Anderson Localization for the Holstein model

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ABSTRACT. *A one-dimensional system of electrons on a lattice, interacting with a periodic potential, with period incommensurate with the lattice spacing and satisfying a Diophantine condition, is considered in the case of strong interaction. The Schwinger functions are computed and their asymptotic behaviour is studied, proving Anderson localization. The decay of the Schwinger functions is shown to depend critically on the value of the chemical potential.*

1. Introduction

1.1. The experimental discovery of a class of anisotropic (one-dimensional) crystals exhibiting a periodic modulation of the atomic positions (called *charge density wave*), superimposed on the Bravais periodic lattice and such that the two periodicities are generally incommensurate (see for instance [AAR]), motivates the theoretical study of systems of electrons (describing the conduction electrons of such crystals) on a lattice subject to an incommensurate periodic potential. This problem is generally faced studying the discrete Schrödinger equation with an incommensurate potential or the Schrödinger equation with a quasi-periodic potential. Also mathematically a lot of efforts was devoted to it; see for instance the review [PF] for references. The solutions of the Schrödinger equation were analysed by a perturbation theory (in the two opposite situations of small and large potential) which is afflicted by a *small divisor problem.* In particular if the potential is small the solutions are *quasi-Bloch* waves, while for large potentials they are exponentially localized.

However the results available for the Schrödinger equation do not apply immediately to the physical situations described above, as they concern single-particle properties and not statistical many-particle ones. It is true that neglecting the interaction between electrons the many-electron wave-functions could be constructed starting from the single-electron ones, but the deduction of statistical observables like the correlation functions could be not straightforward.

A different approach was started in [BGM] studying directly in the thermodynamic limit the correlation functions of the one-dimensional *Holstein model*, describing a system of electrons on a lattice interacting with a periodic potential, in the case of weak interaction and period incommensurate with the lattice spacing. The analysis was performed by using the techniques and methods typical of the Renormalization Group (RG): the correlation functions are written as *Grassmanian integrals* expressed by a series plagued by a small denominator problem. The convergence was proved using techniques similar to the ones adopted for the convergence of the invariant tori of perturbed Hamiltonians in Classical Mechanics, see [G1,GM]. In fact a formulation of the KAM problem in terms of quantum field theory has been recently proposed, see

[FT,G2,GGM,BGK].

The advantage of such method is twofold. While in fact one has directly the correlation functions of the system in the thermodynamic limit, which can be compared in principle with the experiments, the above techniques can take into account also the two-body interaction between electrons, what is important to make the model more realistic. This was done in [M], by considering the *Holstein-Hubbard* model (similar to the Holstein model but in which the electron-electron interaction is added). The non-analytic dependence of the correlation functions on the amplitude of the electron-electron interaction (due to the *Luttinger liquid* nature of $d = 1$ interacting fermions) found in [M] shows that it is not possible to study this model by an analytic continuation from the non-interacting limit, what means that the results found for the Schrödinger equation cannot be used directly in this problem.

In this work we continue our study of the Holstein model considering the opposite limit in which the interaction with the incommensurate potential is very large (near to the *anti-integrable limit*). The analysis is performed perturbatively by considering as the "free Hamiltonian" the interaction of the fermions with the external potential, while the hopping part is treated as a perturbation. We write the *Schwinger functions* (defined in the next section) as functional integrals which will be proved to admit a well defined thermodynamic limit. A large distance exponential decay of the Schwinger functions is also obtained. Our method would allow us in principle to consider also the interaction between electrons.

1.2. The Hamiltonian of the Holstein model is given by

$$H' = \sum_{x,y\in\Lambda} t_{x,y}\, \psi^+_x \psi^-_y + \lambda(\mu+\nu) \sum_{x\in\Lambda} \psi^+_x \psi^-_x - \lambda \sum_{x\in\Lambda} \varphi_x \psi^+_x \psi^-_x \;, \tag{1.1}$$

where x, y are points on the one-dimensional lattice Λ with unit spacing and length L; we shall identify Λ with $\{x \in \mathbb{Z} :\ -[L/2] \le x \le [(L-1)/2]\}$. Moreover the matrix $t_{x,y}$ is defined as $t_{x,y} = \delta_{x,y} - (1/2)[\delta_{x,y+1} + \delta_{x,y-1}]$, where $\delta_{x,y}$ is the Kronecker delta: the term in (1.1) containing the matrix $t_{x,y}$ (the only surviving one for $\lambda = 0$) is usually referred to as *hopping term.* The fields $\psi^\pm_x$ are creation $(+)$ and annihilation $(-)$ fermionic fields. The *potential* φ_x is a smooth function physically representing the interaction with the phonon field. In (1.1) λ is the interaction strength, $\lambda\mu$ is the chemical potential and ν is a *counterterm* to be fixed in a proper way; we will discuss its physical meaning in §2.11 below.

We shall consider potentials φ_x which are of the form $\varphi_x = \bar\varphi(\omega x)$, where $\bar\varphi$ is a real function on the real line periodic of period 1 and ω, the *rotation number*, is an irrational number (satisfying some Diophantine condition; see the statement of the theorem 1.4 below).

We are interested in the behaviour of the Schwinger functions of the model (1.1) for large $|\lambda|$; then it is more natural to consider the Hamiltonian obtained from (1.1) by dividing it by λ:

$$\begin{aligned} H &= H_0 + V \;, \\ H_0 &= \sum_{x\in\Lambda} (\mu - \varphi_x)\, \psi^+_x \psi^-_x \;, \\ V &= -\frac{\varepsilon}{2} \sum_{x\in\Lambda} \left[\psi^+_x \psi^-_{x+1} + \psi^+_x \psi^-_{x-1} - 2\psi^+_x \psi^-_x \right] + \nu \sum_{x\in\Lambda} \psi^+_x \psi^-_x \;, \end{aligned} \tag{1.2}$$

where $\varepsilon \equiv 1/\lambda$ can be considered a small parameter.

Given a Hamiltonian H, the finite-temperature two-point Schwinger function (two-point *Matsubara functions*, [NO]) for H is defined as

$$S_{L,\beta}(\mathbf{x},\mathbf{y}) = \frac{\mathrm{Tr} e^{-\beta H} T \psi^-_{\mathbf{x}} \psi^+_{\mathbf{y}}}{\mathrm{Tr} e^{-\beta H}} \;, \tag{1.3}$$

where $\psi^\pm_{\mathbf{x}} = e^{x_0 H} \psi^\pm_x e^{-x_0 H}$, with $\mathbf{x} = (x, x_0)$, $-\beta/2 \le x_0 \le \beta/2$ for some $\beta > 0$, and T is the time-ordering operator, [NO].

We shall impose free boundary conditions on x and antiperiodic boundary conditions on x_0, [NO]. Note that the free boundary conditions on x allow potentials of the considered form even at finite L.

1.3. If $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the one dimensional torus, $\|\cdot\|_{\mathbb{T}}$ is the distance on $\mathbb{T}$ and, given a C^s function $f(t)$, $\partial^s f(t)$ denotes its s-th derivative with respect to t, *i.e.* $\partial^s f(t) \equiv d^s f(t)/dt^s$; in particular we shall set $f'(t) = \partial f(t)$.

We shall prove the following theorem. The potential φ_x in (1.1) will be chosen to be an *even* C^1 function, essentially for expository clearness; however we stress since now that what really matters is the form of the potential φ_x near the critical points of $\varphi_x - \mu$. Extensions will be discussed in §4.

1.4. Theorem. *Let be $\varphi_x = \bar\varphi(\omega x)$ an even function in $C^1(\mathbb{T})$, i.e. $\varphi_x = \varphi_{-x}$, $\bar\varphi(x) = \bar\varphi(x+1)$, and with ω verifying a Diophantine condition*

$$\|\omega n\|_{\mathbb{T}} \ge C_0 |n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} , \tag{1.4}$$

for some constants $\tau > 1$ and $C_0 > 0$. Let us define $\bar\omega \equiv \omega \bar x$ such that $\mu = \bar\varphi(\bar\omega)$ and assume that there is only one $\bar x \in (0, 1/2)$ satisfying such a condition and that $\bar\varphi'(\bar\omega) \neq 0$.

Then there exists $\varepsilon_0 > 0$, depending on ω and $\bar\omega$, and, for $|\varepsilon| < \varepsilon_0$, a function $\nu \equiv \nu(\varepsilon) \neq 0$, such that
(1) if $\bar\omega \notin \omega\mathbb{Z}$ mod 1 and the additional Diophantine condition

$$\|\omega n \pm 2\bar\omega\|_{\mathbb{T}} \ge C_0 |n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} , \tag{1.5}$$

is verified, then the two-point Schwinger function $S_{L,\beta}(\mathbf{x}, \mathbf{y})$ admits a limit

$$\lim_{\beta\to\infty} \lim_{L\to\infty} S_{L,\beta}(\mathbf{x}, \mathbf{y}) = S(\mathbf{x}, \mathbf{y})$$

bounded by

$$|S(\mathbf{x}, \mathbf{y})| \le \log\left(1 + \min\{|x|, |y|\}\right)^{\tau} \frac{C_N \exp\left\{-4^{-1}|x-y| \log\left|\varepsilon^{-1}\right|\right\}}{1 + \left[(1 + \min\{|x|, |y|\})^{-\tau} |x_0 - y_0|\right]^N} , \tag{1.6}$$

for any $N \ge 1$ and for some constant C_N depending on N;
(2) if $2\bar\omega = (2k+1)\omega$ mod 1, $k \in \mathbb{N}$, then, for $\alpha = 2(k+1)$ and for some constant C'_N depending on N,

$$|S(\mathbf{x}, \mathbf{y})| \le \log\max\{(1 + \min\{|x|, |y|\})^{-\tau}, \sigma\} \frac{C'_N \exp\left\{-(4\alpha)^{-1}|x-y| \log\left|\varepsilon^{-1}\right|\right\}}{1 + \left[\max\{(1 + \min\{|x|, |y|\})^{-\tau}, \sigma\} |x_0 - y_0|\right]^N} , \tag{1.7}$$

with $0 \le \sigma \le C\,|\varepsilon|^{\eta(k)}$, where

$$\eta(k) = \begin{cases} (2k+1)/4 , & k > 1 , \\ 1 , & k = 1 , \end{cases} \tag{1.8}$$

for some constant C.

1.5. Note that both the conditions (1) and (2) in the theorem above exclude $\bar x$ to be a point on the lattice, so avoiding divergences in the free propagator (see (2.3) below). We see that the two-point Schwinger function decays exponentially for large spatial distances with rate $O(|\log|\lambda||^{-1})$, for large λ. This is a consequence of the *Anderson localization* of the solutions of the Schrödinger equation with a large incommensurate potential, see [PF]. However the decay for large values of $|x_0 - y_0|$ is different in the two cases, corresponding to two choices of the chemical potential (hence of the fermion density). This is due to the presence of a (possibly vanishing) gap in the ground state energy of H in correspondence of the choice of the chemical potential done in case (2) of the theorem. We are not able to prove that in general these gaps are nonvanishing and we find only an upper bound for them, except in the case $\bar\omega \equiv \omega/2$ (in which a bound from below is possible).

1.6. The above theorem will be proven in §2 and §3, referring to the Appendices for the technical aspects. In §4 we shall deal with the problem of extending the results to more general potentials: this will lead to theorem 4.6, whose physical relevance will be discussed in §4.9. Also a comparison will be presented therein with the existing literature about the Schrödinger equation, in particular with the results in [S,E].

2. Anomalous integration and effective potential

2.1. For $H \equiv H_0$, the two-point Schwinger function (1.3) is given by

$$g_{L,\beta}(\mathbf{x},\mathbf{y}) = \delta_{x,y} \frac{e^{-(\mu-\varphi_x)(x_0-y_0)} \left[\theta(x_0-y_0) - e^{-\beta(\mu-\varphi_x)}\theta(y_0-x_0)\right]}{1+e^{-\beta(\mu-\varphi_x)}} \,. \tag{2.1}$$

If $g_{L,\beta}(\mathbf{x},\mathbf{y}) \equiv g_{L,\beta}(x,y;\tau)$, with $\tau = x_0 - y_0$, then, for $-\beta \le \tau \le 0$, one has $g_{L,\beta}(x,y;\tau+\beta) = -g_{L,\beta}(x,y;\tau)$. Therefore we can write

$$g_{L,\beta}(x,y;\tau) = \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} e^{-ik_0\tau} \hat g_{L,\beta}(x,y;k_0) \,, \tag{2.2}$$

where $\mathcal{D}_\beta = \{k_0 = (2n+1)\pi\beta^{-1},\, n\in\mathbb{Z}\}$ and

$$\hat g_{L,\beta}(x,y;k_0) = \int_{-\beta/2}^{\beta/2} d\tau\, e^{ik_0\tau} g_{L,\beta}(x,y;\tau) = \delta_{x,y}\hat g(x,k_0) \equiv \frac{\delta_{x,y}}{-ik_0-\varphi_x+\mu} \,. \tag{2.3}$$

Let we introduce a cut-off M so that $k_0 = 2(n+1/2)\pi/\beta, n\in\mathbb{Z}, -M\le n\le M-1$.

The Schwinger function (1.3) can be written as power series in ε, convergent for $|\varepsilon| \le \varepsilon_\beta$, for some constant ε_β (the only trivial bound of ε_β goes to zero, as $\beta\to\infty$). This power expansion can be constructed in the usual way, [NO], in terms of Feynman graphs (in this case only chains, since the interaction is quadratic in the fields), by using as *free propagator* the function (2.2): in the following we shall look for a different expansion which will turn out to be more suitable to find a nontrivial bound for ε_β.

2.2. We introduce a finite set of *Grassmanian variables* $\psi^{\pm}_{x,k_0}$ (one for each of the allowed values for $x\in\Lambda$ and $k_0\in\mathcal{D}_\beta$, provided that the ultraviolet cut-off M has been introduced and L is kept finite) and a linear functional $P(d\psi)$ on the generated Grassmanian algebra, such that

$$\int P(d\psi)\psi^-_{x,k_0}\psi^+_{y,k'_0} = \delta_{x,y}\delta_{k_0,k'_0}\frac{\beta}{-ik_0-\varphi_x+\mu} \equiv \beta\,\delta_{x,y}\delta_{k_0,k'_0}\,\hat g(x,k_0) \,. \tag{2.4}$$

The integration $P(d\psi)$ has a simple representation in terms of the Grassmanian integration $d\psi^- d\psi^+$, defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(\psi^-,\psi^+)$ in the variables $\psi^-_{x,k_0}, \psi^+_{x,k_0}$,

$$\int d\psi^- d\psi^+ Q(\psi^-,\psi^+) = \begin{cases} 1 & \text{if } Q(\psi^-,\psi^+) = \prod_{x,k_0}\psi^-_{x,k_0}\psi^+_{x,k_0} \\ 0 & \text{otherwise} \,. \end{cases} \tag{2.5}$$

We have

$$P(d\psi) = d\psi^- d\psi^+ \Big\{\prod_{x,k_0}(\beta\hat g(x,k_0))\Big\} \exp\Big\{-\sum_{x,k_0}[\beta\hat g(x,k_0)]^{-1}\,\psi^+_{x,k_0}\psi^-_{x,k_0}\Big\} \,. \tag{2.6}$$

Note that, as $\psi^+_{x,k_0}\psi^+_{x,k_0} = \psi^-_{x,k_0}\psi^-_{x,k_0} = 0$, then $e^{-z\psi^+_{x,k_0}\psi^-_{x,k_0}} = 1 - z\psi^+_{x,k_0}\psi^-_{x,k_0}$, for any complex z.

