Diagrammatic techniques in perturbation theory

Guido Gentile

Dipartimento di Matematica, Università di Roma Tre, Roma, I-00146

Introduction

Consider the dynamical system

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t} = G(u) + \varepsilon F(u), \qquad [1]$$

where $F, G : \mathcal{S} \subset \mathbb{R}^d \to \mathbb{R}^d$ are analytic functions and ε a real (small) parameter. Suppose also that for $\varepsilon = 0$ a solution $u_0 : \mathbb{R} \to \mathcal{S}$ (for some initial condition $u_0(0) = \overline{u}$) is known.

We look for a solution of [1] which is a perturbation of u_0, that is for a solution u which can be written in the form $u = u_0 + U$, with $U = O(\varepsilon)$ and $U(0) = \overline{U} \equiv u(0) - \overline{u}$. Then we can consider the variational equation

$$\dot{U} = M(t)\, U + \Phi(t), \qquad M_{ij}(t) = \partial_{u_i} G_j(u_0(t)), \qquad [2]$$

where we have set $\Phi(t) = \tilde{\Phi}(u_0(t), U)$, with $\tilde{\Phi}(u_0, U) = G(u_0 + U) - \partial_u G(u_0)\, U + \varepsilon F(u_0 + U)$. By defining the Wronskian matrix W as the solution of the matrix equation $\dot{W} = M(t)\, W$ such that $W(0) = \mathbb{1}$ (the columns of W are given by d independent solutions of the linear equation $\dot{u} = M(t)\, u$) we can write

$$U(t) = W(t)\, \overline{U} + W(t) \int_0^t \mathrm{d}\tau\, W^{-1}(\tau) \Phi(\tau). \qquad [3]$$

If we expect the solution U to be of order ε we can try to write it as a Taylor series in ε, that is

$$U(t) = \sum_{k=1}^{\infty} \varepsilon^k U^{(k)}, \qquad [4]$$

and, by inserting [4] into [3] and equating the coefficients with the same Taylor order, we obtain

$$U^{(k)}(t) = W(t)\,\overline{U}^{(k)} + W(t) \int_0^t \mathrm{d}\tau\, W^{-1}(\tau)\Phi^{(k)}(\tau), \qquad [5]$$

where $\Phi^{(k)}(t)$ is defined as

$$\begin{aligned} \Phi^{(1)}(t) &= F(u_0(t)), \\ \Phi^{(k)}(t) &= \sum_{p=2}^{\infty} \frac{1}{p!} \frac{\partial^p G}{\partial u^p}(u_0(t)) \sum_{k_1+\ldots+k_p=k} U^{(k_1)} \ldots U^{(k_p)} \\ &\quad + \sum_{p=1}^{\infty} \frac{1}{p!} \frac{\partial^p F}{\partial u^p}(u_0(t)) \sum_{k_1+\ldots+k_p=k-1} U^{(k_1)} \ldots U^{(k_p)}, \qquad k \geq 2, \end{aligned} \qquad [6]$$

hence $\Phi^{(k)}(t)$ depends only on coefficients of orders strictly less than k. In such a way we obtain an algorithm allowing us to construct the solution recursively, so that the problem is solved, up to (essential) convergence problems.

Historical excursus

To study a system like [1] by following the strategy outlined above can be hopeless if we do not make some further assumptions on the kind of motions we are looking for.

We shall see later in a concrete example that the coefficients $U^{(k)}(t)$ can increase in time, in a k-dependent way, so preventing the convergence of the series for large t. This is a general feature of this class of problems: if no care is made in the choice of

the initial datum, the algorithm can provide a reliable description of the dynamics only for a very short time.

Anyway if one looks for solutions having a special dependence on time, things can work better. This happens, for instance, if one looks for quasi-periodic solutions, that is functions which depend on time through the variable $\psi = \omega t$, with $\omega \in \mathbb{R}^N$ a vector with rationally independent components, that is such that $\omega \cdot \nu \neq 0$ for all $\nu \in \mathbb{Z}^N \setminus \{0\}$ (here and henceforth the dot denotes the standard inner product, $\omega \cdot \nu = \omega_1\nu_1 + \ldots + \omega_N\nu_N$). A typical problem of interest is: what happens to a quasi-periodic solution $u_0(t)$ when a perturbation εF is added to the unperturbed vector field G, as in [1]? Situations of this kind arise when considering perturbations of integrable systems: a classical example is provided by planetary motion in celestial mechanics.

Perturbation series like [4] have been extensively studied by astronomers in order to obtain a more accurate description of the celestial motions compared to that following from Kepler's theory (in which all interactions between planets are neglected and the planets themselves are considered as points). In particular we recall the works of Newcomb and Lindstedt (series like [4] are nowadays known as Lindstedt series). At the end of the XIX$^{\text{th}}$ century Poincaré showed that the series describing quasi-periodic motions are well defined to any perturbation order k (at least if the perturbation is a trigonometric polynomial), provided that the components of ω are assumed to be rationally independent: this means that, under this condition, the coefficients $U^{(k)}(t)$ are defined for all $k \in \mathbb{N}$. However Poincaré also showed that, in general, the series are divergent; this is due to the fact that, as we shall see, in the perturbation series there appear small divisors $\omega \cdot \nu$ which, even if they do not vanish, can be arbitrarily close to zero.

