied, so as to obtain that $C_{\bar\varepsilon/2}$ is contained inside the analyticity domain. In this respect Theorem 1.1 extends previous results in the literature [3, 4], where analyticity in a pair of disks was obtained under stronger conditions on $\boldsymbol{\omega}$, such as the standard Diophantine condition

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| > \frac{\gamma}{|\boldsymbol{\nu}|^\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d_*, \tag{1.2}$$

or the Bryuno condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$ If either $d = 1$ or $d = 2$ and $\boldsymbol{\omega}$ satisfies the standard Diophantine condition (1.2) with $\tau = 1$, the response solution is Borel-summable.

In the present letter we remove in Theorem 1.1 the condition for B to be large, by proving the following result.

Theorem 1.2. *Assume Hypotheses 1 and 2 for the system (1.1) and denote by Σ_ξ the strip of analyticity of f. Then for all $B > 0$ there exists $\varepsilon_0 > 0$ such that there is a response solution $x(t) = c_0 + u(\boldsymbol{\omega}t, \varepsilon)$ to (1.1), with $u(\boldsymbol{\psi}, \varepsilon) = O(\varepsilon)$ analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$. The dependence of ε_0 on B is of the form $\varepsilon_0 = \varepsilon_1 B^\alpha$, for some $\alpha > 0$ and ε_1 independent of B.*

The proof of the theorem given in Section 3 yields the value $\alpha = 8$: such a value is non-optimal and could be improved by a more careful analysis. Thanks to Theorem 1.2 we can estimate the domain of analyticity by the union of the domains $\Omega_{\varepsilon_0,B}$, with $\varepsilon_0 = B^\alpha\varepsilon_1$, by letting B varying in $(0, 1]$. This provides a domain that near the origin has boundary of the form $|\mathrm{Re}\,\varepsilon| \approx \varepsilon_1^{-\beta}|\mathrm{Im}\,\varepsilon|^{2+\beta}$, where $\beta = 1/\alpha$.

2 Tree representation

We can rewrite (1.1) as

$$\varepsilon\ddot{x}+\dot{x}+\varepsilon\, a\,(x-c_0)+\mu\varepsilon\sum_{p=2}^{\infty}a_p(x-c_0)^p=\mu\varepsilon\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}f_{\boldsymbol{\nu}}, \tag{2.1}$$

where $a:=a_1$ and $\mu=1$. However, we can consider μ as a free parameter and study (2.1) for $\varepsilon\in\mathbb{C}$ and $\mu\in\mathbb{R}$. Then we look for a quasi-periodic solution to (2.1) of the form

$$x(t,\varepsilon,\mu)=c_0+u(\boldsymbol{\omega}t,\varepsilon,\mu),\qquad u(\boldsymbol{\psi},\varepsilon,\mu)=\sum_{k=1}^{\infty}\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mu^k\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}u^{(k)}_{\boldsymbol{\nu}}(\varepsilon). \tag{2.2}$$

By inserting (2.2) into (2.1) we obtain a recursive definition for the coefficients $u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)$, which admits a natural graphical representation in terms of trees.

A *rooted tree* θ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (root line). All the points in θ except the root are called *nodes*. The orientation of the lines in θ induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes v and w, we shall write $w\prec v$ every time v is along the path (of lines) which connects w to the root. We shall write $w\prec\ell$ if $w\preceq v$, where v is the node which ℓ exits. For any node v denote by p_v the number of lines entering v: v is called and *end node* if $p_v=0$ and an *internal node* if $p_v>0$. We denote by $N(\theta)$ the set of nodes, by $E(\theta)$ the set of end nodes, by $V(\theta)$ the set of internal nodes and by $L(\theta)$ the set of lines; one has $N(\theta)=E(\theta)\amalg V(\theta)$.

We associate with each end node $v\in E(\theta)$ a *mode* label $\boldsymbol{\nu}_v\in\mathbb{Z}^d_*$ and with each internal node an *degree* label $d_v\in\{0,1\}$. With each line $\ell\in L(\theta)$ we associate a *momentum* $\boldsymbol{\nu}_\ell\in\mathbb{Z}^d$. We impose the following constraints on the labels:

1. $\boldsymbol{\nu}_\ell=\sum_{w\in E_\ell(\theta)}\boldsymbol{\nu}_w$, where $E_\ell(\theta):=\{w\in E(\theta):w\prec\ell\}$;
2. $p_v\geq 2\ \forall v\in V(\theta)$;
3. if $d_v=0$ then the line ℓ exiting v has $\boldsymbol{\nu}_\ell=\mathbf{0}$.