By using standard arguments (see, for example, [NO], where a different regularization of the propagator is used), one can show that the Schwinger functions can be calculated as “expectations” of suitable functions of the Grassmanian variables with respect to the "measure" (2.6). In particular, the two-point Schwinger function, which in our case determines the other Schwinger functions through the Wick rule (as the Hamiltonian is quadratic in the fermionic fields), can be written as

$$S_{L,\beta}(\mathbf{x},\mathbf{y}) = \frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta} e^{-ik_0(x_0-y_0)^+}\hat{S}(x,y;k_0)\;, \qquad \hat{S}_{L,\beta}(x,y;k_0) = \lim_{M\to\infty}\frac{\int P(d\psi)\, e^{\mathcal{V}(\psi)}\,\psi^-_{x,k_0}\psi^+_{y,k_0}}{\int P(d\psi)\, e^{\mathcal{V}(\psi)}}\;, \tag{2.7}$$

with

$$\mathcal{V}(\psi) = \frac{\varepsilon}{2}\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\left[\psi^+_{x,k_0}\psi^-_{x+1,k_0} + \psi^+_{x,k_0}\psi^-_{x-1,k_0}\right] - \nu_0\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\psi^+_{x,k_0}\psi^-_{x,k_0}\;, \tag{2.8}$$

where $\nu_0 = \nu + \varepsilon$.

2.3. *Remark.* The *ultraviolet cut-off* M on the k_0 variable was introduced in order to give a precise meaning to the Grassmanian integration (so that Λ and $\mathcal{D}_\beta$ become indeed finite sets, hence the numerator and the denominator in (2.7) are finite sums), but it does not play any essential rôle in this paper, as all bounds will be uniform with respect to M and they easily imply the existence of the limit. Hence, we shall not stress anymore the dependence on M of the various quantities we shall study.

2.4. For the Grassmanian integration (2.6), we can write $P(d\psi) = \prod_{h=h_\beta}^{1} P(d\psi^{(h)})$ for some $h_\beta \le 1$ to be fixed later (after (2.10) below). This can be done by setting

$$\psi^{\pm}_{x,k_0} = \bigoplus_{h=h_\beta}^{1}\psi^{(h)\pm}_{x,k_0}\;, \qquad \hat{g}(x,k_0) = \sum_{h=h_\beta}^{1}\hat{g}^{(h)}(x,k_0)\;, \tag{2.9}$$

where $\psi^{(h)\pm}_{x,k_0}$ are families of Grassman fields with propagators $\hat{g}^{(h)}(x,k_0)$ which are defined in the following way.

Note that, for small $\omega x'$ (mod 1),

$$\varphi_{x'+\rho\bar{x}} - \mu = \rho v_0\omega x' + \Phi^{\rho}_{x'}\;, \qquad v_0 = \partial\bar{\varphi}(\omega\bar{x})\;, \qquad \rho = \pm 1, \tag{2.10}$$

with $\Phi^{\rho}_{x'} = o(\|\omega x'\|_{\mathbb{T}})$; the parity assumptions on φ_x implies $\Phi^{1}_{x'} = \Phi^{-1}_{-x'}$.

Set $\mathbf{r} = (x,k_0)$ and $\bar{\mathbf{x}} = (\bar{x},0)$. Given $\mathbf{r}\in\Lambda\times\mathcal{D}_\beta$ define $\mathbf{r} = (x',k_0)$, where $x' = x-\rho\bar{x}$, with $\rho = \mathrm{sign}\,(\omega x)$, and set $\Lambda' = \{x' : x\in\Lambda \text{ such that } x = \rho\bar{x}+x' \text{ with } \rho = \mathrm{sign}\,(\omega x)\}$.

We introduce a *scaling parameter* $\gamma > 1$ and a function $\chi(x',k_0)\in C^\infty(\mathbb{T}^1\times\mathbb{R})$ such that and $\|\mathbf{r}'\|^2 \equiv k_0^2 + v_0^2\|\omega x'\|^2_{\mathbb{T}}$, then

$$\chi(\mathbf{r}') = \chi(-\mathbf{r}') = \begin{cases} 1 & \text{if } \|\mathbf{r}'\| < t_0 \equiv a_0/\gamma\;, \\ 0 & \text{if } \|\mathbf{r}'\| > a_0\;, \end{cases} \tag{2.11}$$

where a_0 is such that the supports of $\chi(\mathbf{r}-\bar{\mathbf{x}})$ and $\chi(\mathbf{r}+\bar{\mathbf{x}})$ are disjoint. Then define

$$\hat{f}_1(\mathbf{r}) = 1 - \chi(\mathbf{r}-\bar{\mathbf{x}}) - \chi(\mathbf{r}+\bar{\mathbf{x}}) \tag{2.12}$$

and, for any integer $h \le 0$,

$$f_h(\mathbf{r}') = \chi(\gamma^{-h}\mathbf{r}') - \chi(\gamma^{-h+1}\mathbf{r}')\;; \tag{2.13}$$

then, for any $\bar{h} < 0$, we have

$$\chi(\mathbf{r}') = \sum_{h=\bar{h}+1}^{0} f_h(\mathbf{r}') + \chi(\gamma^{-\bar{h}}\mathbf{r}')\;. \tag{2.14}$$

Note that, if $h \le 0$, $f_h(\mathbf{r}') = 0$ for $\|\mathbf{r}'\| < t_0\gamma^{h-1}$ or $\|\mathbf{r}'\| > t_0\gamma^{h+1}$.

We finally define, for any $h \le 0$,

$$\hat{f}_h(\mathbf{r}) = f_h(\mathbf{r}-\bar{\mathbf{x}}) + f_h(\mathbf{r}+\bar{\mathbf{x}}) \;, \tag{2.15}$$

and, for any $h \le 1$,

$$\hat{g}^{(h)}(\mathbf{r}) \equiv \frac{\hat{f}_h(\mathbf{r})}{-ik_0 - \varphi_x + \mu} \;. \tag{2.16}$$

The definition (2.15) also implies that, if $h \le 0$, the support of $\hat{f}_h(\mathbf{r})$ is the union of two disjoint sets, I_h^+ and I_h^-. In I_h^+, ωx (mod 1) is strictly positive and $\|\omega(x-\bar{x})\|_{\mathbb{T}} \le a_0\gamma^h/|v_0|$, while, in I_h^-, ωx (mod 1) is strictly negative and $\|\omega(x+\bar{x})\|_{\mathbb{T}} \le a_0\gamma^h/|v_0|$. Therefore, if $h \le 0$, we can write $\psi_{\mathbf{r}}^{(h)\pm}$ as the sum of two independent Grassman variables $\psi_{\mathbf{r},\rho}^{(h)\pm}$, $\rho = \pm 1$, with propagator

$$\int P(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{r}_1,\rho_1} \psi^{(h)+}_{\mathbf{r}_2,\rho_2} = \beta\, \delta_{\mathbf{r}_1,\mathbf{r}_2}\, \delta_{\rho_1,\rho_2}\, \hat{g}^{(h)}_{\rho_1}(\mathbf{r}_1) \;, \tag{2.17}$$

so that

$$\psi_{\mathbf{r}}^{(h)\pm} = \bigoplus_{\rho=\pm1} \psi_{\mathbf{r},\rho}^{(h)\pm} \;, \qquad \hat{g}^{(h)}(\mathbf{r}) = \sum_{\rho=\pm1} \hat{g}^{(h)}_{\rho}(\mathbf{r}) \;, \tag{2.18}$$

$$\hat{g}^{(h)}_{\rho}(\mathbf{r}) = \frac{\tilde{\theta}(\rho\omega x)\, \hat{f}_h(\mathbf{r})}{-ik_0 - \varphi_x + \mu} \;, \tag{2.19}$$

where $\tilde{\theta}(\cdot)$ is the (periodic) Heaviside function. If $\rho\omega x > 0$, we will write in the following $x = x' + \rho\bar{x}$; note that as $\bar{x} \notin \Lambda$ then $x' \notin \mathbb{Z}$. In order to simplify the notation, it will be useful in the following to denote $\hat{g}^{(1)}(\mathbf{r})$ also as $\hat{g}^{(1)}_1(\mathbf{r})$, with $x = x' + \bar{x}$.

It is easy to prove, by using (2.10), that, for any $h \le 1$ and any ρ,

$$|\hat{g}^{(h)}_{\rho}(\mathbf{r}')| \le G_0 \gamma^{-h} \;, \tag{2.20}$$

for a suitable positive constant G_0.

Finally note that, as $|k_0| \ge \pi/\beta$, one has $\|\mathbf{r}'\| \ge \pi/\beta$, so that $h_\beta = [\log(\pi/\beta)/\log\gamma]$ (where $[\cdot]$ denotes the integer part), *i.e.* $\gamma^{h_\beta} \approx \pi/\beta$.

2.5. In order to prove that the Schwinger functions in (2.7) exist, we start by studying the denominator

$$\int P(d\psi)\, e^{\mathcal{V}(\psi)} \;.$$

We perform the integration $P(d\psi)$ in the following way, defined by induction.

Assume that we have integrated all the fields with scale $1 \ge h' > h$ and we have to integrate the r.h.s. of

$$\int P(d\psi)\, e^{\mathcal{V}(\psi)} = e^{-E_h} \int P(d\psi^{(\le h)})\, e^{\mathcal{V}^{(h)}(\psi^{\le h})} \;, \tag{2.21}$$

where $\mathcal{V}^{(h)}$ is the *effective potential*

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{\rho_1,\rho_2=\pm1} \sum_{m=-\infty}^{\infty} \sum_{x\in\Lambda} \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} \psi^{(\le h)+}_{x'+\rho_1\bar{x},k_0,\rho_1} \psi^{(\le h)-}_{x'+\rho_2\bar{x}+[m+(\rho_1-\rho_2)\bar{x}],k_0,\rho_2} \mathcal{W}^{(h)}(x, x+m; k_0) \tag{2.22}$$

and E_h is defined iteratively, see (2.42) below. For $h = 1$ the integration is just given by (2.6).

The kernels $\mathcal{W}^{(h)}(x,x+m;k_0)$ are expressed as the sum of suitable *Feynman graphs*, see §2.8 below. The integration $P(d\psi^{(\le h)})$ is defined, for $h\le 0$, as

$$P(d\psi^{(\le h)}) = \prod_{\mathbf{r}'\in\Lambda'\times\mathcal{D}_\beta} \frac{1}{\mathcal{N}_h(\mathbf{r}')} \prod_{\rho=\pm1} d\psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} d\psi^{(\le h)-}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} \exp\Big\{ -\sum_{\rho=\pm1}\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta} C_h(\mathbf{r}')\cdot \\ \cdot\Big[\Big(-ik_0-\rho v_0\omega x'-\Phi^\rho_{x'}\Big)\psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho}\psi^{(\le h)-}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} - \sigma_h(\mathbf{r}')\,\psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho}\psi^{(\le h)-}_{\mathbf{r}'-\rho\bar{\mathbf{x}},-\rho}\Big]\Big\}\,, \tag{2.23}$$

where

$$C_h^{-1}(\mathbf{r}') = \sum_{j=h_\beta}^{h} f_j(\mathbf{r}')\,, \tag{2.24}$$

$$\mathcal{N}_h(\mathbf{r}') = \beta^{-1}C_h(\mathbf{r}')\left\{\left[-ik_0-\Phi^1_{x'}\right]\left[-ik_0-\Phi^{-1}_{x'}\right] - v_0^2\|\omega x'\|^2_{\mathbb{T}} - [\sigma_h(\mathbf{r}')]^2\right\} \tag{2.25}$$

and $\sigma_h(\mathbf{r}')$ is also defined iteratively, see (2.31) below.

We write

$$\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}\,, \tag{2.26}$$

where $\mathcal{L}$ is the *localization operator*, a linear operator such that

$$\mathcal{L}\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\psi^{(\le h)+}_{x'+\rho_1\bar{x},k_0,\rho_1}\psi^{(\le h)-}_{x'+\rho_2\bar{x}+[m+(\rho_1-\rho_2)\bar{x}],k_0,\rho_2}\mathcal{W}^{(h)}(x,x+m;k_0) \\ = \tilde{\delta}_{\omega m+(\rho_1-\rho_2)\bar{\omega},0}\,\mathcal{W}^{(h)}(\rho_1\bar{x},\rho_1\bar{x}+m;0)\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\psi^{(\le h)+}_{x'+\rho_1\bar{x},k_0,\rho_1}\psi^{(\le h)-}_{x'+\rho_2\bar{x}+[m+(\rho_1-\rho_2)\bar{x}],k_0,\rho_2}\,, \tag{2.27}$$

if $\tilde{\delta}_{x,y} = \sum_{k\in\mathbb{Z}}\delta_{x,y+k}$, and $\mathcal{R} = \mathbb{1}-\mathcal{L}$ is the *renormalization operator*.

2.6. *Remark.* Note that in case (1) of the theorem the condition defining the delta in (2.27) can be verified only if $\rho_1=\rho_2$ and $m=0$, while in case (2) it can be verified also if $\rho_1=-\rho_2$, $m=-\rho_1\,(2k+1)$.

2.7. Using the parity property of φ_x, we write, for $h\le 0$,

$$\mathcal{L}\mathcal{V}^{(h)} = \gamma^h\nu_h F^{(h)}_\nu + s_h F^{(h)}_\sigma\,, \tag{2.28}$$

where $F^{(h)}_\nu$ and $F^{(h)}_\sigma$ are given by

$$F^{(h)}_\nu = \sum_{\rho=\pm1}\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho}\psi^{(\le h)-}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho}\,, \\ F^{(h)}_\sigma = \sum_{\rho=\pm1}\sum_{x\in\Lambda}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho}\psi^{(\le h)-}_{\mathbf{r}'-\rho\bar{\mathbf{x}},-\rho}\,. \tag{2.29}$$

We write, for $h\le 0$,

$$e^{-E_h}\int P(d\psi^{(\le h)})\,e^{\mathcal{V}^{(h)}(\psi^{(\le h)})} = e^{-E_h-t_h}\int \tilde{P}(d\psi^{(\le h)})\,e^{\tilde{\mathcal{V}}^{(h)}(\psi^{(\le h)})}\,, \tag{2.30}$$

where $\tilde{P}(d\psi^{(\le h)})$ has the same form of $P(d\psi^{(\le h)})$ in (2.24) with $\sigma_h(\mathbf{r}')$ replaced by $\sigma_{h-1}(\mathbf{r}')$, where

$$\begin{cases}\sigma_{h-1}(\mathbf{r}') = \sigma_h(\mathbf{r}') + C_h^{-1}(\mathbf{r}')\,s_h\,, & h<0\,,\\ \sigma_0(\mathbf{r}') = 0\,,\end{cases} \tag{2.31}$$

and $\tilde{\mathcal{V}}^{(h)} = \mathcal{L}\tilde{\mathcal{V}}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, if

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)} = \gamma^h \nu_h F_\nu^{(h)} \tag{2.32}$$

is the *localized effective potential* on scale h. In (2.30) t_h takes into account the different normalizations of the two integrations and it is given, for $h \le -1$, by

$$t_h = -\sum_{\mathbf{r}'\in\Lambda'\times\mathcal{D}_\beta} \log\left[\frac{\left[-ik_0 - \Phi^1_{x'}\right]\left[-ik_0 - \Phi^{-1}_{x'}\right] - v_0^2\|\omega x'\|^2_{\mathbb{T}} - [\sigma_{h-1}(\mathbf{r}')]^2}{\left[-ik_0 - \Phi^1_{x'}\right]\left[-ik_0 - \Phi^{-1}_{x'}\right] - v_0^2\|\omega x'\|^2_{\mathbb{T}} - [\sigma_h(\mathbf{r}')]^2}\right] \; ;$$

a similar expression holds for $h = 0$. The r.h.s of (2.30) can be written as

$$e^{-E_h - t_h}\int P(d\psi^{(\le h-1)})\int \tilde{P}(d\psi^{(h)})\, e^{\tilde{\mathcal{V}}^{(h)}(\psi^{(\le h)})} \;, \tag{2.33}$$

where $P(d\psi^{(\le h-1)})$ and $\tilde{P}(d\psi^{(h)})$ are given by (2.23) with $\sigma_h(\mathbf{r}')$ replaced by $\sigma_{h-1}(\mathbf{r}')$, with $C_h(\mathbf{r}')$ replaced by $C_{h-1}(\mathbf{r}')$ and $f_h^{-1}(\mathbf{r}')$, respectively, and with $\psi^{(\le h)}$ replaced by $\psi^{(\le h-1)}$ and $\psi^{(h)}$, respectively.