The convergence of the series can be proved indeed (more generally for analytic, or even differentiably smooth enough, perturbations) by assuming on ω a stronger non-resonance condition, such as the Diophantine condition

$$|\omega \cdot \nu| > \frac{C_0}{|\nu|^\tau} \qquad \forall \nu \in \mathbb{Z}^N \setminus \{0\}, \tag{7}$$

where $|\nu| = |\nu_1| + \ldots + |\nu_N|$, and C_0 and τ are positive constants. We note that

the set of vectors satisfying [7] for some positive constant C_0 has full measure in $\mathbb{R}^N$ provided one takes $\tau > N - 1$.

Such a result is part of the KAM theorem, and it was first proved by Kolmogorov in 1954, following an approach quite different fom the one described here. New proofs were given in 1962 by Arnol′d and by Moser, but only very recently, in 1988 (in a paper published 8 years later), Eliasson gave a proof in which a bound C^k is explicitly derived for the coefficients $U^{(k)}(t)$, again implying convergence for ε small enough.

Eliasson's work was not immediately widely known, and only after publication of papers by Gallavotti and by Chierchia and Falcolini, in which Eliasson's ideas were revisited, did his work come fully appreciated. The study of perturbation series [4] employs techniques very similar to those typical of a very different field of mathematical physics, quantum field theory, even if such an analogy was stressed and used to full extent only in subsequent papers.

Up to now the techniques have been applied to a wide class of problems of dynamical systems: a list of original results is given at the end.

A paradigmatic example

Consider the case $\mathcal{S} = \mathcal{A} \times \mathbb{T}^N$, with $\mathcal{A}$ an open subset of $\mathbb{R}^N$, and let $\mathcal{H}_0 : \mathcal{A} \to \mathbb{R}$ and $f : \mathcal{A} \times \mathbb{T}^N$ be two analytic functions. Then consider the Hamiltonian system with Hamiltonian $\mathcal{H}(A, \alpha) = \mathcal{H}_0(A) + \varepsilon f(A, \alpha)$. The corresponding equations describe a dynamical system of the form [1], with $u = (A, \alpha)$, which we can write explicitly:

$$\begin{cases} \dot{A} = -\varepsilon \partial_\alpha f(A, \alpha), \\ \dot{\alpha} = \partial_A \mathcal{H}_0(A) + \varepsilon \partial_A f(A, \alpha). \end{cases} \qquad [8]$$

Suppose for simplicity we have $\mathcal{H}_0(A) = A^2/2$ and $f(A, \alpha) = f(\alpha)$, where $A^2 = A \cdot A$. Then we obtain for α the closed equation

$$\ddot{\alpha} = -\varepsilon \partial_\alpha f(\alpha), \qquad [9]$$

while A can be obtained by direct integration once [9] has been solved.

For $\varepsilon = 0$ [9] gives trivially $\alpha = \alpha_0(t) \equiv \alpha_0 + \omega t$, where $\omega = \partial_A \mathcal{H}_0(A_0) = A_0$ is called the rotation (or frequency) vector. Hence for $\varepsilon = 0$ all solutions are quasi-periodic. We are interested in the preservation of quasi-periodic solutions when $\varepsilon \neq 0$.

For $\varepsilon \neq 0$ we can write, as in [3],

$$\alpha = \alpha_0(t) + a(t), \qquad a(t) = \sum_{k=1}^{\infty} \varepsilon^k a^{(k)}(t), \tag{10}$$

where $a^{(k)}$ is determined as the solution of the equation

$$a^{(k)} = t\overline{A}^{(k)} + \overline{a}^{(k)} - \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \, \left[\partial_\alpha f(\alpha(\tau')\right]^{(k-1)}, \tag{11}$$

with $[\partial_\alpha f(\alpha(\tau')]^{(k-1)}$ expressed as in [6].

Since we are looking for quasi-periodic solutions with rotation vector ω we can try to write the solution as a Fourier series, by expanding

$$a^{(k)}(t) = \sum_{\nu \in \mathbb{Z}^N} e^{\mathrm{i}\nu\cdot\omega t} \, a_\nu^{(k)}, \tag{12}$$

with ω as before. If the series [10], with the Taylor coefficients as in [12], exists, it will describe a quasi-periodic solution analytic in ε, and in such a case we say that it is obtained by continuation of the unperturbed one with rotation vector ω, that is $\alpha_0(t)$.

Suppose that the integrand $[\partial_\alpha f(\alpha(\tau')]^{(k-1)}$ in [11] has vanishing average. Then the integral over τ' in [11] produces a quasi-periodic function, which in general have a non-vanishing average, so that the integral over τ produces a quasi-periodic function plus a term linear in t. If we choose $\overline{A}^{(k)}$ in [11] so as to cancel out exactly the term linear in time we end up with a quasi-periodic function. In Fourier space an explicit

calculation gives for all $\nu \neq 0$

$$
\begin{aligned}
a_\nu^{(1)} &= \frac{1}{(\omega\cdot\nu)^2}\,\mathrm{i}\nu f_\nu, \\
a_\nu^{(k)} &= \frac{1}{(\omega\cdot\nu)^2}\sum_{p=1}^{\infty}\sum_{\substack{k_1+\ldots+k_p=k-1\\ \nu_0+\nu_1+\ldots+\nu_p=\nu}}\frac{(\mathrm{i}\nu_0)^{p+1}}{p!}a_{\nu_1}^{(k_1)}\ldots a_{\nu_p}^{(k_p)}, \qquad k\geq 2,
\end{aligned}
\tag{13}
$$

which again is suitable for an iterative construction of the solution.