We shall write $V(\theta)=V_0(\theta)\amalg V_1(\theta)$, where $V_0(\theta):=\{v\in V(\theta):d_v=0\}$. For any discrete set A we denote by $|A|$ its cardinality. Define the *degree* and the *order* of θ as $d(\theta):=|E(\theta)|+|V_1(\theta)|$ and $k(\theta):=|N(\theta)|$, respectively.

We call *equivalent* two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. In the following we shall consider only inequivalent labelled rooted trees, and we shall call them call trees *tout court*, for simplicity.

We associate with each node $v\in N(\theta)$ a *node factor* F_v and with each line $\ell\in L(\theta)$ a *propagator* $\mathcal{G}_\ell$, such that

$$F_v:=\begin{cases}-\varepsilon^{d_v}\,a_{p_v}, & v\in V(\theta),\\ \varepsilon\, f_{\boldsymbol{\nu}_v}, & v\in E(\theta),\end{cases}\qquad\qquad \mathcal{G}_\ell:=\begin{cases}1/D(\varepsilon,\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell\neq\mathbf{0},\\ 1/a, & \boldsymbol{\nu}_\ell=\mathbf{0},\end{cases}$$

where $D(\varepsilon, s) := -\varepsilon s^2 + \mathrm{i}s + \varepsilon\, a$. Then, by defining

$$\mathscr{V}(\theta, \varepsilon) := \left(\prod_{v \in N(\theta)} F_v \right) \left(\prod_{\ell \in L(\theta)} \mathcal{G}_\ell \right) \tag{2.3}$$

one has

$$u_{\boldsymbol{\nu}}^{(k)}(\varepsilon) = \sum_{\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta, \varepsilon), \quad \boldsymbol{\nu} \in \mathbb{Z}^d \tag{2.4}$$

where $\mathcal{T}_{k,\boldsymbol{\nu}}$ is the set of trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line. Note that $u_{\mathbf{0}}^{(1)} = 0$ and $u_{\boldsymbol{\nu}}^{(2)} = 0$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$.

3 Proof of Theorem 1.2

We shall prove Theorem 1.2 in the case in which $N(f) = \infty$. The case of trigonometric polynomials is in fact easier and can be dealt with as shown in [2].

Lemma 3.1. *Set $c_0 = \min\{1/8, B/18, B/8|a|, |a|/8, |a|B/4, \sqrt{|a|}/2\}$. There exists $\varepsilon_1 > 0$ such that one has $|D(\varepsilon, s)| \geq c_0 \max\{\min\{1, s^2\}, |\varepsilon|^2\}$ for all $s \in \mathbb{R}$ and all $\varepsilon \in \Omega_{B,\varepsilon_1}$.*

Proof. Write $\varepsilon = x + \mathrm{i}y$, with $|x| \geq By^2$ and x small enough. By symmetry it is enough to study $y \geq 0$. One has $|D(\varepsilon, s)|^2 = (s + ya - ys^2)^2 + x^2(a - s^2)^2$. If $y = 0$ the bound is straightforward. If $y > 0$ denote by s_1 and s_2 the two roots of $s + ya - ys^2 = 0$: one has $s_1 = -ay + O(y^2)$ and $s_2 = 1/y + ay + O(y^2)$. Let ε_1 be so small that $|s_1 + ay| \leq |a|y/2$, $|s_2 - 1/y| \leq 1/6y$ and $18|a|y^2 \leq 1$ for $|\varepsilon| \leq \varepsilon_1$. The following inequalities are easily checked: (1) if $|s| < 2|a|y$, then $|x|\,|a - s^2| \geq |ax|/2 \geq |a|By^2/2 \geq Bs^2/8|a|$; (2) if $|s - s_2| < 1/2y$, then $|x|\,|a - s^2| \geq |x|s^2/2 \geq |x|/18y^2 \geq B/18$; (3) if $|s| \geq 2|a|y$ and $|s - s_2| \geq 1/2y$, then (3.1) $|s + ya - ys^2| \geq y|s - s_1|\,|s - s_2| \geq |a|y/4$, (3.2) $|s + ya - ys^2| \geq |s - s_1|/2 \geq |s|/8$, (3.3) if either $a < 0$ or $a > 0$ and $|a - s^2| > |a|/2$ one has $|x|\,|a - s^2| > |ax|/2$, while if $a > 0$ and $|a - s^2| \leq |a|/2$ one has $|s + ya - ys^2| \geq |s| - y|a - s^2| \geq \sqrt{a}/2$. By collecting together all the bounds the assertion follows. ∎