Note that $\sigma_h(\mathbf{r}')$ is defined iteratively by (2.31), for all $h \le 0$ provided that $\sigma_0(\mathbf{r}') = 0$; by the k_0-dependence of the propagator one easily check that $\sigma_h(\mathbf{r}')$ is real. In case (1) of the theorem $s_0 = 0$ and also $s_j = 0$ for any $j < 0$ (see the remark 2.6), so that $\sigma_h(\mathbf{r}') = 0$ for any h.

On the other hand in case (2), by defining $\eta(k)$ as in (1.8), one has $|\sigma_h(\mathbf{r}')| \le C\,|\varepsilon|^{\eta(k)}$, for some constant C, as it will be proven in Appendix A2. Then, as a consequence of the change of the Grassmanian integration, (2.16) has to be replaced with

$$\hat{g}^{(h)}(\mathbf{r}) = \sum_{\rho,\rho'=\pm 1} f_h(\mathbf{r}')[T_h^{-1}(\mathbf{r}')]_{\rho,\rho'} \equiv \sum_{\rho,\rho'=\pm 1} \tilde{g}^{(h)}_{\rho,\rho'}(\mathbf{r}') \;, \tag{2.34}$$

where the 2×2 matrix $T_h(\mathbf{r}')$ has elements

$$\begin{cases} [T_h(\mathbf{r}')]_{1,1} = -ik_0 - v_0\omega x' - \Phi^1_{x'} \;, \\ [T_h(\mathbf{r}')]_{1,2} = [T_h(\mathbf{r}')]_{2,1} = -\sigma_h(\mathbf{r}') \;, \\ [T_h(\mathbf{r}')]_{2,2} = -ik_0 + v_0\omega x' - \Phi^{-1}_{x'} \;, \end{cases} \tag{2.35}$$

which is well defined on the support of $f_h(\mathbf{r}')$, so that, if we set

$$A_h(\mathbf{r}') \equiv \det T_h(\mathbf{r}') = [-ik_0 - \Phi^1_{x'}][-ik_0 - \Phi^{-1}_{x'}] - v_0^2\|\omega x'\|^2_{\mathbb{T}} - [\sigma_h(\mathbf{r}')]^2 \;, \tag{2.36}$$

then

$$T_h^{-1}(\mathbf{r}') = \frac{1}{A_h(\mathbf{r}')}\begin{pmatrix} [\tau_h(\mathbf{r}')]_{1,1} & [\tau_h(\mathbf{r}')]_{1,2} \\ [\tau_h(\mathbf{r}')]_{2,1} & [\tau_h(\mathbf{r}')]_{2,2} \end{pmatrix} \;, \tag{2.37}$$

with

$$\begin{cases} [\tau_h(\mathbf{r}')]_{1,1} = -ik_0 - \Phi^{-1}_{x'} + v_0\omega x' \;, \\ [\tau_h(\mathbf{r}')]_{1,2} = [\tau_0(\mathbf{r}')]_{2,1} = \sigma_h(\mathbf{r}') \;, \\ [\tau_h(\mathbf{r}')]_{2,2} = -ik_0 - \Phi^1_{x'} - v_0\omega x' \;. \end{cases} \tag{2.38}$$

For $h = 1$ we set $\tilde{g}^{(1)}_{1,1} = \hat{g}^{(1)}_1(\mathbf{r})$.

Moreover

$$\sigma_h(\mathbf{r}') = \sum_{j=h}^{0} C_j^{-1}(\mathbf{r}')\, s_j \;. \tag{2.39}$$

Note that there exists a constant G_1, such that

$$|\tilde{g}^{(h)}_{\rho,\rho'}(\mathbf{r}')| \le G_1\gamma^{-h} \;, \tag{2.40}$$

which can be proven as (2.20).

Integrating the $\psi^{(h)}$ field we find that (2.33) becomes

$$e^{-E_{h-1}} \int P(d\psi^{(\leq h-1)})\, e^{\mathcal{V}^{(h-1)}(\psi^{(\leq h-1)})} \;, \tag{2.41}$$

with

$$E_{h-1} = E_h + t_h + \tilde{E}_h; , \tag{2.42}$$

where $\tilde{E}_h = -\log \int \tilde{P}(d\psi^{(h)})\, e^{\tilde{\mathcal{V}}^{(h)}(\psi^{(h)})}$; we can consider (2.42) defined for any $h \leq 1$, provided that we set $E_1 = 0$ and $t_1 = 0$.

2.8. In order to perform some estimates it is convenient to introduce a diagrammatic representation of the effective potential $\tilde{\mathcal{V}}^{(h)}$, in terms of chain graphs described below.

A graph ϑ of order n is a chain of $n+1$ lines $\ell_1, \ldots, \ell_{n+1}$ connecting a set of n ordered points (*vertices*) $v_1, \ldots, v_n$, so that ℓ_i enters v_i and ℓ_{i+1} exits from v_i; ℓ_1 and ℓ_{n+1} are the *external lines* of the graph and both have a free extreme, while the others are the *internal lines*; we shall denote $\mathrm{int}(\vartheta)$ the set of all internal lines. We say that $v_i < v_j$ if v_i precedes v_j and we denote v_j' the vertex immediately following v_j, if $j < n$. We denote also by ℓ_v the line entering the vertex v, so that $\ell_i \equiv \ell_{v_i}$, $1 \leq i \leq n$. We say that a line ℓ emerges from a vertex v if ℓ either enters v ($\ell = \ell_v$) or exits from v ($\ell = \ell_{v'}$).

We shall say that ϑ is a *labelled graph* of order n and *external scale* h, if ϑ is a graph of order n, to which the following *labels* are associated:

- a label $\delta_v = 0, \pm 1$ for each vertex,
- a *scale* label h for both the external lines and a scale label $h_\ell \geq h+1$ for each $\ell \in \mathrm{int}(\vartheta)$,
- two labels $\rho^1_\ell, \rho^2_\ell = \pm 1$ for each line ℓ, setting $\rho^1_{\ell_1} = 0$ and $\rho^2_{\ell_{q+1}} = 0$,
- a momentum k_0 on each line, and
- a coordinate $x'_{\ell_1} = x'$, with $x = x' + \rho_1 \bar{x}$, for the first line and a coordinate

$$x'_{\ell_{v'}} = x' + \sum_{w \leq v} \left[\delta_w + \left(\rho^2_{\ell_w} - \rho^1_{\ell_{w'}} \right) \bar{x} \right]$$

for each other line $\ell_{v'}$.

Moreover, $h(\vartheta) \equiv \min_{\ell \in \mathrm{int}(\vartheta)} h_\ell$ will be called the *internal scale* or simply the *scale* of ϑ.

A graph of order n can be obtained from n graph elements formed by a vertex with two emerging half-lines (representing the left one a ψ^+ field and the right one a ψ^- field), by pairing the half-lines (*contractions*) in such a way that a line $\psi^\pm$ can be paired only with a line $\psi^\mp$ and the resulting graph turns out to be connected with only two half-lines left not contracted (the external lines of the graph).

Given a line ℓ, we associate to it a propagator $\tilde{g}^{(h_\ell)}_{\rho^1_\ell, \rho^2_\ell}(\mathbf{r}'_\ell)$. Given a vertex v one has $\delta_v = 0$ only if $h_{\ell_v} \leq 0$ and we associate to it a factor $\gamma^h \nu_h$, if $h = \min\{h_v, h_{v'}\}$; we say that such a vertex is a ν-vertex. If a vertex v has $\delta_v = \pm 1$, we associate to it simply a factor ε.

Given a labelled graph ϑ, we can consider a maximal connected subset T of lines ℓ in ϑ with scales $h_\ell \geq h_T$ and with at least one line on scale h_T. Then the external lines of T (*i.e.* the lines that have only one vertex inside T) have scale labels smaller than h_T. We shall say that T is a *cluster* of scale h_T. The vertices connected by the lines internal to T are said to belong to T.

An inclusion relation can be established between the clusters, in such a way that the innermost clusters are the clusters with the highest scale (*minimal clusters*), and so on. Note that ϑ itself is a cluster (of scale $h(\vartheta)$).

Each cluster T has an incoming line ℓ^i_T and an outgoing line ℓ^o_T; we set $x'_{\ell^o_T} - x'_{\ell^i_T} \equiv m_T + (\rho^2_{\ell^i_T} - \rho^1_{\ell^o_T})\bar{x}$, where

$$m_T = \sum_{v \in T} \delta_v + \sum_{\ell \in T} \left(\rho^2_\ell - \rho^1_\ell \right) \bar{x} \tag{2.43}$$

is an integer. The maximum between $h_{\ell_T^i}$ and $h_{\ell_T^o}$ will be called the *external scale* of T.

Note that, for $m_T \neq 0$,

$$2a_0\gamma^{h_{T'}}/|v_0| \geq \|\omega x'_{\ell_T^o}\|_{\mathbb{T}} + \|\omega x'_{\ell_T^i}\|_{\mathbb{T}} \geq \|\omega(x'_{\ell_T^o} - x'_{\ell_T^i})\|_{\mathbb{T}} = \|\omega m_T + (\rho^2_{\ell_T^i} - \rho^1_{\ell_T^o})\bar{\omega}\|_{\mathbb{T}} \;, \tag{2.44}$$

where $h_{T'}$ is the scale of the cluster immediately containing T (*i.e.* $h_{T'}$ is the external scale of T), so that

$$2a_0\gamma^{h_{T'}}/|v_0| \geq \begin{cases} C_0|m_T|^{-\tau} \;, & \text{case (1)} \;, \\ C_0\left(|m_T| + (2k+1)\right)^{-\tau} \;, & \text{case (2)} \;, \end{cases} \tag{2.45}$$

a key inequality which will be deeply used in the proof of Lemma 2.10 below (see Appendix A1).

We say that V is a *resonance* (or a *resonant cluster*) of ϑ, if $x'_{\ell_V^i} = x'_{\ell_V^o}$, *i.e.* if the Kronecker delta in the r.h.s of (2.28) is verified. On each resonance the $\mathcal{R}$ operation acts.

We define $\mathcal{T}^h_{n,m}$ the set of the graphs ϑ of order n and with external scale h, such that the difference between the coordinate x_{ℓ_1} of the entering and the coordinate $x_{\ell_{n+1}}$ of the exiting line is m, *i.e.*

$$\sum_{v\in\vartheta}\delta_v + \sum_{\ell\in\mathrm{int}(\vartheta)}\left(\rho^2_\ell - \rho^1_\ell\right)\bar{x} = m \;, \tag{2.46}$$

$h(\vartheta) = h+1$ and on each resonance the $\mathcal{R}$ operator acts.

Then we can write

$$\begin{aligned}
\mathcal{W}^{(h)}(x, x+m; k_0) &= \sum_{n=1}^{\infty}\mathcal{W}^{(h)}_n(x, x+m; k_0) \;, \\
\mathcal{W}^{(h)}_n(x, x+m; k_0) &= \sum_{n=1}^{\infty}\sum_{\vartheta\in\mathcal{T}^h_{n,m}}\mathrm{Val}(\vartheta) \;, \\
\mathrm{Val}(\vartheta) &= \varepsilon^n\left[\mathcal{R}\prod_{\ell\in\mathrm{int}(\vartheta)}\tilde{g}^{(h_\ell)}_{\rho^1_\ell,\rho^2_\ell}(\mathbf{r}'_\ell)\right]\left(\prod_{T\in\mathbf{T}}\left(\frac{\gamma^{h_T}\nu_{h_T}}{\varepsilon}\right)^{M^{(\nu)}_{T_0}}\right) \;,
\end{aligned} \tag{2.47}$$

where $\mathbf{T}$ is the set of clusters in ϑ, T_0 is the set of lines and vertices inside T and outside the clusters internal to T and $M^{(\nu)}_{T_0}$ is the number of ν-vertices in T_0. A resonance can be seen as a tree with external lines carrying the same coordinate labels $x'_{\ell_1} = x'_{\ell_{p+1}}$; in such a case we shall call $\mathcal{W}^{(h)}_n(\bar{x}, \pm\bar{x}; k_0)$ the *resonance value.*

Note that the $\mathcal{R}$ operator in (2.47) produces derivatives of the propagators: one can easily show that, for any values $\mathbf{r}'_1, \mathbf{r}'_2$,

$$\left|\frac{d}{dt}\tilde{g}^{(h_\ell)}_{\rho^1_\ell,\rho^2_\ell}(t\mathbf{r}'_1 + \mathbf{r}'_2)\right| \leq G_2\|\mathbf{r}'_2\|\gamma^{-2h} \;, \qquad t\in[0,1] \;, \tag{2.48}$$

for some constant G_2, a property that will be used in Appendix A1 to prove Lemma 2.10 below.

2.9. Let we define (G_1 is defined in (2.40))

$$h^* = \inf\{h \geq h_\beta : G_1\gamma^h \geq |\sigma_h|\} \;. \tag{2.49}$$

In case (1) of the theorem of course $h^* \equiv h_\beta$. In case (2) however $h^* \neq h_\beta$ generically (we cannot exclude that for some potential φ_x one has $\sigma_h = 0$ identically: no lower bounds for γ^{h^*} can be in general given). If one defines

$$\tilde{g}^{(\leq h^*)}_{\rho,\rho'}(\mathbf{r}') = \sum_{j=h_\beta}^{h^*}\tilde{g}^{(j)}_{\rho,\rho'}(\mathbf{r}') \;, \tag{2.50}$$

then

$$\tilde{g}^{(\le h^*)}_{\rho,\rho'}(\mathbf{r}') \le G_1 \gamma^{-h^*} \; ; \tag{2.51}$$

this means that, if $h^* > h_\beta$, the scales $\le h^*$ can be integrated all together.

The convergence of the effective potential is a consequence of the following lemma, proved in Appendices A1 and A2.

2.10. LEMMA. *If $\gamma > 2^\tau$, there exists ε_0 such that, for $|\varepsilon| \le \varepsilon_0$ and $h^* \le h \le 0$ ($h^* = h_\beta$ in case (1)), one has*

$$\left|\mathcal{W}^{(h)}_n(x, x+m; k_0)\right| \le C D^{n-1} |\varepsilon|^{n/4} e^{-\frac{1}{4\alpha}\log\left|\varepsilon^{-1}\right| |m|} \; , \tag{2.52}$$

for some constants C, D and $\alpha = 1$ in case (1), $\alpha = 2(k+1)$ in case (2).

2.11. Let us make some comments on the elaborate integration procedure described above. The series for the effective potential are plagued by a problem of small divisors similar to the one in the Lindstedt series for KAM tori of Hamiltonian systems close to integrable ones.

Retaining only the terms in the series with no resonances, it would be possible to show that, as a consequence of the Diophantine condition, a bound $O(|\varepsilon|^{n/4})$ for the graphs with n vertices could be obtained. According to RG approach, the resonance values are written as sums of two terms, using the decomposition $\mathbb{1} = \mathcal{L} + \mathcal{R}$ given by (2.27), and one considers a renormalized expansion in terms of graphs such that (1) on all the clusters the $\mathcal{R}$ operation acts and (2) the graph values depend also on a set of *running coupling constants*, which take into account the local part of the resonances; of course the action of $\mathcal{R}$ is trivial ($\mathcal{R} = \mathbb{1}$) except than for the resonances.

It is then possible to show that, if the running coupling constants admit a bound $O(|\varepsilon|)$, the renormalized graphs still admit a bound $O(|\varepsilon|^{n/4})$ (Appendix A1). However one has still to show that the running coupling constants are bounded (Appendix A2).