Of course the property that the integrand in [11] has zero average is fundamental, otherwise, terms increasing as powers of t would appear (the so called secular terms). Indeed it is easy to realize that to order k, terms proportional to t^{2k} could be present, so requiring, at best, $|\varepsilon| < |t|^{-2}$ for convergence up to time t, and so excluding a fortiori the possibility of quasi-periodic solutions.

The aforementioned property of zero average can be verified only if the rotation vector is non-resonant, that is if its components are rationally independent, or more particularly if the Diophantine condition [7] is satisfied. Such a result was first proved by Poincaré, and it holds however the parameters $\overline{a}^{(k)}$ appearing in [11] are fixed. This reflects the fact that quasi-periodic motions take place on invariant surfaces (KAM tori), which can be parameterized in terms of the angle variables $\alpha(t)$, so that the values $\overline{a}^{(k)}$ contribute to the initial phases, and the latter can be arbitrarily fixed.

The recursive equations [13] can be suitably studied by introducing a diagrammatic representation, as explained below.

Graphs and trees

A (connected) graph $\mathbb{G}$ is a collection of points (called vertices) and lines connecting all of them. We denote with $V(\mathbb{G})$ and $L(\mathbb{G})$ the set of vertices and the set of lines, respectively. A path between two vertices is a subset of $L(\mathbb{G})$ connecting the two vertices. A graph is planar if it can be drawn in a plane without graph lines crossing.

A tree is a planar graph $\mathbb{G}$ containing no closed loops (cycles); in other words, it is a connected acyclic graph. One can consider a tree $\mathbb{G}$ with a single special vertex $\mathbb{v}_0$: this introduces a natural partial ordering on the set of lines and vertices, and one can imagine that each line carries an arrow pointing toward the vertex $\mathbb{v}_0$. We can add an extra oriented line ℓ_0 connecting the special vertex $\mathbb{v}_0$ to another point which will be called the root of the tree; the added line will be called the root line. In this way we obtain a rooted tree θ defined by $V(\theta) = V(\mathbb{G})$ and $L(\theta) = L(\mathbb{G}) \cup \ell_0$. A labelled tree is a rooted tree θ together with a label function defined on the sets $V(\theta)$ and $L(\theta)$.

We shall call equivalent two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labelled trees, simply by considering equivalent two labelled trees if they can be transformed into each other in such a way that also the labels match.

Given two vertices $\mathbb{v}, \mathbb{w} \in P(\theta)$, we say that $\mathbb{w} \prec \mathbb{v}$ if $\mathbb{v}$ is on the path connecting $\mathbb{w}$ to the root line. We can identify a line with the vertices it connects;given a line $\ell = (\mathbb{v}, \mathbb{w})$ we say that ℓ enters $\mathbb{v}$ and exits $\mathbb{w}$. For each vertex $\mathbb{v}$ we define the branching number as the number $p_{\mathbb{v}}$ of lines entering $\mathbb{v}$.

The number of unlabelled trees with k vertices can be bounded by the number of random walks with $2k$ steps, that is by 4^k.

The labels are as follows: with each vertex $\mathbb{v}$ we associate a mode label $\nu_{\mathbb{v}} \in \mathbb{Z}^N$, and with each line we associate a momentum $\nu_\ell \in \mathbb{Z}^N$, such that the momentum of the line leaving the vertex $\mathbb{v}$ is given by the sum of the mode labels of all vertices preceding $\mathbb{v}$ (with $\mathbb{v}$ being included): if $\ell = (\mathbb{v}', \mathbb{v})$ then $\nu_\ell = \sum_{\mathbb{w} \preceq \mathbb{v}} \nu_{\mathbb{w}}$. Note that for a fixed unlabelled tree the branching labels are uniquely determined, and, for a given assignment of the mode labels, the momenta of the lines are uniquely determined too.

Define

$$V_{\mathbb{v}} = \frac{(\mathrm{i}\nu_{\mathbb{v}})^{p_{\mathbb{v}}}}{p_{\mathbb{v}}!} f_{\nu_{\mathbb{v}}}, \qquad g_\ell = \frac{1}{(\omega \cdot \nu_\ell)^2}, \tag{14}$$

where we refer to the tensor $V_{\mathbb{v}}$ as the node factor of $\mathbb{v}$ and to the scalar g_ℓ as the

propagator of the line ℓ. One has $|f_\nu| \leq F\, e^{-\kappa|\nu|}$ for suitable positive constants F and κ, by the analyticity assumption. Then one can check that the coefficients $a_\nu^{(k)}$, defined in [12], for $\nu \neq 0$ can be expressed in terms of trees as

$$a_\nu^{(k)} = \sum_{\theta \in \Theta_\nu^{(k)}} \mathrm{Val}(\theta), \qquad \mathrm{Val}(\theta) = \Big(\prod_{\mathbb{v} \in V(\theta)} V_{\mathbb{v}} \Big) \Big(\prod_{\ell \in L(\theta)} g_\ell \Big), \qquad [15]$$

where $\Theta_\nu^{(k)}$ denotes the set of all inequivalent trees with k vertices and with momentum ν associated with the root line, while the coefficients $a_0^{(k)}$ can be fixed $a_0^{(k)} = 0$ for all $k \geq 1$, by the arbitrariness of the initial phases previously remarked.