Lemma 3.2. *For any tree θ one has $|E(\theta)| \geq |V(\theta)| + 1$ and hence $2|E(\theta)| \geq k(\theta) + 1$.*

Proof. By induction on the order $k(\theta)$. ∎

For $v \in V_1(\theta)$ define $E(\theta, v) := \{w \in E(\theta) :$ the line exiting w enters $v\}$ and set $r_v := |E(\theta, v)|$, $s_v := p_v - r_v$, $\boldsymbol{\mu}_v := \sum_{w \in E(\theta,v)} \boldsymbol{\nu}_w$ and $\mu_v := |\boldsymbol{\mu}_v|$. Define $V_2(\theta) := \{v \in V(\theta) : s_v = 0\}$ and $V_3(\theta) := \{v \in V(\theta) : r_v = s_v = 1\}$. For $v \in V_2(\theta)$ call ℓ_v the line exiting v, and for $v \in V_3(\theta)$ call ℓ_v the line exiting v and ℓ'_v the line entering v which does not exits an end node. Define $\overline{V}_2(\theta) := \{v \in V_2(\theta) : \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}\}$ and $\overline{V}_3(\theta) := \{v \in V_3(\theta) : \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}$ and $\boldsymbol{\nu}_{\ell'_v} \neq \mathbf{0}\}$, and set $\overline{V}_1(\theta) = \overline{V}_2(\theta) \amalg \overline{V}_3(\theta)$. By construction one has $\overline{V}_1(\theta) \subset V_1(\theta)$.

Lemma 3.3. There exists $C_0 > 0$ such that $C_0|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \mathrm{e}^{-\xi|\boldsymbol{\nu}|/16}$ $\forall \boldsymbol{\nu} \in \mathbb{Z}^d_*$.

Proof. It follows from Hypothesis 2 by using that $\beta_n(\boldsymbol{\omega}) = \alpha_n(\boldsymbol{\omega})$ if $N(f) = \infty$. ∎

Lemma 3.4. *One has* $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| \ge \mathrm{e}^{-\xi\mu_v/16}$ *for* $v \in \overline{V}_2(\theta)$ *and* $2C_0 \max\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}|, |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}|\} \ge \mathrm{e}^{-\xi\mu_v/16}$ *for* $v \in \overline{V}_3(\theta)$.

Proof. For $v \in \overline{V}_2(\theta)$ one has $\boldsymbol{\nu}_{\ell_v} = \boldsymbol{\mu}_v$, so that one can use Lemma 3.3. For $v \in \overline{V}_3(\theta)$ one proceeds by contradiction. Suppose that the assertion is false: this would imply

$$\mathrm{e}^{-\xi\mu_v/16} > C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| + C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}| \ge C_0|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_{\ell_v} - \boldsymbol{\nu}_{\ell'_v})| = C_0|\boldsymbol{\omega}\cdot\boldsymbol{\mu}_v| \ge \mathrm{e}^{-\xi\mu_v/16},$$

where we have used that $E(\theta, v)$ contains only one node w and hence $\boldsymbol{\mu}_v = \boldsymbol{\nu}_w \neq \mathbf{0}$. ∎

Define $L_1(\theta, v) := \{\ell_v\}$ for $v \in \overline{V}_2(\theta)$ and $L_1(\theta, v) := \{\ell \in \{\ell_v, \ell'_v\} : 2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \ge \mathrm{e}^{-\xi\mu_v/16}\}$ for $v \in \overline{V}_3(\theta)$. Lemma 3.4 yields $L_1(\theta, v) \neq \emptyset$ for all $v \in \overline{V}_1(\theta)$. Set also $L_1(\theta) := \{\ell \in L(\theta) : \exists v \in \overline{V}_1(\theta) \text{ such that } \ell \in L_1(\theta, v)\}$, $L_{\mathrm{int}}(\theta) := \{\ell \in L(\theta) : \ell \text{ exits a node } v \in V_1(\theta)\}$ and $L_0(\theta) := L_{\mathrm{int}}(\theta) \setminus L_1(\theta)$.