The running coupling constants have a clear physical meaning: the ν_h represent the renormalization of the chemical potential with respect to the $\varepsilon = 0$ case, while the σ_h, present only in the case (2), take into account the opening of a gap in the single-particle spectrum. The flow of the ν_h is controlled by adding a counterterm $\nu = \nu(\varepsilon)$ in the Hamiltonian. Note that also in the case with gaps it is necessary to fix the chemical potential as the gap is quite smaller than the chemical potential renormalization. The flow of σ_h is controlled by putting it in the Grassmanian integration; this is often referred to by saying that the "free measure" is changed as an effect of the interaction (*anomalous Grassmanian integration*).

3. The two-point Schwinger function

3.1. In this section we define a perturbative expansion, similar to the one discussed for the effective potential in the previous section, for the two-point Schwinger function, defined by (2.7), which can be rewritten, at finite L, β,

$$S_{L,\beta}(\mathbf{x},\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} \frac{1}{N_1} \int P(d\psi)\, e^{\mathcal{V}(\psi) + \int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} \Big|_{\phi^+ = \phi^- = 0} \; , \tag{3.1}$$

where $\int d\mathbf{x}$ is a shortcut for $\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2} dx_0$, $N_1 = \int P(d\psi)\, e^{\mathcal{V}(\psi)}$ and $\{\phi^\pm_{\mathbf{x}}\}$ are Grassmanian variables (the *external field*), anticommuting with $\{\psi^\pm_{\mathbf{x}}\}$.

Setting $\psi = \psi^{(\le 0)} + \psi^{(1)}$ and performing the integration over the field $\psi^{(1)}$ (ultraviolet integration), which can be easily performed as in [BGM], to which we refer for more details, we find

$$\begin{aligned} S_{L,\beta}(\mathbf{x},\mathbf{y}) = & \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} e^{\int d\mathbf{x} d\mathbf{y}\, \phi^+_{\mathbf{x}}\, V^{(0)}_{\phi,\phi}(\mathbf{x},\mathbf{y})\, \phi^-_{\mathbf{y}}} \cdot \\ & \cdot \frac{1}{N_0} \int P(d\psi^{(\le 0)})\, e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}} + \psi^{(\le 0)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(0)}(\psi^{(\le 0)}) + W^{(0)}(\psi^{(\le 0)},\phi)} \Big|_{\phi^+ = \phi^- = 0} \; , \end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
N_0 &= \int P(d\psi^{(\le 0)})\, e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})} \;, \\
W^{(0)}(\psi^{(\le 0)},\phi) &= \int d\mathbf{x}d\mathbf{y}\Big(\phi^+_{\mathbf{x}}\, K^{(0)}_{\phi,\psi}(\mathbf{x},\mathbf{y})\, \psi^{(\le 0)-}_{\mathbf{y}} + \psi^{(\le 0)+}_{\mathbf{x}}\, K^{(0)}_{\psi,\phi}(\mathbf{x},\mathbf{y})\, \phi^-_{\mathbf{y}}\Big) \;, \\
V^{(0)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) &= g^{(1)}(\mathbf{x},\mathbf{y}) + K^{(0)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) \;,
\end{aligned}
\tag{3.3}
$$

with $g^{(1)}(\mathbf{x},\mathbf{y})$ given by

$$
g^{(1)}(\mathbf{x},\mathbf{y}) = \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} e^{-ik_0(x_0-y_0)}\, \frac{\hat f_1(x-y,k_0)}{-ik_0-\varphi_x+\mu} \;. \tag{3.4}
$$

We have, in particular,

$$
K^{(0)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) = \sum_{n=3}^{\infty} \sum_{\vartheta\in\mathcal{T}^{\phi\phi,0}_{n,m}} \sum_{k_0\in\mathcal{D}_\beta} e^{-ik_0(x_0-y_0)}\, \mathrm{Val}(\vartheta) \;, \tag{3.5}
$$

where $\mathcal{T}^{\phi\phi,0}_{n,m}$ is the set of all labelled graphs of order n with two external propagators (corresponding to the lines ℓ_1 and ℓ_{n+1}), such that

$$
\sum_{v\in\vartheta} \delta_v + \sum_{\ell\in\mathrm{int}(\vartheta)} 1 = m \;, \tag{3.6}
$$

with $h_\ell = 1\ \forall \ell\in\vartheta$; moreover, $\mathrm{Val}(\vartheta)$ is obtained from (2.47) by adding two external propagators with argument (x,k_0) and (y,k_0) and scale 1. In the same way are defined the kernels $K^{(0)}_{\phi,\psi}(\mathbf{x},\mathbf{y})$ or $K^{(0)}_{\psi,\phi}(\mathbf{x},\mathbf{y})$, with the only difference that only to one external line a propagator is associated.

Then (3.2) can be written

$$
S_{L,\beta}(\mathbf{x},\mathbf{y}) = V^{(0)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) + S^{(0)}(\mathbf{x},\mathbf{y}) \;, \tag{3.7}
$$

where

$$
\begin{aligned}
S^{(0)}(\mathbf{x};\mathbf{y}) = {} & \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} \frac{1}{N_0} \int P(d\psi^{(\le 0)}) \cdot \\
& \cdot e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}} + \psi^{(\le 0)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(0)}(\psi^{(\le 0)}) + W^{(0)}(\psi^{(\le 0)},\phi)} \Big|_{\phi^+=\phi^-=0} \;.
\end{aligned}
\tag{3.8}
$$

3.2. Proceeding as above, after integrating $\psi^{(0)},\ldots,\psi^{(h+1)}$ we find

$$
S_{L,\beta}(\mathbf{x},\mathbf{y}) = \sum_{h'=h}^{0} V^{(h')}_{\phi,\phi}(\mathbf{x};\mathbf{y}) + S^{(h)}(\mathbf{x},\mathbf{y}) \tag{3.9}
$$

where

$$
\begin{aligned}
S^{(h)}(\mathbf{x},\mathbf{y}) &= \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} \frac{1}{N_h} \int P(d\psi^{(\le h)}) \cdot \\
&\qquad \cdot e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le h)-}_{\mathbf{x}} + \psi^{(\le h)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(h)}(\psi^{(\le h)}) + W^{(h)}(\psi^{(\le h)},\phi)} \Big|_{\phi^+=\phi^-=0} \;, \\
W^{(h)}(\psi^{(\le h)},\phi) &= \int d\mathbf{x}d\mathbf{y}\Big(\phi^+_{\mathbf{x}}\, K^{(h)}_{\phi,\psi}(\mathbf{x},\mathbf{y})\, \psi^{(\le h)-}_{\mathbf{y}} + \psi^{(\le h)+}_{\mathbf{y}}\, K^{(h)}_{\psi,\phi}(\mathbf{x},\mathbf{y})\, \phi^-_{\mathbf{x}}\Big) \;, \\
V^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) &= g^{(h+1)}(\mathbf{x},\mathbf{y}) + K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) \;.
\end{aligned}
\tag{3.10}
$$

The kernels $K^{(h)}_{\chi^{(1)},\chi^{(2)}}(\mathbf{x},\mathbf{y})$ can be represented as sums of graphs of the same type of those appearing in the graph expansion of the effective potential $\mathcal{V}^{(h)}$; the new graphs differ only in the following respects:

- if $\chi^{(2)}=\phi$, the right external line is associated to the ϕ^- field and the graph ends with a vertex carrying no ε factor;

- if $\chi^{(1)} = \phi$, the left external line is associated to the ϕ^+ field and the graph begins with a vertex carrying no ε factor;
- $\mathcal{R} \equiv 1\!\!1$ on resonances containing an external propagator (defined as after (3.5));
- $h(\vartheta) = h+1$ for all graphs, if $\chi^{(1)} = \chi^{(2)} = \phi$.

Then the functional derivatives in (4.8) give

$$S_{L,\beta}(\mathbf{x},\mathbf{y}) = \sum_{h=h^*+1}^{0} \Big(g^{(h+1)}(\mathbf{x};\mathbf{y}) + K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) \Big) + \bar{g}^{(h^*)}(\mathbf{x},\mathbf{y}) + \bar{K}^{(h^*)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) \;, \tag{3.11}$$

where h^* is defined in (2.49), $\bar{g}^{(h^*)}(\mathbf{x},\mathbf{y})$ and $\bar{K}^{(h^*)}_{\phi,\phi}(\mathbf{x},\mathbf{y})$ have a different meaning in case (1) or (2) of the theorem (see §3.3 and §3.4 below) and

$$K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) = \sum_{n=3}^{\infty} \sum_{\vartheta \in \mathcal{T}^{\phi\phi,h}_{n,m}} \frac{1}{\beta} \sum_{k_0 \in \mathcal{D}_\beta} e^{-ik_0(x_0-y_0)}\, \mathrm{Val}(\vartheta) \;. \tag{3.12}$$

if $\mathcal{T}^{\phi\phi,h}_{n,m}$, with $x-y=m$, is the set of all labelled graphs of order n with two external propagators, such that

$$\sum_{v\in\vartheta} \delta_v + \sum_{\ell \in \mathrm{int}(\vartheta)} \left(\rho^1_\ell - \rho^2_\ell\right) \bar{x} = m \;. \tag{3.13}$$

and $\mathrm{Val}(\vartheta)$ is computed with the rules explained after (3.5) and (3.10).

The discussion will proceed from now on in a different way depending on case (1) or (2) of the theorem.

3.3. In the case (1) in (3.11) one has $\bar{g}^{(h^*)}(\mathbf{x},\mathbf{y}) = g^{(h^*)}(\mathbf{x},\mathbf{y})$ and $\bar{K}^{(h^*)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) = K^{(h^*)}_{\phi,\phi}(\mathbf{x},\mathbf{y})$, with $h^* = h_\beta$.

Note that, from (3.13),

$$|v_0| t_0^{-1} \gamma^{-h-1} \le \sup_{0\le k\le 2n} \max_{\rho=\pm1} \frac{1}{\|\omega(x+k)+\rho\bar{\omega}\|} \le 4C_0^{-1} 2^\tau \left[\min\{|x|,|y|\}+2n\right]^\tau \;, \tag{3.14}$$

where in the last inequality we have used the Diophantine condition (1.5) for $\|\omega(x+k)+\rho\bar{\omega}\| \le 1/4$ (in such a case one can write $\|\omega(x+k)+\rho\bar{\omega}\| = \|\omega 2(x+k)+\rho 2\bar{\omega}\|/2$), while for $\|\omega(x+k)+\rho\bar{\omega}\| > 1/4$, the bound in (3.14) is of course trivial. Then

$$|v_0| t_0 \gamma^{-h-1} \le 4C_0^{-1} 4^\tau \left(1+\min\{|x|,|y|\}\right)^\tau \left(1 + \frac{n}{1+\min\{|x|,|y|\}}\right)^\tau \equiv |v_0| t_0 \gamma^{-\bar{h}(n)-1} \;. \tag{3.15}$$

In order to bound (3.11) we use the following result, proven in Appendix A3:

$$|\mathrm{Val}(\vartheta)| \le \gamma^{-h} C D^{n-1} |\varepsilon|^{(n-2)/4}\, e^{-\frac{1}{4\alpha}\log|\varepsilon^{-1}| m} \qquad \forall \vartheta \in \mathcal{T}^{\phi\phi,h}_{n,m} \;, \tag{3.16}$$

with the same notations used in (2.52); in particular $\alpha = 1$. The integral over k_0 is over a domain at most of order $\gamma^{h(\vartheta)}$, with $h(\vartheta) = h+1$, as k_0 is constrained to be on the compact support of the propagator with scale $h(\vartheta)$, so that

$$\sum_{h=h_\beta}^{0} \left| K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) \right| \le \sum_{h=h_\beta}^{0} \sum_{n=3}^{\infty} \sum_{\vartheta \in \mathcal{T}^{\phi\phi,h}_{n,m}} C_3 \gamma^h \sup_{k_0 \in \mathcal{D}_\beta} |\mathrm{Val}(\vartheta)| \;, \tag{3.17}$$

where $\mathrm{Val}(\vartheta)$ is bounded as in (3.16). By (3.15) we see that the sums over h and n are not independent; in particular we can exchange the order of the sums writing

$$\begin{aligned}\sum_{h=h_\beta}^{0}\left|K_{\phi,\phi}^{(h)}(\mathbf{x},\mathbf{y})\right| &\leq \sum_{n=3}^{\infty}\sum_{h=\bar{h}(n)}^{0}\sum_{\vartheta\in\mathcal{T}_{n,m}^{\phi\phi,h}} C_3\gamma^h \sup_{k_0\in\mathcal{D}_\beta}|\mathrm{Val}(\vartheta)| \\ &\leq \sum_{n=[|x-y|/2]}^{\infty} C_3 C D^n |\varepsilon|^{(n-2)/4}\, e^{-\frac{1}{4}\log|\varepsilon^{-1}||x-y|}. \\ &\quad\cdot\log\left[(1+\min\{|x|,|y|\})\left(1+\frac{n}{1+\min\{|x|,|y|\}}\right)\right]^\tau \\ &\leq C_5\log\left(1+\min\{|x|,|y|\}\right)^\tau \exp\left\{-\frac{|x-y|}{4}\log\left|\varepsilon^{-1}\right|\right\}\ ,\end{aligned} \tag{3.18}$$

where C_5 depends on ω and $\bar{\omega}$, the sum over the scales of the tree is controlled by (3.15) and we have used (3.14) and the fact that $n\geq|x-y|/2$.

We can obtain another bound, which is better for large $|x_0-y_0|$; by using

$$K_{\phi,\phi}^{(h)}(\mathbf{x},\mathbf{y}) = \frac{1}{|x_0-y_0|^N}\sum_{n=3}^{\infty}\sum_{\vartheta\in\mathcal{T}_{n,m}^{\phi\phi,h}}\int_{\infty}^{\infty}\frac{dk_0}{2\pi}e^{-ik_0(x_0-y_0)}\,D_0^N\mathrm{Val}(\vartheta)\ , \tag{3.19}$$

where D_0 denotes the discrete derivative with respect to k_0, (see also [BGM]), and proceeding as above we find the bound

$$\sum_{h=h_\beta}^{0}\left|K_{\phi,\phi}^{(h)}(\mathbf{x},\mathbf{y})\right| \leq C_6\frac{1}{(|x_0-y_0|\,|x|^{-\tau})^N}\exp\left\{-\frac{|x-y|}{4}\log\left|\varepsilon^{-1}\right|\right\}\ , \tag{3.20}$$

which is better than the bound (3.18) for $|x_0-y_0|\geq|x|^\tau$. So the bound (1.6) follows.

3.4. In case (2) of the theorem, one has $\bar{g}^{(h^*)}(\mathbf{x},\mathbf{y}) = g^{(\leq h^*)}(\mathbf{x},\mathbf{y})$ and $\bar{K}_{\phi,\phi}^{(h^*)}(\mathbf{x},\mathbf{y}) = K_{\phi,\phi}^{(\leq h^*)}(\mathbf{x},\mathbf{y})$, with $h^*>h_\beta$, generically, as one can integrate all the scales $\leq h^*$ in a single step, namely

$$S_{L,\beta}(\mathbf{x},\mathbf{y}) = \sum_{h=h^*+1}^{0}\left(g^{(h+1)}(\mathbf{x},\mathbf{y}) + K_{\phi,\phi}^{(h)}(\mathbf{x},\mathbf{y})\right) + g^{(\leq h^*)}(\mathbf{x},\mathbf{y}) + K_{\phi,\phi}^{(<h^*)}(\mathbf{x},\mathbf{y})\ , \tag{3.21}$$

and an expression similar to (3.12) for $K_{\phi,\phi}^{(h)}(\mathbf{x},\mathbf{y})$ is valid for $K_{\phi,\phi}^{(\leq h^*)}(\mathbf{x},\mathbf{y})$, with $g^{(\leq h^*)}$ in place of $g^{(h+1)}$. Proceeding as above the bound (1.7) in the theorem 1.4 follows.