The proof of such an assertion can be performed by induction on k. Alternatively we can start from the recursive definition [13], whereby the trees naturally arise in the following way.

Represent graphically the coefficient $a_\nu^{(k)}$ as in Figure 1; to keep track of the labels k and ν we assign k to the black bullet and ν to the line. For $k = 1$ the black bullet is meant as a grey vertex.

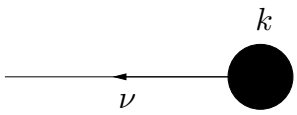

FIGURE 1

Then recursive equation [13] can be graphically represented as in Figure 2, provided that we associate with the (grey) vertex $\mathbb{v}_0$ the node factor $V_{\mathbb{v}_0}$, with $\nu_{\mathbb{v}_0} = \nu_0$ and $p_{\mathbb{v}_0} = p$ denoting the number of lines entering $\mathbb{v}_0$, and with the lines ℓ_i, $i = 1, \ldots, p$, entering $\mathbb{v}_0$ the momenta ν_{ℓ_i}, respectively. Of course the sums over p and over the possible assignments of the labels $\{k_i\}_{i=1}^p$ and $\{\nu_i\}_{i=0}^p$ are understood. Each black bullet on the right hand side of Figure 2, together with its exiting line looks like the diagram on the left hand side, so that it represents $a_{\nu_i}^{(k_i)}$, $i = 1, \ldots, p$. Note that Figure 2 has to be interpreted in the following way: if one associates with the diagram as drawn in the right hand side a numerical value (as described above)

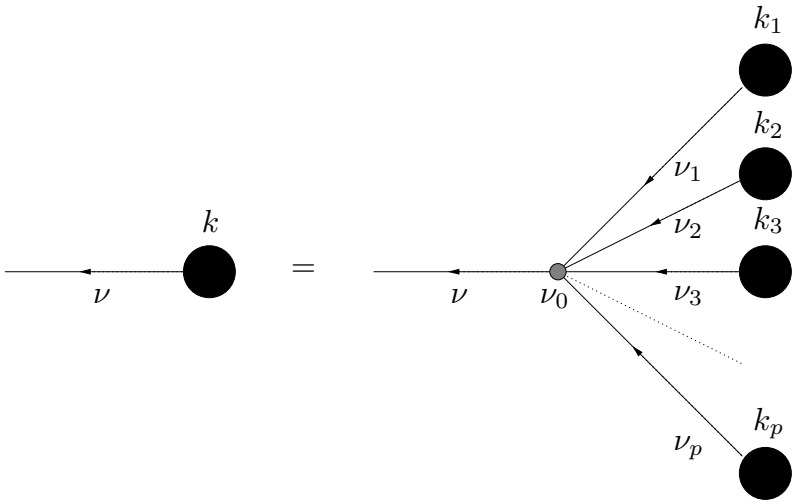

FIGURE 2

and one sums all the values over the assignments of the labels, then the resulting quantity is precisely $a_\nu^{(k)}$.

The (fundamental) difference between the black bullets on the right and left hand side is that the labels k_i of the latter are strictly less than k, hence we can iterate the diagrammatic decomposition simply by expressing again each $a_{\nu_i}^{(k_i)}$ as $a_\nu^{(k)}$ in [13], and so on, until one obtains a tree with k grey vertices and no black bullets; see Figure 3, where the labels are not explicitly written. This corresponds to the tree expansion [15].

The property that $[\partial_\alpha f(\alpha(\tau'))]^{(k-1)}$ in [11] has zero average for all $k \geq 1$ implies that for all lines $\ell \in L(\theta)$ one has $\nu_\ell \neq 0$, so that the propagators g_ℓ, hence the numerical values $\mathrm{Val}(\theta)$, are well defined for all trees θ.

Any tree appearing in [15] is an example of what is called in physics a Feynman graph, while the diagrammatic rules one has to follow in order to associate to the tree θ its right numerical value $\mathrm{Val}(\theta)$ are usually called the Feynman rules for the model under consideration. Such a terminology is borrowed from quantum field theory.

Multiscale analysis and clusters

Suppose we replace [9] with $\alpha = \varepsilon \partial_\alpha f(\alpha)$, so that no small divisors appear (that is

