Renormalization Group for One-Dimensional Fermions. A Review on Mathematical Results

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sec.1

1. Introduction

p.1.1 **1.1.** *A general overview of the state of the art.* The study of one-dimensional non-relativistic interacting Fermi systems has attracted a vaste interest over the years, among physicists and mathematicians. The mathematical interest is motivated by the possibility, due to the low dimensionality, to obtain some rigorous non trivial results about such systems (conversely up to now this is almost impossible in higher dimensions). The physical motivations arise from the fact that such systems can modelize some real materials, like organic anysotropic compounds. A new wind of interest among physicists was generated in the '90 by the Anderson theory of high T_c superconductivity [A], which relies on the assumption that the physics of two-dimensional interacting Fermi systems is somehow similar to the physics of one-dimensional ones.

As far as rigorous results are concerned, they can be distinguished mainly into two classes: results obtained by exact solutions and results obtained by the study of the Schrödinger equation. An excellent rewiev of exact solutions is in [Ma1]: we can recall the classical *exact* solutions of the *Luttinger model*, [ML], of the *Hubbard model*, [LW], and of the spin chains (which can be seen as interacting Fermi systems by performing a Schwinger-Dyson transformation) like the XY *model*, [LSM], the XXY *model*, [YY], and the XYZ model, [B]. Such solutions are really nonperturbative, as they hold also for large coupling and are based mainly on (rigorous) bosonization, Bethe Ansatz or transfer matrix method. However a limitation of such solutions is that they can not be extended to other models, even to very similar ones, as they are crucially dependent on the fine details of the models. Moreover – with the remarkable exception of the Luttinger model – the exact solutions provide a detailed information of the Hamiltonian spectrum, but it is generally not possible to derive from them the correlations (in terms of which the physical observables are expressed).

In a completely different framework other rigorous results about one-dimensional Fermi systems in an external field can be derived by the analysis of the one-dimensional single particle Schrödinger equation (see for instance [T] or [PF] for reviews). We can mention [K] for the periodic potential, [FS] and [AM] for the stochastic potential and [DS], [MP] and [E1] for the quasi-periodic potential. From such results about the spectrum of the Schrödinger equation one can obtain in principle the asymptotic behaviour of the correlation functions for a system of fermions in an external field; this is however nontrivial in general (one has to use some properties of the wave functions in the complex plane) and, as far as we know, it has been done only in the case of random potentials in [AG] and in [BM1] in the case of periodic potentials.

It is very difficult to resume the large number of works in the physics literature about one-dimensional Fermi systems (we can refer to the classical [So] or to the more recent [V], [SCP], [MCD]). Many results are found by *third order* multiplicative Renormalization Group, [So], but it is not clear the relevance of the higher order terms and the validity of the third order approximation. Moreover such methods can be applied only to models with linear dispersion relations (so not really fundamental ones) and only if there is no lattice and if the volume is infinite. Such limitations are particularly annoying as they make difficult a detailed comparison with numerical simulations. Other results are found by the "bosonization" techniques, in which the ultraviolet problem is not treated in a consistent way so that an extra parameter – not present in the original model – appears in the expressions found for the correlations, see [LP]; this means that such expressions can be in any case only approximately true. While it is likely that many of the physical conclusions are valid, the lack of distinction between rigorous results and results not really proved at a mathematical level makes generally very difficult the dialogue between theoretical physicists, mathematical physicists and mathematicians working more or less on the same problems.

We mention finally the approach based on conformal quantum field theory, see for instance [FK]. This approach is quite powerful as it can provide the critical indices, but it can be generally applied only to models for which the exact solutions are possible.

p.1.2 **1.2.** *More recent results.* In this work we shall review what is known at a rigorous level about the correlation functions of many (generally *not soluble*) models of interacting one-dimensional Fermi systems, with emphasis

about the new results obtained starting from the '90. A main novelty (with respect to the framework briefly described in §1.1) was the application (started from [BG1] and [FT]) to solid state models of the techniques based on the rigorous implementation of Wilsonian Renormalization Group, [W], developed in the context of *constructive Quantum Field Theory* (see [BG2], [Br], [GK2] and [R] for reviews): this was a quite natural development, as field theory methods were applied to solid state physics since many years (see for instance [AGD]). Such techniques allow in principle to express the correlation functions of a quantum field theory describing Fermi systems as convergent series (even if they are generally non-analytic in the perturbative parameters). One of the first realization of this was the theory of the Gross-Neveu model (a system of relativistic one-dimensional fermions) developed in [GK1], [FMRT1] and [Le]. The application of such techniques to one-dimensional non-relativistic Fermi systems was originally discussed in [BG1], [BeGM], [GS], [BGPS], [BM1], [BM2], [BM3], [BGM1], [BGM2], [M1], [M2], [M3], [BeM] and [GeM], and it will be the main content of the present review. The result is that the correlation functions of many *not soluble* models can be written as convergent series, in the weak coupling regime, and such expressions provide all the informations one is interested in.

p.1.3 **1.3.** *Contents.* Aim of this paper is from one side to review in a systematic way results spreaded out in a number of works and from the other to provide the technical tools necessary to read the original papers. The physical observables are expressed in terms of *Schwinger functions*, which in turn are expressed by functional integrals defined in terms of *Grassman variables*; in §4 we resume some properties of the fermionic functional integrals which will be used to define a constructive algorithm for the computation of the Schwinger functions. The Renormalization group ideas are implemented by writing each integration as product of many integrations "on different scales" and the integration of a scale leads to a new effective interaction; the technical tools for defining the expansions (trees, clusters, Feynman diagrams, and so on) are defined in §5. This leads to a sequence of effective interactions whose expansion converges provided that the previous scale interaction is small, due to cancellations based on the Fermi statistics, see §6.