4. Generalizations and final comments

4.1. We compare now our results with the result about the Schrödinger equation with a large quasiperiodic potential

$$-\psi(x-1)-\psi(x+1)+2\psi(x)+\lambda V(x)\psi(x) = E\psi(x)\ , \tag{4.1}$$

where $V(x)=\bar{V}(\alpha+\omega x)$, with $\bar{V}(\cdot)$ periodic of period 1 and $\alpha\in\mathbb{R}$. Note the presence of the free parameter α in $V(x)$, while we take α fixed. The results obtained for $|\lambda|$ large depend crucially on α; in fact, under "reasonable" assumptions on $\bar{V}$, see [PF], it is proved that the spectrum is pure point and the eigenfunctions decay exponentially for *almost all* α. The search for exponentially localized solutions of the above Schrödinger equation was done through the use of perturbation theory considering the second difference operator to be the perturbation; such a perturbation theory is afflicted by a small divisors problem. In [BLS], $V(x)$ was assumed strictly monotone in the period; a typical example is $V(x) = \tan(2\pi(\alpha + \omega x))$, with ω verifying a Diophantine condition. Functions like $V(x) = \cos(2\pi(\alpha + \omega x))$, which are the more interesting ones, were excluded by [BLS], but were considered later in [BLT,S]. In particular [S] considered C^2 functions $\bar V(\alpha+\omega x)$, having exactly one nondegenerate maximum and minimum and strictly monotone with nonzero derivative between them (and some assumptions – slightly different from the usual Diophantine one – on the continued fraction expansion of ω was made). In particular, imposing some conditions on α and E (satisfied on a full measure set), in [S] the existence of Anderson localization was proved. In [E] further generalizations were considered, in particular relaxing the hypothesis of monotonicity done in [S]; see also §4.4 and §4.5 below.

In order to compare the above results, especially [S], with our work, we can study the following problem. We replace in (1.1) the function φ_x with $\bar V(\alpha + \omega x)$ and we consider $\mu = \bar V(\alpha + \omega \bar n)$, with $\bar n$ integer. This corresponds to choose μ in correspondence of an eigenvalue of the spectrum of the Schrödinger equation.

We assume as Diophantine condition, besides (1.3), also

$$\|\omega n \pm 2\alpha\|_{\mathbb{T}} \ge C_0 |n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} . \tag{4.2}$$

Then we can proceed as in the proof of theorem 1.4, with the following differences (let C_1, C_2, C_3, C_4 denote suitable constants):

- we replace (2.44) and (2.45), case (1), with

$$C_3 \gamma^{h_{T'}} \ge \left| \bar V(\alpha + \omega x_{\ell^o_T}) - \bar V(\alpha + \omega x_{\ell^i_T}) \right| \ge C_2 \sup_{\varepsilon = 0, \pm 1} \|\omega m + 2\varepsilon\alpha\|_{\mathbb{T}} \ge C_1 |m|^{-\tau} ; \tag{4.3}$$

- we replace (3.14) with

$$C_4 t_0 \gamma^{-h-1} \le \sup_{0 \le k \le 2n} \frac{1}{\left| \bar V(\alpha + \omega(x+k)) - \bar V(\alpha + \omega \bar n) \right|} \le C_1 \left[\min\{|x|, |y|\} + 2n \right]^{\tau} . \tag{4.4}$$

By these substitutions everything is essentially unchanged with respect to the case (1) of the theorem 1.4: so we can prove the convergence of the Schwinger functions, *at least if we make the extra assumption that the potential $V(x)$ is an even function* (as theorem 1.4 was just stated under such an assumption). We shall come in a moment (see in particular §4.9) on the relevance of the parity assumption.

We prefer to study the model (1.1), in which the phase α is fixed and the chemical potential can vary, as physically it corresponds to change the number of electrons. Moreover we are interested also in fixing the chemical potential within gaps of order $O(\sigma)$ in the spectrum of the Schrödinger equation, which could be important from a physical point of view: this corresponds to a zero measure set of $\bar\omega = \omega \bar x$ (equivalently of α), hence it is irrelevant in the analysis performed in [S,E], where only properties holding almost everywhere were explicitly considered.

4.2. We consider now the problem of extending the results to more general potentials. In fact we shall see that, if we allow the potential φ_x to be modified by the perturbation (*i.e.* if we choose φ_x as a function of ε), then the case of any potential with only one nondegenerate maximum and minimum and strictly monotone between them, as the one considered in [S], can be easily recovered. The same holds also if we want to consider potentials in the class defined in [E]; see §4.6÷§4.7 below. The physical meaning of such a modification will be discussed in §4.9.

4.3. We start by considering in the Hamiltonian (1.1) the class of the potentials considered in [S]. First of all the analysis of the previous sections can be easily extended to the case in which the two roots of the equation $\bar\varphi(\omega x) = \mu$ (which we shall call the *critical points* of the potential φ_x) are $\bar x_1$ and $\bar x_2$, with $\bar x_2 \ne -\bar x_1$, and φ_x is even with respect to $(\bar x_1 + \bar x_2)/2$; if we have such a function $\bar\varphi(\omega x)$ it is enough to define $\tilde\varphi(\omega(x + x_0))$, with $x_0 = (\bar x_1 + \bar x_2)/2$, in order to obtain a potential even with respect to x_0, so that its roots are opposite to each other. So it is always possible to have a potential $\varphi_x = \bar\varphi(\omega x)$ such that $\varphi(\pm\omega\bar x) = \mu$.

In general, if φ_x is not necessarily even, not only $\bar{x}_1 \neq -\bar{x}_2$, but also, setting $\bar{\varphi}'(\omega\bar{x}_1) = v_0^{(1)}$ and $\bar{\varphi}'(\omega\bar{x}_2) = v_0^{(2)}$, one has $v_0^{(1)} \neq -v_0^{(2)}$; as $\bar{x}_1 \neq -\bar{x}_2$ we prefer to denote $1,2$ the values of the label ρ, instead of ± 1 as in §2.

One property used in the analysis of the previous sections due to the parity assumption on φ_x was indeed that $\bar{\varphi}'(\pm\omega\bar{x}) = \pm v_0$, but it is immediate to note that the analysis can be adapted to the case in which $v_0^{(1)} \neq -v_0^{(2)}$. Simply one has to use a different scale unit for the compact support functions (appearing in the multiscale decomposition of the propagator) near the two points $\bar{x}_1$ and $\bar{x}_2$, *i.e.* one replaces in (2.11) $a_0/|v_0|$ with $a_0/|v_0^{(1)}|$ near $\bar{x}_1$ and with $a_0/|v_0^{(2)}|$ near $\bar{x}_2$, where a_0 is such that the supports of $\chi(\mathbf{r} - \bar{\mathbf{x}}_1)$ and $\chi(\mathbf{r} - \bar{\mathbf{x}}_2)$ are disjoint (here $\bar{\mathbf{x}}_j = (\bar{x}_j, 0)$, for $j = 1, 2$).

Another property that fails in the non-even case is the equality between $\Phi^1_{x'}$ and $\Phi^2_{-x'}$; this implies that (2.28) has to be replaced by

$$\mathcal{L}\mathcal{V}^{(h)} = \sum_{\rho=1}^{2} \left(\gamma^h \nu_{\rho h} F^{(h)}_{\nu,\rho} + s_{\rho h} F^{(h)}_{\sigma,\rho} \right) , \tag{4.5}$$

where $F^{(h)}_{\nu,\rho}$ and $F^{(h)}_{\sigma,\rho}$ are given by

$$\begin{aligned} F^{(h)}_{\nu,\rho} &= \sum_{x\in\Lambda} \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} \psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} \psi^{(\le h)-}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} , \\ F^{(h)}_{\sigma,\rho} &= \sum_{x\in\Lambda} \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} \psi^{(\le h)+}_{\mathbf{r}'+\rho\bar{\mathbf{x}},\rho} \psi^{(\le h)-}_{\mathbf{r}'-\rho\bar{\mathbf{x}},-\rho} , \end{aligned} \tag{4.6}$$

so that four running coupling constants turn out to be involved: $\nu_{1h}, \nu_{2h}, \sigma_{1h}, \sigma_{2h}$. This does not introduce any extra problems as the flow of the running coupling constants can still be controlled as in the case considered in the even case.

The main difference is that now a counterterm of the form in (1.2) is not sufficient to control the flow: one has to introduce in the Hamiltonian a term

$$\sum_{x\in\Lambda} \nu_x \psi^+_x \psi^-_x , \tag{4.7}$$

where $\nu_x = \bar{\nu}(\omega x)$ is any function in $C^1(\mathbb{T})$ such that $\bar{\nu}(\omega\bar{x}_1) = \nu_1$ in a neighbourhood of radius a_0 and center $\omega\bar{x}_1$ and $\bar{\nu}(\omega\bar{x}_2) = \nu_2$ in a neighbourhood of radius a_0 and center $\omega\bar{x}_2$, if ν_1 and ν_2 are fixed in such a way that the flow of the running coupling constants ν_{1h}, ν_{2h} is controlled as in the even case (see §2.11); we refer to §A2.11 for more technical details.

Note that (4.7) corresponds to a counterterm depending on x. A similar situation (in Fourier space) arose in [FST], where the modification of the Fermi surface is studied in nonspherical two-dimensional fermion systems with quartic interaction.

By using the above comments one finds that for potentials like the ones of [S] a result analogous to the theorem 1.4 holds, *provided that the potential* φ_x *is modified into* $\varphi_x + \nu_x$; see §4.9 below.

We also note that what really matters is the behaviour of the potential φ_x near the two points $\pm\bar{x}$, so that, far from them, no regularity assumption is required on the potential (which can even be a very irregular function far enough from the critical points $\bar{x}_1$ and $\bar{x}_2$ of the potential). Of course if one is interested in properties holding almost everywhere in the spectrum, as in [S,E], a global regularity has to be required.

4.4. For the case (1) of the theorem, also potentials with more than two points $\bar{x}_1, \ldots, \bar{x}_p$, $p \ge 3$, such that

$$\bar{\varphi}(\omega\bar{x}_j) = \mu , \qquad j = 1, \ldots, p , \tag{4.8}$$

like the ones in [E], could be considered with our methods, essentially without any real extra difficulties (however see §4.9). Of course some additional Diophantine conditions should be imposed on ω. Instead of (1.5) one should require the $p(p-1)/2$ conditions

$$\|\omega n + \omega\,(\bar{x}_i - \bar{x}_j)\,\| > C_0 |n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} , \qquad \forall i \neq j = 1, \ldots, p , \tag{4.9}$$

which slightly narrow the set of admissible ω, but always leaving as possible a full measure set, provided that $\bar{x}_i - \bar{x}_j$ does not remain constant for $i \neq j$ when varying μ.

If the interaction potential is a function of class C^s, with $s \geq 2$, also the case in which $\partial^{s'} \varphi_{\bar{x}} \neq 0$, for some $1 \leq s' \leq s$, can be dealt with with our techniques (this would allow us to recover the *first transversality condition* in [E]). Anyway, if s_0 is the minimum s' such that $\partial^{s'} \varphi_{\bar{x}} \neq 0$, a renormalization to order s_0 (and not just to first order except for the case $s_0 = 1$) would be required, so that the flow of s_0 running coupling constants ought to be controlled. The analysis turns out to be more involved but not out of reach; see §4.5 below.

Concerning the *second transversality condition* in [E], in the case $s_0 = 1$, it is trivially satisfied near the critical points $\bar{x}_1$ and $\bar{x}_2$, as $v_0^{(1)} \neq v_0^{(2)}$. If the second tranversality condition is assumed to hold for any x as in [E], then it follows that the set of μ for which the condition (1.5) is satisfied has full measure.

Also the case $s \geq 2$ can be treated along the lines sketched in the above paragraph. In fact the second transversality condition in [E] automatically excludes the potential φ_x to be locally translation invariant, so that it imposes that, when varying μ, the quantities $\bar{x}_i - \bar{x}_j$ are not constants for any $i \neq j$: then the set of ω satisfying (4.9) is of full measure (see comments after (4.9)).

In conclusion when varying μ, if φ_x satisfies the two transversality conditions in [E], we have that (4.9) holds almost everywhere in μ.

4.5. If there are p critical points $\bar{x}_1, \ldots, \bar{x}_p$ satisfying (4.8), for fixed μ, and ω fulfils the Diophantine condition (4.9), then we have to renormalize the theory to order s_0 (see §4.4), by introducing, for any point $\bar{x}_j$, $j = 1, \ldots, p$, s_0 running coupling constants

$$\gamma^h \nu_{jh}^{(0)},\ \gamma^{(s_0-1)h/s_0} \nu_{jh}^{(1)},\ \ldots,\ \gamma^{(s_0-2)h/s_0} \nu_{jh}^{(s_0-2)},\ \gamma^{h/s_0} \nu_{jh}^{(s_0-1)}, \tag{4.10}$$

such that

$$\mathcal{L}\mathcal{V}^{(h)} = \sum_{j=1}^{p} \sum_{r=1}^{s_0} \gamma^{(s_0-r)h/s_0} \nu_{jh}^{(r)} F_{\nu,j,r}^{(h)} , \tag{4.11}$$

where

$$F_{\nu,j,r}^{(h)} = \sum_{x \in \Lambda} \frac{1}{\beta} \sum_{k_0 \in \Delta_\beta} \psi^{(\leq h)+}_{\mathbf{r}'+\bar{\mathbf{x}}_j, j} \left(\omega x'\right)^r \psi^{(\leq h)-}_{\mathbf{r}'+\bar{\mathbf{x}}_j, j} ; \tag{4.12}$$

recall that no σ-type running coupling constants appear in the case in which ω is incommensurate with respect to $\omega(\bar{x}_i - \bar{x}_j)$ for all $i \neq j$.

The flow of the running coupling constants (4.10) can be discussed as in the case of theorem 1.4 and it turns out to be controlled if suitable counterterms are introduced in the Hamiltonian. Instead of only one counterterm (4.7) one has to introduce s_0 counterterms, which leads to add to the Hamiltonian

$$\sum_{x \in \Lambda} (\mu - \varphi_x)\, \psi_x^+ \psi_x^- - \frac{\varepsilon}{2} \sum_{x \in \Lambda} \left[\psi_x^+ \psi_{x+1}^- + \psi_x^+ \psi_{x-1}^- - 2 \psi_x^+ \psi_x^- \right] \tag{4.13}$$

a term

$$\sum_{x \in \Lambda} \nu_x\, \psi_x^+ \psi_x^- = \sum_{x \in \Lambda} \left(\tilde{\nu}_x^{(0)} + \tilde{\nu}_x^{(1)} + \ldots + \tilde{\nu}_x^{(s_0-1)} \right) \psi_x^+ \psi_x^- , \tag{4.14}$$

where $\tilde{\nu}_x^{(r)} = \bar{\nu}^{(r)}(\omega x)$ are any functions in $C^{s_0}(\mathbb{T})$ such that $\bar{\nu}^{(r)}(\omega x) = \left(\omega(x - \bar{x}_j)\right)^r \nu_j^{(r)}$ in a neighbour of radius a_0 and center $\omega \bar{x}_j$, if a_0 is so chosen that the supports of the functions $\chi(\mathbf{r} - \bar{\mathbf{x}}_j)$, with $j = 1, \ldots, p$, are disjoint and the constants $\nu_j^{(0)}, \ldots, \nu_j^{(s_0-1)}$ are fixed in such a way that the running coupling constants remain bounded; we refer to Appendix A4 for details. Here we confine ourselves to state the final result, which is a generalization of theorem 1.4, case (1).