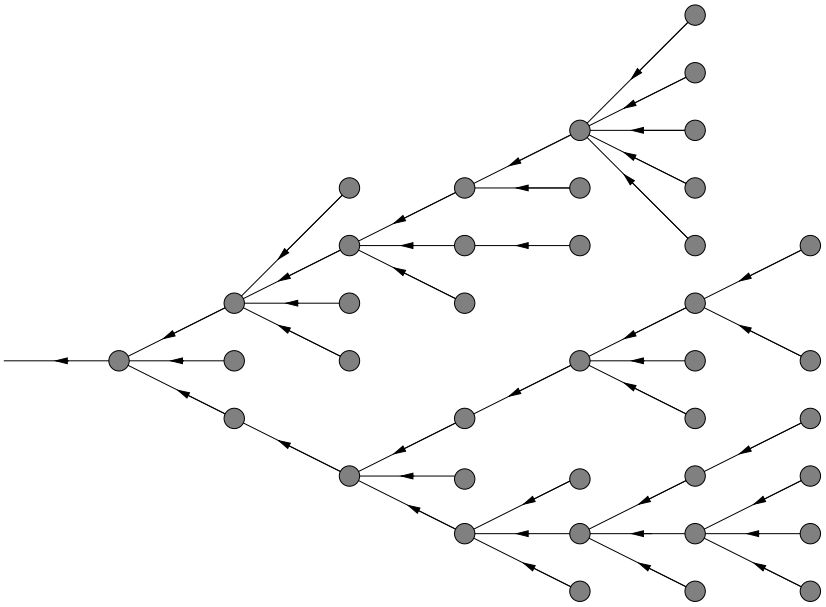

FIGURE 3

$g_\ell = 1$ in [14]). Then convergence is easily proved for ε small enough, since (by using $e^{-x}x^k/k! \leq 1$ for all $x \in \mathbb{R}_+$ and all $k \in \mathbb{N}$) one finds

$$\prod_{\mathbb{v}\in V(\theta)} |V_{\mathbb{v}}| \leq \left(\frac{4^3 F}{\kappa^3}\right)^k e^{-\kappa|\nu|/2} \Big(\prod_{\mathbb{v}\in V(\theta)} e^{-\kappa|\nu_{\mathbb{v}}|/2} \Big), \qquad [16]$$

and the sum over the mode labels can be performed by using the exponential decay factors $e^{-\kappa|\nu_{\mathbb{v}}|/2}$, while the sum over all possible unlabelled trees gives 4^k.

Of course the interesting case is when the propagators are there. In such a case, even if no division by zero occur, as $\omega\cdot\nu_\ell \neq 0$ (by the assumed Diophantine condition [13] and the absence of secular terms previously discussed), the quantities $\omega \cdot \nu_\ell$ in [14] can be very small.

Then we can introduce a scale h characterising the size of each propagator: we say that a line ℓ has scale $h_\ell = h \geq 0$ if $\omega \cdot \nu_\ell$ is of order $2^{-h}C_0$ and scale $h_\ell = -1$ if $\omega \cdot \nu_\ell$ is greater than C_0 (of course a more formal definition can be easily envisaged,

but we refer to the original papers for this). Then we can write

$$\prod_{\ell \in L(\theta)} |g_\ell| \leq C_0^{-2k} \prod_{h=0}^{\infty} 2^{2hN_h(\theta)} \leq C_0^{-2k} 2^{2h_0 k} \exp\Big(\sum_{h=h_0}^{\infty} \log 2\, 2hN_h(\theta) \Big), \qquad [17]$$

where $N_h(\theta)$ is the number of lines in $L(\theta)$ with scale h and h_0 is a (so far arbitrary) positive integer. The problem is then reduced to that of finding an estimate for $N_h(\theta)$.

To identify which kinds of tree are the source of problems, we introduce the notion of a cluster and a self-energy graph. A cluster T with scale h_T is a connected set of nodes which are linked by a continuous path of lines with the same scale label h_T or a lower one and which is maximal, namely all the lines not belonging to T but connected to it have scales higher than h_T. An inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. Each cluster T can have an arbitrary number of lines coming in it (entering lines), but only one or zero lines coming out from it (exiting line): we call external lines of T the lines which either enter or exit T. A cluster T with only one entering line ℓ_T^2 and with one exiting line ℓ_T^1 such that one has $\nu_{\ell_T^1} = \nu_{\ell_T^2}$ will be called a self-energy graph (SEG) or resonance. In such a case we shall call a resonant line the line ℓ_T^1. Examples of clusters and SEGs are suggested by the bubbles in Figure 4; the mode labels are not represented, whereas the scales are explicitly written. Note that in Figure 4 the tree itself is a cluster (with scale 6), and each of the two clusters with one entering and one exiting lines is a SEG only if the momenta of its external lines are equal to each other.

If we call $S_h(\theta)$ the number of SEGs whose resonant lines have scales h, then $N_h^*(\theta) = N_h(\theta) - S_h(\theta)$ will denote the number of non-resonant lines with scale h.

A fundamental result, known as Siegel-Bryuno lemma, shows that, for some positive constant c, one has

$$N_h^*(\theta) \leq 2^{h/\tau} c \sum_{\mathbb{v} \in V(\theta)} |\nu_{\mathbb{v}}|, \qquad [18]$$