Lemma 3.5. *For any tree* θ *one has* $4\,|L_0(\theta)| \le 3|E(\theta)| - 4$.

Proof. By induction on $V(\theta)$. If $|V(\theta)| = 1$ then either $V(\theta) = V_0(\theta)$ or $V(\theta) = \overline{V}_2(\theta)$ and hence $|L_0(\theta)| = 0$, so that the bound holds. If $|V(\theta)| \ge 2$ the root line ℓ_0 of θ exits a node $v_0 \in V(\theta)$ with $s_{v_0} + r_{v_0} \ge 2$ and $s_{v_0} \ge 1$. Call $\theta_1, \ldots, \theta_{s_{v_0}}$ the trees whose respective root lines $\ell_1, \ldots, \ell_{s_{v_0}}$ enter v_0: one has $|E(\theta)| = |E(\theta_1)| + \ldots + |E(\theta_{s_{v_0}})| + r_{v_0}$. If $\ell_0 \notin L_0(\theta)$ then $|L_0(\theta)| = |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$ and the bound follows from the inductive hypothesis.

If $\ell_0 \in L_0(\theta)$ then one has $|L_0(\theta)| = 1 + |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$, so that, again by the inductive hypothesis, $4|L_0(\theta)| \le 3|E(\theta)| - 3r_{v_0} - 4\,(s_{v_0} - 1)$. If either $r_{v_0} + s_{v_0} \ge 3$ or $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 2$, the bound follows. If $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 1$, then $v_0 \in V_3(\theta)$, so that either $\boldsymbol{\nu}_{\ell_1} = \mathbf{0}$ or $2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| \ge \mathrm{e}^{-\xi\mu_{v_0}/16}$, by Lemma 3.4, because $\ell_0 \in L_0(\theta)$ and hence $2C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_0}| < \mathrm{e}^{-\xi\mu_{v_0}/16}$. Therefore $\ell_1 \notin L_0(\theta)$. If v_1 is the node which ℓ_1 exits, call $\theta'_1, \ldots, \theta'_{s_{v_1}}$ the trees whose root lines enter v_1: one has $|L_0(\theta)| = 1 + |L(\theta'_1)| + \ldots + |L_0(\theta'_{s_{v_1}})|$ and hence, by the inductive hypothesis, $4|L_0(\theta)| \le 3|E(\theta)| - 3r_{v_0} - 3r_{v_1} - 4\,(s_{v_1} - 1)$, where $3r_{v_0} + 3r_{v_1} + 4s_{v_1} - 4 \ge 5$, so that the bound follows in this case too. ∎

Lemma 3.6. *For any* $k \ge 1$ *and* $\boldsymbol{\nu} \in \mathbb{Z}^d$ *and any tree* $\theta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ *one has*

$$|\mathscr{V}(\theta, \varepsilon)| \le A_0^k c_0^{-k} |\varepsilon|^{1 + \frac{k+1}{8}} \prod_{v \in E(\theta)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_v|/8},$$

with A_0 *a positive constant depending on* Φ, Γ *and* ρ, *and* c_0 *as in Lemma 3.1.*

Proof. One bounds (2.3) as

$$|\mathscr{V}(\theta, \varepsilon)| \le |\varepsilon|^{d(\theta)} \Bigg(\prod_{v \in V(\theta)} |a_{p_v}| \Bigg) \Bigg(\prod_{v \in E(\theta)} |f_{\boldsymbol{\nu}_v}| \Bigg) \Bigg(\prod_{\ell \in L(\theta)} |\mathcal{G}_\ell| \Bigg).$$