4.6. Theorem. *Consider the fermion system described by the Hamiltonian*

$$\mathcal{H}_0 = \sum_{x \in \Lambda} (\mu - \varphi_x)\, \psi_x^+ \psi_x^- - \frac{\varepsilon}{2} \sum_{x \in \Lambda} \left[\psi_x^+ \psi_{x+1}^- + \psi_x^+ \psi_{x-1}^- - 2 \psi_x^+ \psi_x^- \right] , \tag{4.15}$$

where $\varphi_x = \bar\varphi(\omega x)$ *a function in* $C^s(\mathbb{T})$*, of period 1 and with* ω *satisfying a Diophantine condition*

$$\|\omega n\|_{\mathbb{T}} \geq C_0 |n|^{-\tau}\;, \qquad \forall n \in \mathbb{Z}\setminus\{0\}\;, \tag{4.16}$$

for some constants $\tau > 1$ *and* $C_0 > 0$*. Let us define* $\bar\omega_j \equiv \omega \bar x_j$ *such that* $\mu = \bar\varphi(\bar\omega_j)$ *and assume that there exists* $s_0 \leq s$ *such that*

$$\max_{r\leq s_0} \{|\partial^r \bar\varphi(\omega x)|\} \geq \xi\;, \qquad \forall x \in \{\bar x_1, \ldots, \bar x_p\}\;, \tag{4.17}$$

for some constant ξ*.*

Then there exists $\varepsilon_0 > 0$*, depending on* $\omega, \bar\omega_1, \ldots, \bar\omega_p$ *and* ξ*, and, for* $|\varepsilon| < \varepsilon_0$*,* $s_0 p$ *functions* $\nu_j^{(r)} \equiv \nu_j^{(r)}(\varepsilon)$ *(some of which possibly vanishing) with* $j = 1, \ldots, p$ *and* $r = 0, \ldots, s_0 - 1$*, such that, if* $\bar\omega \notin \omega_j \mathbb{Z}$ *mod 1 and the additional Diophantine condition*

$$\|\omega n \pm (\bar\omega_i - \bar\omega_j)\|_{\mathbb{T}} \geq C_0 |n|^{-\tau}\;, \qquad \forall n \in \mathbb{Z}\setminus\{0\}\;, \qquad \forall i \neq j = 1, \ldots, p\;, \tag{4.18}$$

is verified, then the two-point Schwinger function $S_{L,\beta}(\mathbf{x},\mathbf{y})$ *for the system described by the Hamiltonian*

$$\mathcal{H}_\nu = \mathcal{H}_0 + \sum_{x\in\Lambda} \left(\tilde\nu_x^{(0)} + \tilde\nu_x^{(1)} + \ldots + \tilde\nu_x^{(s_0-1)}\right) \psi_x^+ \psi_x^- \tag{4.19}$$

admits a limit

$$\lim_{L\to\infty}\lim_{\beta\to\infty} S_{L,\beta}(\mathbf{x},\mathbf{y}) = S(\mathbf{x},\mathbf{y})$$

bounded by

$$|S(\mathbf{x},\mathbf{y})| \leq \log\left(1 + \min\{|x|,|y|\}\right)^\tau \frac{C_N' \exp\left\{-4^{-1}|x-y| \log\left|\varepsilon^{-1}\right|\right\}}{1 + \left[(1+\min\{|x|,|y|\})^{-\tau}|x_0 - y_0|\right]^N}\;, \tag{4.20}$$

for any $N \geq 1$ *and for some constant* C_N' *depending on* N*.*

4.7. Let us make some minor comments on the statement of the theorem above, deferring to §4.9 the main one.

(1) The number of running coupling constants is in general $s_1 + \ldots + s_p$, of s_j is first nonvanishing derivative of $\bar\varphi(\omega x)$ in $x = \bar x_j$; as $s_j \leq s_0$ for some $s_0 \leq s$, the number of nonvanishing running coupling constants is $\leq ps$: this motivates the statement about the number of running coupling constants in the theorem.

(2) In the case of the theorem 1.4 the term (4.7) takes into account the shifting of the singularity of the propagator with respect to the free case ($\varepsilon = 0$). If the first $s_j - 1$ derivatives are vanishing in some critical points $\bar x_j$ such a counterterm is not sufficient and one has to make nonvanishing also the first $s_j - 1$ derivatives of the potential by modifying it, near the critical points $\bar x_j$, in the following way:

$$\begin{aligned} &\varphi_x - \mu \equiv (\omega(x - \bar x_j))^{s_j}\, \Psi_x \\ &\quad\to\quad \nu_x^{(0)} + \nu_x^{(1)} (\omega(x-\bar x_j)) + \ldots + \nu_x^{(s_j-1)} (\omega(x - \bar x_j))^{s_j - 1} + (\omega(x-\bar x_j))^{s_j}\, \Psi_x\;, \end{aligned} \tag{4.21}$$

Of course if $s_j = 1$ we recover the result of the theorem 1.4.

4.8. As to the case (2) generalizations could be possible, but several possible cases should be discussed, depending on the possible commensurability relations between the values $\bar\omega_1, \ldots, \bar\omega_p$, so we prefer to not explicitly consider further extensions in this direction, also in view of the observations in the following section.

4.9. Let us discuss the physical meaning of the above results. In general, fixed the chemical potential to an ε-independent value, we expect the singularities of the Schwinger functions in the two cases $\varepsilon = 0$ and

$\varepsilon \neq 0$ to be different. As such singularities are physically observable, it is reasonable to fix them to some ε-independent value (accessible to the experiments), by introducing a suitable counterterm which leads to add a term (4.7) to the Hamiltonian, (if the formal "Luttinger theorem" held in this case, this would correspond to fix the density of the fermion system). In the Renormalization Group language, we want to fix the *dressed quantities* to some ε-independent value.

However, as the symmetries of H_0 and V are different, *except in the case discussed in theorem 1.4*, the counterterm will depend on the coordinate x; in other words we have to choose an ε-dependent potential in such a way that the singularities of the Schwinger functions are ε-independent.

This is not surprising: we want to prove that the Schwinger function is a perturbation of (2.3), in which the potential φ_x appears, but we know that the eigenvalues of the Schödinger equation are of the form $\varphi_x + \varepsilon\delta\varphi_x$, for some suitable function $\delta\varphi_x$; see [S,E]. Note that in the particular case of theorem 1.4 the counterterm is a constant (so that (4.7) becomes proportional to the fermion density). So in such a case another interpretation is possible: one has to fix the chemical potential to an ε-dependent value so that the singularities of the Schwinger functions are ε-independent.

Having nonconstant counterterms is what usually happens in interacting fermion systems. For instance in order to study a system of two-dimensional fermions with a non-rotation invariant dispersion relation and interacting with a weak two-body potential, one has to introduce an angle-dependent counterterm, see [FST].

If one is interested in studying a Hamiltonian with no counterterms (but it is questionable if this is the right thing to do: this correspond to consider as physically observable the *bare* instead of the *dressed quantities*), one has an implicit function to solve, which is nontrivial in our case because of the small divisors problem.

Appendix A1. Convergence of the effective potential

A1.1. Let us consider the quantity $\mathcal{W}_n^{(h)}(x, x+m; k_0)$ introduced in §3:

$$\mathcal{W}_n^{(h)}(x, x+m; k_0) = \sum_{\vartheta \in \mathcal{T}^h_{n,m}} \mathrm{Val}(\vartheta) \ . \tag{A1.1}$$

We define the *depth* of a cluster T inductively by setting $D_{T'} = D_T + 1$, if T' is the cluster immediately containing T, and considering the vertices as clusters with depth 0; the graph ϑ is a cluster with maximal depth D_ϑ.

Given a cluster T we shall say that a cluster $\tilde{T}$ is *maximal* in T if it is contained inside T but not in any other cluster inside T.

We introduce the following notations:

- $\mathbf{T}$ is the set of clusters contained in ϑ (including ϑ);
- $\mathbf{V}$ is the set of resonances;
- T_0 the set of lines and vertices inside T and outside the clusters internal to T;
- M_T is the number of vertices in T;
- M_{T_0} is the number of vertices in T_0;
- $M_{T_0}^{(\nu)}$ is the number of ν-vertices in T_0;
- L_{T_0} is the number of lines in T_0;
- k_T^0 is the number of maximal nonresonant clusters in T;
- $k_T^{\mathcal{R}}$ is the number of maximal renormalized resonant clusters in T;
- k_T is the number of maximal clusters in T;
- D_T is the depth of the cluster T.
- $\mathbf{T}_k$ as the set of clusters with depth $D_T = k$, $\mathbf{T}_k = \{T \in \mathbf{T} \ : \ D_T = k\}$

Then, in (A1.1), one can write

$$\sum_{\vartheta\in\mathcal{T}^h_{n,m}} \mathrm{Val}(\vartheta) = \sum_{\substack{\delta_1,\ldots,\delta_n \\ \delta_1+\ldots+\delta_n=m}} \varepsilon^n \sum_{\{h_\ell\}} \left[\mathcal{R} \prod_{\ell\in\mathrm{int}(\vartheta)} \tilde{g}_\ell \right] \left(\prod_{T\in\mathbf{T}} \left(\frac{\gamma^{h_T}\nu_{h_T}}{\varepsilon} \right)^{M^{(\nu)}_{T_0}} \right) , \tag{A1.2}$$

with the notations introduced in §3 (in particular $\tilde{g}_\ell$ is a shorthand for $\tilde{g}^{(h_\ell)}_{\rho^1_\ell,\rho^2_\ell}(\mathbf{r}'_\ell)$).

Recall (2.44) and (2.45); by using in case (1) of the theorem that $|m_T| \le M_T$, one has

$$2a_0\gamma^{h_{T'}}/|v_0| \ge C_0|m_T|^{-\tau} \ge C_0 M_T^{-\tau} , \tag{A1.3}$$

by (1.5).

In case (2) of the theorem one finds, by using that $|m_T| \le M_T + (2k-1)(M_T-1)$

$$2a_0\gamma^{h_{T'}}/|v_0| \ge C_0\left(|m_T| + (2k+1)\right)^{-\tau} \ge C_0\left[M_T(2k+2)\right]^{-\tau} , \tag{A1.4}$$

as $2\omega = (2k+1)\bar{\omega}$ mod 1. We can express both (A1.3) and (A1.4) through the only inequality

$$2a_0\gamma^{h_{T'}}/|v_0| \ge C_0\left(\alpha M_T\right)^{-\tau} , \qquad \begin{cases} \alpha = 1 , & \text{case (1)} , \\ \alpha = 2k+2 , & \text{case (2)} . \end{cases} \tag{A1.5}$$

In particular, if $T=\vartheta$, one finds $|m| \le \alpha n$.

A1.2. Lemma. *For any $\vartheta\in\mathcal{T}^h_{n,m}$, one has*

$$|\varepsilon|^{n/2} = \prod_{v\in\vartheta}|\varepsilon|^{1/2} \le \prod_{T\in\mathbf{T}} |\varepsilon|^{2^{-(D_T+2)}M_T} , \tag{A1.6}$$

if M_T is the number of vertices in T and D_T is the depth of T.

A1.3. *Proof.* We prove by induction on the depth $k\in[0,D_\vartheta]$ the following bound:

$$\prod_{T\in\mathbf{T}_k}\prod_{v\in T}|\varepsilon|^{1/2} \le \left(\prod_{p=0}^{k-1}\prod_{T\in\mathbf{T}_p} |\varepsilon|^{2^{-(p+2)}M_T} \right) \left(\prod_{T\in\mathbf{T}_k} |\varepsilon|^{2^{-(k+1)}M_T} \right) , \tag{A1.7}$$

where the product in the first parentheses has to be thought as 1 for $k=0$.

Then, for $k=0$, (A1.6) is a trivial identity. Suppose that (A1.6) holds for $k-1$; then we show that it holds also for k. In fact, by denoting by M_{T_0} the number of vertices in T_0 (*i.e.* internal to T and external to all clusters contained inside T), one has

$$\begin{aligned} \prod_{T\in\mathbf{T}_k}\prod_{v\in T}|\varepsilon|^{1/2} &= \left(\prod_{T\in\mathbf{T}_k}\prod_{v\in T_0}|\varepsilon|^{1/2} \right) \left(\prod_{T\in\mathbf{T}_{k-1}}\prod_{v\in T}|\varepsilon|^{1/2} \right) \\ &\le \left(\prod_{T\in\mathbf{T}_k} |\varepsilon|^{M_{T_0}/2} \right) \left(\prod_{p=0}^{k-2}\prod_{T\in\mathbf{T}_p} |\varepsilon|^{2^{-(p+2)}M_T} \right) \left(\prod_{T\in\mathbf{T}_{k-1}} |\varepsilon|^{2^{-k}M_T} \right) \\ &\le \left(\prod_{p=0}^{k-1}\prod_{T\in\mathbf{T}_p} |\varepsilon|^{2^{-(p+2)}M_T} \right) \left[\left(\prod_{T\in\mathbf{T}_k} |\varepsilon|^{M_{T_0}/2} \right) \left(\prod_{T\in\mathbf{T}_{k-1}} |\varepsilon|^{2^{-(k+1)}M_T} \right) \right] \\ &\le \left(\prod_{p=0}^{k-1}\prod_{T\in\mathbf{T}_p} |\varepsilon|^{2^{-(p+2)}M_T} \right) \left(\prod_{T\in\mathbf{T}_k} |\varepsilon|^{2^{-(k+1)}M_T} \right) , \end{aligned} \tag{A1.8}$$

so proving (A1.7). By taking $k = D_\vartheta$, (A1.6) follows. ■

A1.4. Lemma. *For any tree ϑ one has*

$$\prod_{\substack{T\in\mathbf{T}\\ D_T\ge 0}} |\varepsilon|^{2^{-(D_T+2)}M_T} \le \prod_{\substack{T\in\mathbf{T}\\ D_T\ge 1}} |\varepsilon|^{2^{-(D_T+1)}C_1\gamma^{-h_T/\tau}k_T^0} , \tag{A1.9}$$

where $C_1 = (C_0|v_0|/2a_0)^{1/\tau}\alpha^{-1}$ and k_T^0 is the number of nonresonant clusters in T_0.

A1.5. *Proof.* One has

$$\prod_{\substack{T\in\mathbf{T}\\ D_T\ge 0}} |\varepsilon|^{2^{-(D_T+2)}M_T} = \prod_{\substack{T\in\mathbf{T}\\ D_T\ge 1}} \prod_{\substack{T\in\mathbf{T}\\ \tilde T'=T}} |\varepsilon|^{2^{-(D_{\tilde T}+2)}M_{\tilde T}} , \tag{A1.10}$$

where we denote by T' the cluster immediately containing T. By taking into account that $\min\{h_{\ell_T^o}, h_{\ell_T^i}\} = h_{T'}$ by construction and $\tilde T' = T$, one has

$$\alpha M_{\tilde T} \ge |m_{\tilde T}| \ge C_2\gamma^{-h_T/\tau} , \qquad C_2 = (C_0|v_0|/2a_0)^{1/\tau} , \tag{A1.11}$$

by (2.44), provided that $m_{\tilde T}\ne 0$. Note that $D_{\tilde T}+1 = D_T$ and

$$\prod_{\substack{T\in\mathbf{T}\\ \tilde T'=T}} |\varepsilon|^{2^{-(D_{\tilde T}+2)}M_{\tilde T}} = \left(|\varepsilon|^{2^{-(D_{\tilde T}+2)}M_{\tilde T}}\right)^{k_T} , \tag{A1.12}$$

if k_T^0 is the number of maximal clusters in T. For any nonresonant cluster $\tilde T$ (*i.e.* with $m_{\tilde T}\ne 0$) one can use (A1.10). Then (A1.9) follows. ■

A1.6. Lemma. *For any tree $\vartheta\in\mathcal{T}^h_{n,m}$ one has*

$$\Big|\mathcal{R}\prod_{\ell\in\mathrm{int}(\vartheta)}\tilde g_\ell\Big| \le 2^{n-1}\left[\prod_{T\in\mathbf{T}}(G_3\gamma^{-h_T})^{L_{T_0}}\right]\left[\prod_{V\in\mathbf{V}}\gamma^{h_{V'}-h_V}\right] , \tag{A1.13}$$

where $G_3 = \max\{G_1, a_0G_2\}$, with G_1 and G_2 being defined, respectively, in (2.40) and in (2.48).