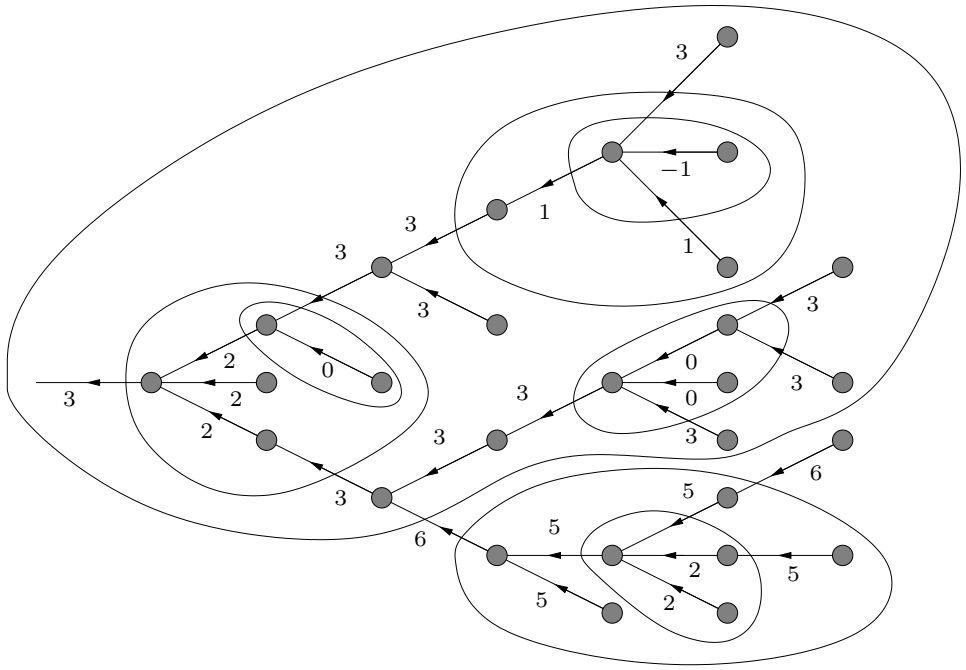

FIGURE 4

which, if inserted into [17] instead of $N_h(\theta)$, would give a convergent series; then h_0 should be chosen in such a way that the sum of the series in [17] is less than, say, $\kappa/4$.

The bound [18] is a very deep one, and was originally proved by Siegel for a related problem (Siegel's problem), in which, in the formalism we are following, SEGs do not occur; such a bound essentially shows that accumulation of small divisors is possible only in the presence of SEGs. A possible tree with k vertices whose value can be proportional to some power of $k!$ is represented in Figure 5, where a chain of $(k-1)/2$ SEGs, k odd, is drawn with external lines carrying a momentum ν such that $\omega \cdot \nu \approx |\nu|^{-\tau}$.

In order to take into account the resonant lines, we have to add a factor $(\omega \cdot \nu_\ell)^{-2}$ for each resonant line ℓ. It is a remarkable fact that, even if there are trees whose value cannot be bounded as a constant to the power k, there are compensations (that is partial cancellations) between the values of all trees with the same number of vertices, such that the sum of all such trees admits a bound of this kind.

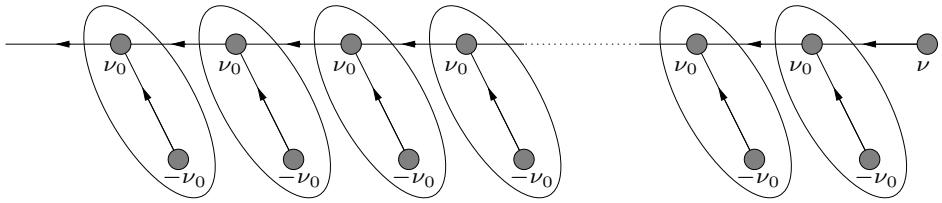

FIGURE 5

The cancellations can be described graphically as follows. Suppose we have a tree θ with a SEG T. Consider all trees which can be obtained by shifting the external lines of T, that is by attaching such lines to all possible vertices internal to T, and sum together the values of all such trees. An example is given in Figure 6. The corresponding sum turns out to be proportional to $(\omega \cdot \nu)^2$, if ν is the momentum of the resonant line of T, and such a factor compensates exactly the propagator of this line. The argument above can be repeated for all SEGs: this requires a little care because there are SEGs which are inside some other SEGs. Again we refer to the original papers for a more formal discussion.

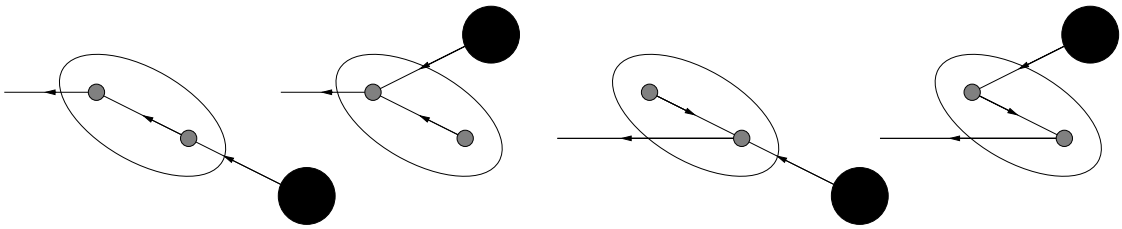

FIGURE 6

The conclusion is that we can take into account the resonant lines: this simply adds an extra constant to the power k, so that an overall estimate C^k, for some $C > 0$, follows for $U^{(k)}(t)$, and the convergence of the series follows.

Other examples and applications

The discussion carried out so far proves a version of the KAM theorem, for the system described by [9], and it is inspired by the original papers by Eliasson (1988) and, mostly, by Gallavotti (1994).

Here we list some problems in which original results have been proved by means of the diagrammatic techniques described above, or by some variants of them.