We deal with the propagators by using Lemma 3.1 as follows. If ℓ exits a node $v \in \overline{V}_2(\theta)$, then we have

$$|\mathcal{G}_\ell| \prod_{w \in E(\theta, v)} |f_{\boldsymbol{\nu}_w}|\, |\mathcal{G}_{\ell_w}| \le \frac{1}{c_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^2} \prod_{w \in E(\theta, v)} \frac{|f_{\boldsymbol{\nu}_w}|}{c_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_w|^2} \le c_0^{-1} C_0^2 (c_0^{-1} C_0^2 \Phi)^{|E(\theta, v)|} \prod_{w \in E(\theta, v)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_w|/4},$$

where ℓ_w denotes the line exiting w. For the other lines in $L_1(\theta)$ we distinguish three cases: given a node $v \in V_3(\theta)$ and denoting by v' the node which the line ℓ'_v exits, (1) if either $\ell'_v \notin L_1(\theta, v)$ or $\ell'_v \in L_1(\theta, v')$, we proceed as for the nodes $v \in \overline{V}_2(\theta)$ with $\ell = \ell_v$ and obtain the same bound; (2) if $L_1(\theta, v) = \{\ell'_v\}$ and $\ell'_v \notin L_1(\theta, v')$, we proceed as for the nodes $v \in \overline{V}_2(\theta)$ with $\ell = \ell'_v$ and we obtain the same bound once more; (3) if both lines ℓ_v, ℓ'_v belong to $L_1(\theta, v)$ and $\ell'_v \notin L_1(\theta, w)$, we bound

$$\left|\mathcal{G}_{\ell_v}\mathcal{G}_{\ell'_v}\right| \prod_{w\in E(\theta,v)} |f_{\boldsymbol{\nu}_v}|\,|\mathcal{G}_{\ell_w}| \le c_0^{-2}C_0^4(c_0^{-1}C_0^2\Phi)^{|E(\theta,v)|} \prod_{w\in E(\theta,v)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_w|/8}.$$

For all the other propagators we bound (1) $|\mathcal{G}_\ell| \le 1/|a|$ if ℓ exits a node $v \in V_0(\theta)$, (2) $|\mathcal{G}_\ell| \le c_0^{-1}|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^{-2}$ if ℓ exits an end node and has not been already used in the bounds above for the lines $\ell \in L_1(\theta)$, and (3) $|\mathcal{G}_\ell| \le c_0^{-1}|\varepsilon|^{-2}$ if $\ell \in L_0(\theta)$. Then we obtain

$$|\mathscr{V}(\theta,\varepsilon)| \le |\varepsilon|^{d(\theta)-2|L_0(\theta)|}\Gamma^{|V(\theta)|}\rho^{|N(\theta)|}(c_0^{-1}C_0^2)^{|V_1(\theta)|}(c_0^{-1}C_0^2\Phi)^{|E_1(\theta)|}|a|^{-|V_0(\theta)|}\mathrm{e}^{-5\xi|\boldsymbol{\nu}|/8},$$

where we can bound, by using Lemma 3.2 and Lemma 3.5, $d(\theta) - 2|L_0(\theta)| = |E(\theta)| + |V_1(\theta)| - 2|L_0(\theta)| \ge |E(\theta)| - |L_0(\theta)| \ge 1 + |E(\theta)|/4 \ge 1 + (k(\theta)+1)/8$, so that the assertion follows. ∎

Lemma 3.7. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d$ one has*

$$\left|u_{\boldsymbol{\nu}}^{(k)}(\varepsilon)\right| \le A_1^k c_0^{-k}\mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}|\varepsilon|^{1+\frac{k+1}{8}},$$

with A_1 a positive constant C depending on Φ, Γ, ξ and ρ, and c_0 as in Lemma 3.1.

Proof. The coefficients $u_{\boldsymbol{\nu}}^{(k)}$ are given by (2.4). Each value $\mathscr{V}(\theta,\varepsilon)$ is bounded through Lemma 3.6. The sum over the Fourier labels is performed by using a factor $\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/8}$ for each end node $v \in E(\theta)$. The sum over the other labels is easily bounded by a constant to the power k. ∎

Lemma 3.7 implies that for ε small enough the series (2.2) converges uniformly to a function analytic in $\boldsymbol{\psi} \in \Sigma_{\xi'}$, with $\xi' < \xi/2$. Moreover such a function is analytic in $\varepsilon \in \Omega_{\varepsilon_0,B}$, provided $A_1^8\varepsilon_0/c_0^8$ is small enough. This completes the proof of Theorem 1.2.

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