A1.7. *Proof.* Let $\vartheta\in\mathcal{T}^h_{n,m}$ and $n>1$ (the case $n=1$ is trivial, except for the ν-vertex). Let us consider the collection $\mathbf{V}_1$ of *maximal resonances*, *i.e.* resonances which are not strictly contained in any other resonance. If V is such a resonance, ℓ_V^i and ℓ_V^o are its external lines, and $x'_{\ell_V^i} = x'_{\ell_V^o}$. Then

$$\mathcal{R}\prod_{\ell\in\mathrm{int}\vartheta}\tilde g_\ell = \Big(\prod_{\ell\cap\mathbf{V}_1=\emptyset}\tilde g_\ell\Big)\prod_{V\in\mathbf{V}_1}\Big[\tilde g_{\ell_V^i}\,\tilde g_{\ell_V^o}\,\mathcal{R}\Xi_V^{h_V}(\mathbf{r}'_{\ell_V^i})\Big] , \tag{A1.14}$$

where

- $\tilde g_\ell$ is a shorthand for $\tilde g^{(h_\ell)}_{\rho_\ell^1,\rho_\ell^2}(\mathbf{r}'_\ell)$, if ℓ is an internal line of ϑ, $\tilde g_\ell = 1$ otherwise;
- $\prod_{\ell\cap\mathbf{V}_1=\emptyset}\tilde g_\ell = 1$, if ϑ itself is a resonance (so that all lines intersect $\mathbf{V}_1$);
- the *resonance value* (see the definition at the end of §2.5) $\Xi_V^{h_V}(\mathbf{r}'_{\ell_V^i})$ is given by

$$\Xi_V^{h_V}(\mathbf{r}'_{\ell_V^i}) = \Big(\prod_{\ell\in V\,:\,\ell\cap\mathbf{V}_2=\emptyset}\tilde g_\ell\Big)\prod_{W\in\mathbf{V}_2\cap V}\Big[\tilde g_{\ell_W^i}\,\tilde g_{\ell_W^o}\,\mathcal{R}\Xi_W^{h_W}(\mathbf{r}'_{\ell_W^i})\Big] , \tag{A1.15}$$

  where $\mathbf{V}_2$ is the collection of resonances which are strictly contained inside some resonance in $\mathbf{V}_1$, and which are maximal, and $\mathbf{V}_2\cap V$ is the subset of resonances in $\mathbf{V}_2$ which are contained in V.

We can write $\mathcal{R}\Xi_V^{h_V}(\mathbf{r}'_{\ell^i_V})$ as

$$\mathcal{R}\Xi_V^{h_V}(\mathbf{r}'_{\ell^i_V}) \equiv \Xi_V^{h_V}(x'_{\ell^i_V}, k_0) - \Xi_V^h(0,0) = \int_0^1 dt_V \left[\frac{d}{dt_V}\Xi_V^h(t_V x'_{\ell^i_V}, t_V k_0))\right] . \tag{A1.16}$$

Note that $\Xi_V^h(t_V x'_{\ell^i_V}, t_V k_0)$ can be written as in (A1.15), by substituting the argument x'_ℓ of any line ℓ with $t x'_{\ell^i_V} + \tilde{r}_\ell$, for suitable values of $\tilde{r}_\ell$. Therefore the r.h.s. of (A1.16) can be written as a sum of terms of the form (A1.15) with a derivative d/dt_V acting either
(1) on one of the propagators corresponding to a line outside $\mathbf{V}_2$, or
(2) on one of the $\mathcal{R}\Xi_W^{h_W}$.
In case (2), we write

$$\frac{d}{dt_V}\mathcal{R}\Xi_W^{h_W}(t_V x'_{\ell^i_V}, t_V k_0) = \frac{d}{dt_V}\left[\Xi_W^{h_W}(t_V x'_{\ell^i_V}, t_V k_0) - \Xi_W^{h_W}(0,0)\right] = \frac{d}{dt_V}\Xi_W^{h_W}(t_V x'_{\ell^o_V}, t_V k_0) , \tag{A1.17}$$

so that, if the derivative corresponding to a resonance V acts on the value of some resonance $W \subset V$, one can replace with $\mathbb{1}$ the $\mathcal{R}$ operator corresponding to W.

We can now iterate this procedure, by applying to $\Xi_W^{h_W}(t_V x'_{\ell^i_V}, t_V k_0)$ the equation (A1.14), with $\mathbf{V}_3$ (the family of resonances which are strictly contained inside some resonance belonging to $\mathbf{V}_2$ in place of $\mathbf{V}_2$), and so on. At the end (A1.14) can be written as a sum of $M_V - 1$ terms, if M_V denotes the number of vertices contained in V, which can be described in the following way.
(1) There is one term for each line $\bar{\ell} \in V$;
(2) if $\bar{\ell} \in T_0$, where T is a cluster contained in V (note that T can be equal to V), and $T = T_r \subset T_{r-1} \ldots \subset T_1 = V$ is the chain of r clusters containing T and contained in V, then the graph value can be computed by replacing with $\mathbb{1}$ the $\mathcal{R}$ operator acting on T_i, $i = 1, \ldots, r$, even if T_i is a resonance, because of the comments after (A1.16);
(3) the $\mathcal{R}$ operation acts on all other resonances contained in V;
(4) the derivative d/dt_V acts on the propagator of $\bar{\ell}$, whose argument is of the form $(t_V x'_{\ell^i_V} + \tilde{r}_{\bar{\ell}}, k_0)$.

A similar decomposition of the resonance value is now applied, for each term of the previous sum, to all resonant clusters, which are still affected by the $\mathcal{R}$ operation. This procedure is iterated, until no $\mathcal{R}$ operation is explicitly present; it is easy to see that we end with an expression of the form

$$\mathcal{R}\prod_{\ell \in \mathrm{int}(\vartheta)} \tilde{g}_\ell = \sum \int dt_1 \ldots dt_s \left[\prod_{T \in \mathbf{T}} \prod_{\ell \in T_0} \Big(\frac{d}{dt_{i(\ell)}}\Big)^{d_\ell} \tilde{g}_\ell\right] , \tag{A1.18}$$

where the sum is over all possible choices of s, $\{d_\ell\}$ and $\{i(\ell)\}$, which satisfy the following conditions:
(1) d_ℓ is equal to 0 or 1;
(2) if $d_\ell = 0$, $i(\ell)$ is arbitrarily defined, otherwise $i(\ell) \in \{1, \ldots, s\}$ and $i(\ell) \neq i(\ell')$, if $\ell \neq \ell'$;
(3) the number of lines for which $d_\ell = 1$ is equal to the number of interpolating parameters s;
(4) for each derived line ℓ there is a chain of r clusters $T = T_r \subset T_{r-1} \ldots \subset T_1 = V$, such that $\ell \in T_0$ and V is a resonance;
(5) no cluster can belong to more than one chain of clusters;
(6) each resonance belongs to one of the chains of clusters;
(7) the argument of the derived line is of the form $t_{i(\ell)} x' + \tilde{r}_\ell$, with $|x'| \le a_o \gamma^{-h_{V'}}$ (in general x' is not $x'_{\ell^0_V}$, but it can depend also on the interpolation parameters corresponding to resonances containing V, if any), where $h_{V'}$ is the scale of the smaller cluster containing it.

The item (7) above implies that, for each derived line,

$$\left|\frac{d}{dt_{i(\ell)}}\tilde{g}_\ell\right| \le a_0 G_2 \gamma^{h_{V'} - h_\ell} \gamma^{-h_\ell} , \tag{A1.19}$$

(see also (2.48)). Note that

$$h_{V'} - h_\ell = \sum_{i=1}^{r} \Big[h_{T_i'} - h_{T_i} \Big] \;, \tag{A1.20}$$

with the notations of item (4); hence the "gain" $\gamma^{h_{V'}-h_\ell}$ in the bound (A1.24), with respect to the bound of a nonderived propagator, can be divided between the clusters of the chain associated to the derived line ℓ, so that each cluster has a factor $\gamma^{h_{T_i'}-h_T} \le 1$ associated with it; in particular we have a factor of this type associated with each resonance, for each term in the sum of (A1.18). Since the number of terms in this sum is bounded by 2^{n-1}, we obtain the bound (A1.13). ∎

A1.8. By collecting together the results of the previous lemmata, one has

$$\begin{aligned} \Big|\mathrm{Val}(\vartheta)\Big| \le 2^{n-1} |\varepsilon|^{n/2} & \left[\prod_{\substack{T\subset \mathbf{T} \\ D_T \ge 1}} |\varepsilon|^{2^{-(D_T+1)} C_1 \gamma^{-h_T/\tau} k_T^0} \right] \cdot \\ \cdot \left[\prod_{T\in\mathbf{T}} (G_3 \gamma^{-h_T})^{L_{T_0}} \right] & \left[\prod_{V\in\mathbf{V}} \gamma^{h_{V'}-h_V} \right] \left[\prod_{T\in\mathbf{T}} \left(\frac{\gamma^{h_T} \nu_{h_T}}{|\varepsilon|} \right)^{M_{T_0}^{(\nu)}} \right] \;, \end{aligned} \tag{A1.21}$$

where L_{T_0} and $M_{T_0}^{(\nu)}$ denote, respectively, the number of lines and the number of ν-vertices in T_0.

A1.9. Lemma. *If $|\nu_h| \le B|\varepsilon|$ for any $h \le 1$ and for some constant B, then one has*

$$\left| \sum_{\vartheta\in\mathcal{T}_{n,m}^h} \mathrm{Val}(\vartheta) \right| \le \gamma^{h(\vartheta)} C D^{n-1} |\varepsilon|^{n/2} \;, \tag{A1.22}$$

for some constants C, D.

A1.10. *Proof.* One has the following (obvious) relations:

$$\begin{aligned} h_T &\le -D_T + 2 \;, \qquad \forall T \in \mathbf{T} \;, \\ L_{T_0} &= k_T - 1 \;, \\ k_T &= M_{T_0}^{(\nu)} + k_T^{\mathcal{R}} + k_T^0 \;, \end{aligned} \tag{A1.23}$$

with the notations listed at the beginning of this section.

One can write in (A1.21)

$$\gamma^{-h_T L_{T_0}} = \gamma^{h_T} \gamma^{-h_T M_{T_0}^{(\nu)}} \gamma^{-h_T k_T^0} \gamma^{-h_T k_T^{\mathcal{R}}} \;, \tag{A1.24}$$

and use that

$$\left[\prod_{T\in\mathbf{T}} \gamma^{-h_T k_T^{\mathcal{R}}} \right] \left[\prod_{V\in\mathbf{V}} \gamma^{h_{V'}-h_V} \right] = \prod_{V\in\mathbf{V}} \gamma^{-h_V} \;, \tag{A1.25}$$

as $k_T^{\mathcal{R}}$ is the number of resonances contained inside T, *i.e.* the number of resonances $V \in \mathbf{V}$ such that $V' = T$.

Then one can bound in (A1.21)

$$\begin{aligned} & \left[\prod_{\substack{T\subset \mathbf{T} \\ D_T \ge 1}} |\varepsilon|^{2^{-(D_T+1)} C_1 \gamma^{-h_T/\tau} k_T^0} \right] \left[\prod_{T\in\mathbf{T}} (\gamma^{-h_T})^{L_{T_0}} \right] \left[\prod_{V\in\mathbf{V}} \gamma^{h_{V'}-h_V} \right] \left[\prod_{T\in\mathbf{T}} \gamma^{h_T M_{T_0}^{(\nu)}} \right] \\ & \qquad \le \gamma^{h(\vartheta)} \left[\prod_{\substack{T\subset \mathbf{T} \\ D_T \ge 1}} \gamma^{-h_T k_T^0} \, |\varepsilon|^{2^{-(D_T+1)} C_1 \gamma^{-h_T/\tau} k_T^0} \right] \;, \end{aligned} \tag{A1.26}$$

as $\vartheta \notin \mathbf{V}$ so that, by using the first relation in (A1.23), one sees that each sum over $h_T \le -D_T + 2$ can be easily performed in (A1.2), if $k_T^0 \neq 0$. In fact if γ is so large that $\tilde\gamma \equiv \gamma^{1/\tau}/2 > 1$, if, $\forall N > 0$, C_N is such that

$$\exp\left\{-\log\left|\varepsilon^{-1}\right| 2^{-3} C_1 \tilde\gamma^r\right\} \le \frac{C_N}{1 + (2^{-3} C_1 \log|\varepsilon^{-1}|\, \tilde\gamma^r)^N} \ , \tag{A1.27}$$

(one can take $C_N = 1 + N!$) and if N is so that $\tilde\gamma^N \ge 2\gamma$, then

$$\begin{aligned}
&\sum_{h_T \le -D_T+2} \left(\gamma^{-h_T} |\varepsilon|^{2^{-(D_T+1)} C_1 \gamma^{-h_T/\tau}}\right)^{k_T^0} \le \sum_{r=D_T-2}^{\infty} \left(\gamma^r e^{-2^{-3} C_1 \log\left|\varepsilon^{-1}\right| (\gamma^{1/\tau}/2)^r}\right)^{k_T^0} \\
\le &\sum_{r=D_T-2}^{\infty} \left(\gamma^r e^{-2^{-3} \log\left|\varepsilon^{-1}\right| C_1 \tilde\gamma^r}\right)^{k_T^0} \le \Big(\sum_{r=D_T-2}^{\infty} \frac{C_N \gamma^r}{1 + (2^{-3} C_1 \log|\varepsilon^{-1}|)^N (2\gamma)^r}\Big)^{k_T^0} \\
&\le \left(C_4 2^{-D_T}\right)^{k_T^0} \ ,
\end{aligned} \tag{A1.28}$$

where $C_4 = 8C_N/(2^{-3} C_1 \log\left|\varepsilon^{-1}\right|)^N$.

The sum over $\{h_T\}$ would give some bad factor, when $k_T^0 = 0$, but it turns out that there is indeed no sum in this case. In fact, if all the clusters and vertices strictly contained in T are resonant (*i.e.* if $k_T^0 = 0$), then T itself must be a resonance and all its internal lines have the same x as the external ones, implying, by support properties of the f_h functions, that the scale label of the external lines is equal to $h_T - 1$. ∎

A1.11. As $n \ge |m|/\alpha$ then

$$|\varepsilon|^{n/4} \le \exp\left\{-\frac{|m|}{4\alpha} \log\left|\varepsilon^{-1}\right|\right\} \ , \tag{A1.29}$$

so that Lemma 2.10 follows from Lemma A1.9.

Appendix A2. The flow of the running coupling constants

A2.1. We still have to check that the bound on $|\nu_h|$ stated in Lemma A1.9 of the previous section is satisfied. Note that, $\forall h < 0$,

$$\begin{aligned}
s_h &= \sigma_h - \sigma_{h+1} = \sum_{q=2}^{\infty} \overline{\mathcal{W}}_q^{(h)}(\bar x, -\bar x; 0) \ , \\
\nu_h &= \gamma \nu_{h+1} + \gamma^{-h} \sum_{q=2}^{\infty} \overline{\mathcal{W}}_q^{(h)}(\bar x, \bar x; 0) \ .
\end{aligned} \tag{A2.1}$$

where $\overline{\mathcal{W}}_q^{(h)}(\bar x, \pm\bar x; 0)$ admit an expansion in terms of graphs ϑ, differing from the corresponding expansion of $\mathcal{W}_q^{(h)}(\bar x, \pm\bar x; 0)$ in the following respects:
(1) the $\mathcal{R}$ operation on the whole graph, which is necessarily a resonance, is substituted with the localization operation, hence in the previous analysis ϑ must not be included in the set $\mathbf{V}$;
(2) the internal scale of ϑ is equal to $h+1$, that is there is in the graph at least one line of frequency $h+1$.