An expansion like the one considered so far can be envisaged also for the motions occurring on the stable and unstable manifolds of hyperbolic lower-dimensional tori for perturbations of Hamiltonians describing a system of rotators (as in the previous case) plus n pendulum-like systems. In such a case the function $G(u)$ has a less simple form. For $n = 1$ one can look for solutions which depend on time through two variables, $\psi = \omega t$ and $x = e^{-gt}$, with $(\omega, g) \in \mathbb{R}^{N+n}$, and ω Diophantine as before and g related to the time-scale of the pendulum. This has been worked out by Gallavotti (1994), and then used by Gallavotti, Gentile and Mastropietro (1999) to study a class of three time-scale systems, in order to obtain a lower bound on the homoclinic angles, that is on the angles between the stable and unstable manifolds of hyperbolic tori which are preserved by the perturbation. The formalism becomes a little more involved, essentially because of the entries of the Wronskian matrix appearing in [5]. In such a case the unperturbed solution $u_0(t)$ corresponds to the rotators moving linearly with rotation vector ω and the pendulum moving along its separatrix; a nontrivial fact is that if g_0 denotes the Lyapunov exponent of the pendulum in the absence of the perturbation, then one has to look for an expansion in $x = e^{-gt}$ with $g = g_0 + O(\varepsilon)$, because the perturbation changes the value of such an exponent.

The same techniques have also been applied to study the relation of the radius of convergence of the standard map, an area-preserving diffeomorphism from the cylinder to itself which has been widely studied in literature since the original papers by Greene and by Chirikov (1979), with the arithmetical properties of the rotation vector (which is in this case just a number). In particular it has been proved that the radius of convergence is naturally interpolated through a function of the rotation number known as Bryuno function (which has been introduced by Yoccoz as the

solution of a suitable functional equation completely independent of the dynamics); we refer to a paper by Berretti and Gentile (2001) for a review of results of this and related problems.

Also the generalized Riccati equation $\dot{u} - \mathrm{i}u^2 - 2\mathrm{i}f(\omega t) + \mathrm{i}\varepsilon^2 = 0$, where $\omega \in \mathbb{T}^d$ is Diophantine and f is an analytic periodic function of $\psi = \omega t$, has been studied with the diagrammatic technique by Gentile (2003). The interest of such an equation relies on the fact that it is related to quantum two-level systems (as first used by Barata), and existence of quasi-periodic solutions of the generalized Riccati equation for a large measure set $\mathcal{E}$ of values of ε can be exploited to prove that the spectrum of the corresponding two-level system is pure point for those values of ε; analogously one can prove that, for fixed ε, one can impose some further non-resonance conditons on ω, still leaving a full measure set, in such a way that the spectrum is pure point. We note also, as a side remark, that, technically, such a problem is very similar to that of studying conservation of elliptic lower-dimensional tori with one normal frequency

Finally we mention a problem of partial differential equations, where, of course, the scheme described above has to be suitably adapted: this is the study of periodic solutions for the nonlinear wave equation $u_{tt} - u_{xx} + mu = \varphi(u)$, with Dirichlet boundary conditions, where m is a real parameter and $\varphi(u)$ is a strictly nonlinear analytic function. Gentile and Mastropietro (2004) reproduced the result of Craig and Wayne of existence of periodic solutions for a large measure set of periods, and, in a subsequent paper by the same authors with Procesi, an analogous result was proved in the case $m = 0$, which was an open problem in literature.

See also

KAM theory and celestial Mechanics. Stability theory and KAM. Stationary solutions of PDEs and heterocline/homocline connections of dynamical systems. Integrable systems and differential geometry. Perturbations of integrable Hamiltonian systems.

Further Reading

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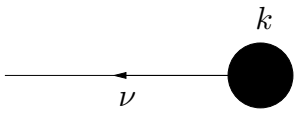