A2.2. *Remark.* If all maximal clusters strictly contained in ϑ are resonant, as well as the vertices belonging to ϑ_0, that is if $k_\vartheta^0 = 0$, then $\mathrm{Val}(\vartheta) = 0$; the same holds if $k_\vartheta^0 = 1$. This follows from the support properties of the propagators, from the definition of resonance and from the observation that all lines $\ell \in \vartheta$ would have $x' = 0$, if $k_\vartheta^0 = 0$, since $x'_{\ell_1} = x'_{\ell_{n+1}} = 0$ for the external lines.

A2.3. Lemma. *If $|\nu_h| \le B|\varepsilon|$ for any $h \le 1$ and for some constant B, then, for any N, one has*

$$\left|\overline{\mathcal{W}}_n^{(h)}(\bar x, \pm\bar x; 0)\right| \le \gamma^{Nh} C_N D_N^{n-1} |\varepsilon|^{n/2} \ , \tag{A2.2}$$

for suitable constants C_N, D_N.

A2.4. *Proof.* Item (2) in §A2.1 implies that ϑ is a cluster on scale $h+1$; by the remark §A2.2 one has $k^0_{\vartheta_0} \neq 0$. Then we can bound the value of each graph contributing to $\overline{\mathcal{W}}_n^{(h)}(\bar{x}, \pm\bar{x}; 0)$ as in the proof of Lemma A1.9. The only difference is that now, in (A1.26), the product in the right hand side is also on ϑ itself, and there is no sum on $h(\vartheta)$ as $h(\vartheta) = h+1$. Therefore we can bound the factor corresponding to $T = \vartheta$ as

$$\begin{aligned} \gamma^{h(\vartheta)} \Big(\gamma^{-h(\vartheta)} |\varepsilon|^{2^{-(D_\vartheta+1)} C_1 \gamma^{-h(\vartheta)/\tau}} \Big)^{k^0_{\vartheta_0}} &\le e^{-2^{-3} C_1 \log|\varepsilon^{-1}| (\gamma^{1/\tau}/2)^{-(h+1)}} \\ &\le \frac{A_N}{1 + (2^{-3} C_1 \log|\varepsilon^{-1}|)^N \tilde{\gamma}^{-(h+1)N}} \le B_N \gamma^{hN} \; , \end{aligned} \tag{A2.3}$$

for suitable constants A_N, B_N. As all other factors can be bounded as before, the bound (A2.2) follows. ■

A2.5. Let us find a bound for σ_h. In case (1) of the theorem there is nothing to prove as by definition $\sigma_h \equiv 0$. In the second case note that in (A2.1) $\overline{\mathcal{W}}_n^{(h)}(\bar{x}, \pm\bar{x}; 0) = 0$ for all $n \le 2k+1$, as it is easy to see by using (2.47), (3.13) and (A2.1). Using (A2.1) and (A2.2) then for $k \ge 1$

$$|\sigma_h| \le C |\varepsilon|^{(2k+1)/4} \; ,$$

for some constant C. Note that no lower bounds are in general found. Nevertheless in the case $k=0$, the first contribution to σ arises from the perturbation, so that the value σ is explicitly computable: $\sigma = -\varepsilon/2 + o(\varepsilon)$.

A2.6. We discuss now the flow of the running coupling constant ν_h. The discussion is identical in case (1) or (2) of the theorem, for the lacking of lower bounds for σ_h (we cannot use the infrared cut-off for the flow of ν_h).

Define in (A2.1)

$$\beta_{h+1}(\varepsilon; [\nu_{h+1}, \nu_1]) = \gamma^{-h} \sum_{q=2}^{\infty} \overline{\mathcal{W}}_q^{(h)}(\bar{x}, \bar{x}; 0) \; , \qquad [\nu_{h+1}, \nu_1] \equiv (\nu_{h+1}, \nu_{h+2}, \ldots, \nu_0, \nu_1) \; , \tag{A2.4}$$

so that, by iteration, one finds $\forall h \le 0$

$$\nu_h = \gamma^{-h+1} \Big(\nu_1 + \sum_{k=h+1}^{1} \gamma^{k-2} \beta_k(\varepsilon; [\nu_k, \nu_1]) \Big) \; . \tag{A2.5}$$

A2.7. *Remarks.* (i) Note that, in any contribution to $\beta_k(\varepsilon; [\nu_{h+1}, \nu_1])$ containing at east one vertex $\nu_{h'}$, for some $h' \ge k+1$, there must be at least two nonresonant vertices (see the Remark A2.2).
(ii) There are contributions to $\beta_k(\varepsilon; [\nu_{h+1}, \nu_1])$ containing only nonresonant vertices (at least two of them).

A2.8. Introduce a sequence $\{\nu_h^{(n)}\}$, with $n \ge 0$, defined recursively, for any $h \le 0$, as

$$\begin{aligned} &\nu_h^{(0)} = 0 \; , \\ &\nu_h^{(n)} = \gamma^{-h+1} \left(\nu_1^{(n-1)} + \sum_{k=h+1}^{1} \gamma^{k-2} \beta_k^{(n-1)} \right) \; , \qquad \beta_k^{(n-1)} = \beta_k(\varepsilon; [\nu_k^{(n-1)}, \nu_1^{(n-1)}]) \; , \end{aligned} \tag{A2.6}$$

and set $\beta_k^{(0)} = 0$.

A2.9. Lemma. *If for any $n \le 0$ one formally sets*

$$\nu_1^{(n)} = - \sum_{k=-\infty}^{1} \gamma^{k-2} \beta_k^{(n)} \; , \tag{A2.7}$$

then the sequence $\{\nu_h^{(n)}\}$ *converges uniformly to a limit* ν_h *such that*

$$|\nu_h| \le B\,|\varepsilon|\,\gamma^{(N-1)h}\ , \tag{A2.8}$$

for all $h \le 1$ *and for some constant* B.

A2.10. *Proof.* We show by induction that $\forall h \le 1$

$$\left|\nu_h^{(n)} - \nu_h^{(n-1)}\right| \le B_0 \gamma^{(N-1)h}\,|\varepsilon|^n\ , \qquad \left|\beta_h^{(n)} - \beta_h^{(n-1)}\right| \le B_1 \gamma^{(N-1)h}\,|\varepsilon|^n\ , \tag{A2.9}$$

for some constants B_0 and B_1.

For $n = 1$, by considering that $\nu_h^{(0)} = 0$ for any h, the bound (A2.2) and Remark (ii) above give

$$\left|\beta_h^{(1)}\right| \le B_1 \gamma^{(N-1)h}\,|\varepsilon|\ , \tag{A2.10}$$

for some constant B_1, so that by defining $\nu_1^{(1)}$ as in (A2.6) for $n = 1$, one obtains

$$\left|\nu_h^{(1)}\right| \le B_0 \gamma^{(N-1)h}\,|\varepsilon|^n \qquad \forall h \le 0\ , \qquad\qquad \left|\nu_1^{(1)}\right| \le B_0 \gamma^{(N-1)}\,|\varepsilon|^n\ , \tag{A2.11}$$

so proving (A2.9) for $n = 1$.

If $n > 1$ and (A2.9) holds for n, then $\beta_h^{(n+1)} - \beta_h^{(n)}$ can be written as sum of values of graphs in which there is at least one vertex with $\nu_{h'}^{(n)} - \nu_{h'}^{(n-1)}$, for some $h' \ge h$ (and at least two other vertices; see Remark A2.7, (i), above). Therefore, as $|\nu_{h'}^{(n)} - \nu_{h'}^{(n-1)}| \le B_0 \gamma^{(N-1)h'}\,|\varepsilon|^n$ and any graph contributing to $\beta_h^{(n+1)} - \beta_h^{(n)}$ has to contain two nonresonant vertices (see Remarks A2.7), the second bound in (A2.9) follows by using (A2.2). Then (A2.6) and (A2.7) imply the first one.

Therefore $\{\nu_h^{(n)}\}$ converges uniformly to a limit ν_h, and ν_h verifies the bound (A2.8) for $h \le 0$. Moreover ν_1, which is given by the limit of (A2.6), for $n \to \infty$ verifies (A2.7) for $h = 1$. ■

A2.11. If φ_x is not even one has four running coupling constants (see §4.2). The flow of the constants σ_{1h} and σ_{2h} is controlled exactly as in §A2.5.

As far as the constants ν_{1h} and ν_{2h} are concerned, one can define the functions

$$\beta_{jh}(\varepsilon; [\nu_{1,h}, \nu_{11}], [\nu_{2,h}, \nu_{21}])\ , \tag{4.12}$$

with $j = 1, 2$ and with the notations of (A2.4); it is easy to check that, formally setting

$$\nu_{1j}^{(n)} = -\sum_{k=-\infty}^{1} \gamma^{k-2} \beta_{jk}^{(n-1)}\ , \qquad j = 1, 2\ , \tag{A2.13}$$

with $\beta_{jk}^{(n)} = \beta_{jk}(\varepsilon; [\nu_{1,k}^{(n)}, \nu_{11}^{(n)}], [\nu_{2,k}^{(n)}, \nu_{21}^{(n)}])$, the sequences $\{\nu_{1h}^{(n)}\}$ and $\{\nu_{2h}^{(n)}\}$, defined as in (A2.6), with the obvious modifications, converge uniformly to two limits ν_{1h} and ν_{2h}, respectively, such that

$$|\nu_{1h}|\,, |\nu_{2h}| \le B\,|\varepsilon|\,\gamma^{(N-1)h}\ , \tag{A2.14}$$

so that the same conclusions as in the even case are obtained.

Appendix A3. Convergence of the Schwinger functions

A3.1. Let ϑ be one of the graphs contributing to the kernel $K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y})$, and let us consider the two vertices, v_1 and v_q, connected to the external lines (which are associated with the external field): such vertices will be called the *external vertices*.

Suppose first that neither v_1 nor v_q are contained in any cluster, different from ϑ itself. In this case, we can bound $\mathrm{Val}(\vartheta)$ as in Appendix A1, by taking into account that
(1) there is no factor associated to the external vertices;
(2) $h(\vartheta) = h+1$;
(3) there are at least two lines of scale $h+1$, the external propagators.
Hence we get a bound differing from (A1.29) only because the power of $|\varepsilon|$ is $n-2$ instead of n and each external propagator gives a contribution proportional to $\gamma^{-h(\vartheta)}$

$$|\mathrm{Val}(\vartheta)| \le \gamma^{-h(\vartheta)} |\varepsilon|^{\frac{n}{2}} C_2^{4M_{\mathbf{T}}} \left\{ \prod_{T\in\mathbf{T}} \left[\gamma^{-h_T} \, e^{-2^{-3}\log|\varepsilon^{-1}| C_1 \gamma^{-h_T/\tau}} \right]^{k^0_{T_0}} \right\} , \tag{A3.1}$$

where the same notation of Appendix A1 is used, except for the definition of M_T, which differs from the previous one, since we do not consider the external vertices in the calculation of $M_T^{(2)}$; moreover we assigned a label $n_v = 0$ to the external vertices.

A3.2. Suppose now that v_1 is contained in some cluster strictly contained in ϑ and that the scale of the external propagator emerging from v_1 is h_1. In this case, there is a chain of clusters $T^{(1)} \subset T^{(2)} \ldots \subset T^{(r)} = \vartheta$, such that $v_1 \in T^{(i)}$ and $h_{T^{(1)}} = h_1$; moreover $\mathcal{R} = 1\!\!1$ on $T^{(i)}$, $i = 1, \ldots, r$, even if $T^{(i)}$ is a resonance.

We proceed again as in Appendix A1, but we have to take into account the lack of the factor $\gamma^{h^e_{T^{(i)}} - h_{T^{(i)}}}$, which was present before, when $T^{(i)}$ is a resonance. Since ϑ is not a resonance (by definition) and $h^e_{T^{(i)}} = h_{T^{(i+1)}}$, we loose at most a factor $\gamma^{h(\vartheta)-h_{T^{(1)}}} = \gamma^{h+1-h_1}$. If we also consider the bound of the external propagator emerging from v_1, we see that the overall effect of the vertex v_1 in the bound of $\mathrm{Val}(\vartheta)$ is to add a factor γ^{-h-1} to the expression in the r.h.s. of (A1.29), that is the same effect that we should get, if the only cluster containing v_1 was ϑ.

A similar argument can be used for studying the effect of the vertex v_q. Hence we get the bound (A2.6) for all graphs contributing to $K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y})$.

Appendix A4. Proof of the theorem 4.6

A4.1. As the proof of the theorem 4.6 proceeds very similar to that of the theorem 1.4, we only outline the main differences and show how they can be dealt with.

A4.2. Suppose that $\bar\varphi(\omega\bar x_j) = \mu$ for p points $\bar x_1, \ldots, \bar x_p$. For each j one introduce s_j running coupling constants $\nu^{(0)}_{hj}, \ldots, \nu^{(s_j-1)}_{hj}$, if s_j is the first nonvanishing derivative of $\bar\varphi(\omega x)$ at $x = \bar x_j$ Then we can set $s_0 = \max\{s_1, \ldots, s_p\}$.

In defining the scales, we set

$$\|\mathbf{r}'\|^2 \equiv k_0^2 + (v_0^{(j)})^2 \|\omega x'\|_{\mathbb{T}}^{2s_j} , \tag{A4.1}$$

where $x = \bar x_j + x'$ when $\|\omega x - \bar\omega_j\|_{\mathbb{T}}$ is small, so that, if

$$\varphi_{x'+\varepsilon\bar x_j} - \mu = v_0^{(j)} \, (\omega x')^{s_j} + \Phi^j_x , \qquad v_0^{(j)} = \frac{1}{s_j!} \partial^{s_j} \bar\varphi(\omega\bar x_j) , \tag{A4.2}$$

then, by setting $t_0 = a_0/\gamma$, one has that

$$|v_0^{(j)}| \, \|\omega x'\|_{\mathbb{T}} \le t_0 \gamma^{(h+1)/s_j} , \tag{4.3}$$

when the corresponding propagator is on scale h.

Then we introduce the running coupling constants as in (4.10). Taking into account the bound (4.3) on $\|\omega x'\|_{\mathbb{T}}$, one proves a result like Lemma A1.9, provided, again, all running coupling constants remain bounded by $B|\varepsilon|$, for some constant B. Note that it has been aiming at having the condition expressed in such a way that the writing (4.10) for the running coupling constants imposes itself as the natural one.

The only difference is that now each resonance has to be renormalized to s_j order: this means that one has to derive up to $s_j - 1$ times the propagators, but a condition

$$\left| \frac{d^r}{dt^r} \tilde{g}^{(h_\ell)}_{\varepsilon^1_\ell,\varepsilon^2_\ell}(t\mathbf{r}'_1 + \mathbf{r}'_2) \right| \le G_2 \|\mathbf{r}'_2\|^{2s_j} \gamma^{-2h} \qquad t \in [0,1] \;, \tag{A4.4}$$

holds, for some constant G_2 (in general G_2 grows as $s_0!$).

A4.3. One can then proceed like in the proof of the theorem 1.4, by using that the bound in Lemma A2.3 can be extended to the quantity

$$\frac{\partial^r}{\partial(\omega x')^r} \overline{W}(\bar{x}_j, \bar{x}_j; 0) \;, \qquad 1 \le r \le s_j \;, \qquad j = 1, \ldots, p \;. \tag{A4.5}$$

One finds that it is possible to choose the running coupling constants on scale $h = 1$, $\nu^{(r)}_{1j}$, by proceeding exactly as in §A2.8, and setting

$$\nu^{(r,n)}_{j1} = - \sum_{k=-\infty}^{1} \gamma^{k-2} \beta^{(r,n)}_{jk} \;, \tag{A4.6}$$

with obvious meaning of the symbols. The analysis is as in §A2.8 and the same conclusions are obtained. Then the running coupling constants on scale $h = 1$ define the counterterms appearing in (4.14).

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