FIGURE 1

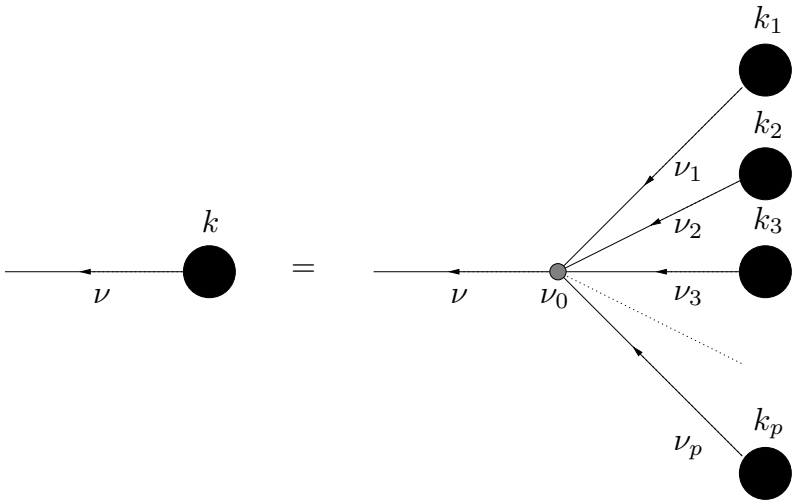

FIGURE 2

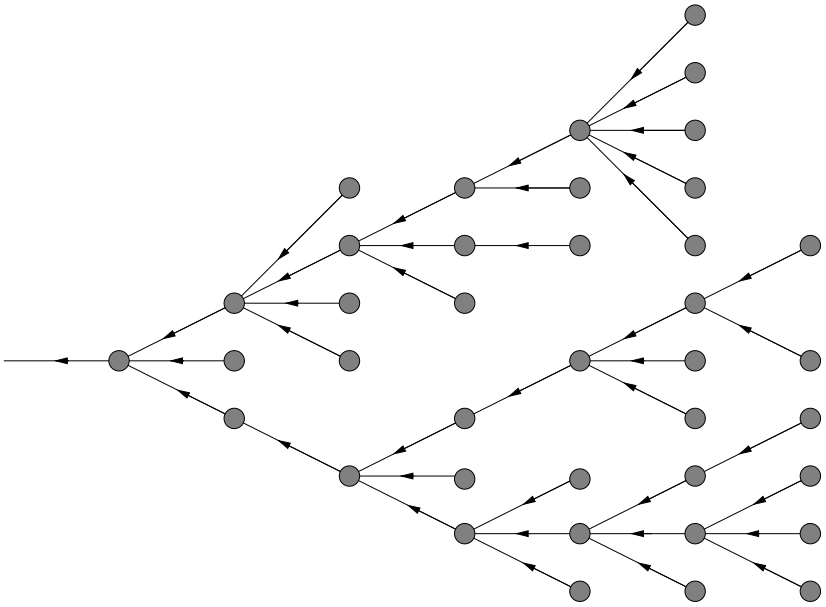

Figure 3

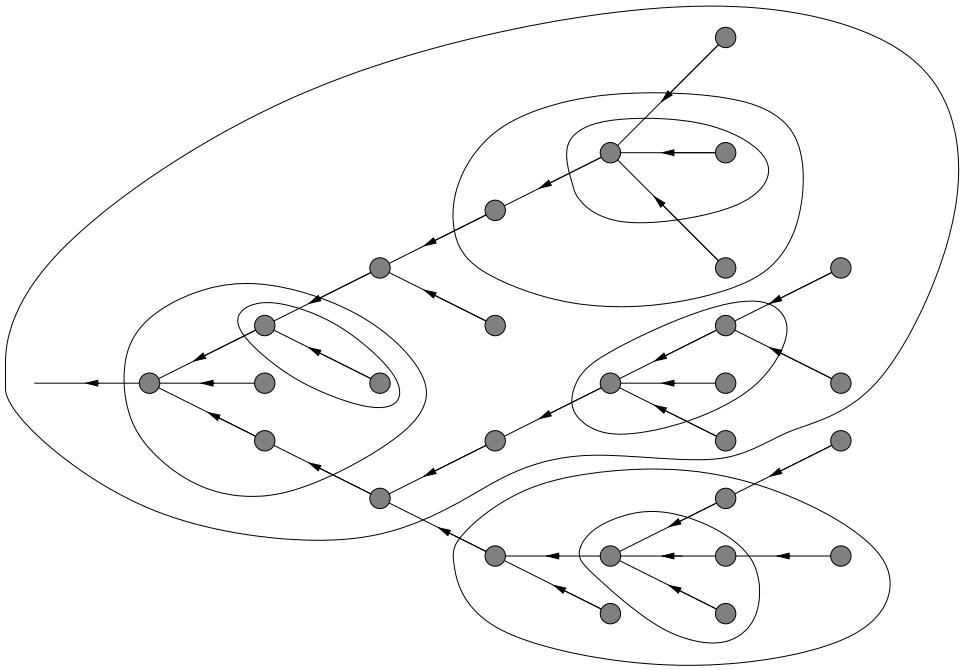

Figure 4

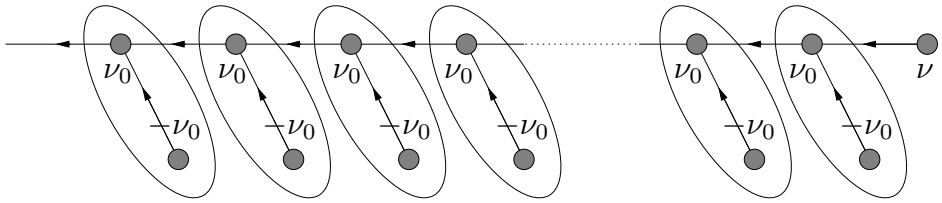

FIGURE 5

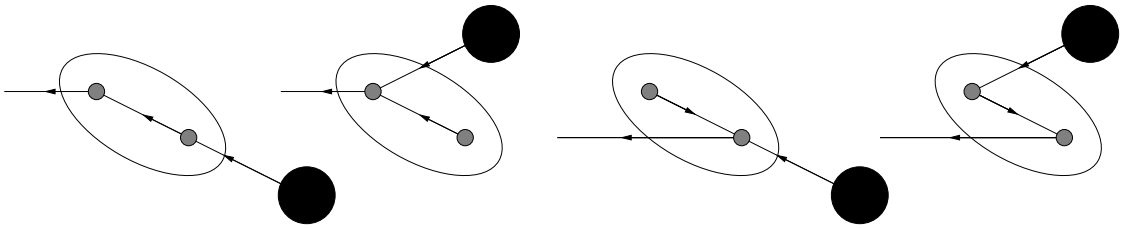

FIGURE 6