In §8 for fixing ideas we consider a particular model, and we define an *anomalous expansion* for the Schwinger function of it: as a *paradigmatic* model we choose the *Holstein-Hubbard* model for spinless fermions, as it contains essentially all the possible difficulties encountered for spinless fermions; it describes in fact fermions subject to a quasi-periodic potential and interacting through a short range two body potential. We start by defining an expansion for the *effective potential*. The presence of a quasi-periodic potential has the effect that the expansion is afflicted by a small divisor problem, so that a comparison for the series appearing in classical mechanics is natural. The theory has an *anomalous dimension* and the bare parameters are modified by *critical indices*. The flow of the running coupling constants is controlled using some *hidden symmetry* of this model, see §10. In particular one exploits remarkable cancellations in the "beta function", proved by a non-perturbative argument based on the exact solution of the Luttinger model, see §16; in other words we extract from the exact solution of the Luttinger model informations for not exactly solvable models, using the fact that they are "close" in a renormalization group sense. An expansion for the two-point Schwinger function is defined in §11, while an expansion for the density-density correlation function is defined in §13. In order to compute the asymptotic behaviour of the density-density response function one has to prove an *approximate Ward identity*, see §15. In §12 we collect the results about the Schwinger functions for a number of *spinless* models, discussing briefly how the above scheme has to be adapted for each of them. Such models are on a lattice or on the continuum, they interact with a*periodic* or *quasi-periodic* external potential, and include a two-body short range interaction. Our results are limited to the case of small external and two body potential, with the exception of the case of large external quasi-periodic field (considered in absence of the two-body interaction) in which the phenomenon of *Anderson localization* is found. The XYZ Heisenberg spin chain is incuded in the class of models we can treat, as it can be written as a interacting fermionic model with an anomalous potential and the spin-spin correlation function is related to the density-density response function, see §13.

We then consider the presence of the spin: the number of running coupling constants increases, see §17

and it turns out that only if the two-body interation is *repulsive* the running coupling costants remain small. This is due to cancellations in the beta function based on the solution of the *Mattis model*, the analogue of the Luttinger model with spinning fermions. In the repulsive case the behaviour of the Schwinger function is similar to the one in the spinless case. In the attractive case only mean field approximations are possible at the moment (with the remarkable exception of the Hubbard model, which was solved in [LW]). If the fermions are on the lattice even the mean field theory is not trivial: we show, see §19, that a mean field theory foresees the formation of collective excitations called *density waves* for any *rational* fermionic density, but there are no results for an *irration density* (in the weak coupling region). We discuss also a mean field theory for a *two chain* system exchanging Cooper pairs, in which a version of the BCS equation for Luttinger liquids is found. Finally we discuss some results for finite temperature bidimensional fermions, see [DR1] and [DR2].

The reader willing to go immediately to the main results before reading the technical parts can read directly §12, §17 and §19.

sec.2

2. One-dimensional interacting Fermi systems

p.2.1 **2.1.** *Free systems.* Let $\psi^{\pm}_{x,\sigma}$ be fermionic creation or annihilation operators defined in the standard fermion Fock space, [NO]. If $\sigma = \pm 1/2$ we say that the fermions are *spinning* (so such operators can describe real electrons), while if $\sigma = 0$ we say that the fermions are *spinless*. Despite the fact that spinless fermions have no physical meaning, they are widely studied in the literature; one can say (tautologically) that they are easier to study. Furthermore the results for spinless systems can be used to understanding phenomena in which the spin does not play any rôle.

The physical systems one aims to modelize are crystals so anysotropic that they can be approximatively described by one-dimensional systems: the conduction electrons are supposed to be confined on a segment and they interact with each other, with the periodic or quasi-periodic background potential generated by the ions of the crystal, with phonons, with stochastic impurities and so on (for physical motivations see [S], [BJ], [SCP] and [V]).

There are two main classes of models describing one-dimensional fermion systems. The first class are the *lattice models* and are such that x is an integer, say between $-[L/2]$ and $[(L-1)/2]$: we shall write $x \in \Lambda$ in such a case, if $\Lambda = \{x \in \mathbb{Z} \ : \ -[L/2] \le x \le [(L-1)/2]\}$. One describes in this way fermions on a chain with length L and step $a = 1$, thinking that the electrons are localized on atomic sites and they can hop to neighbouring sites. Considering only the possibility of hopping between nearest neighbour sites (*i.e.* neglecting the interaction of the electrons with themselves and with the environment) the *hopping Hamiltonian* (by setting $S = 0$ if the fermions are spinless and $S = 1/2$ if they are spinning) is given by

1.2
$$\begin{aligned} H_0 &= T_0 - \mu_0 N_0 \ , \\ T_0 &= \frac{1}{2S+1} \sum_{\sigma=\pm S} \sum_{x\in\Lambda} \left[\frac{1}{2} \left(-\psi^+_{x,\sigma}\psi^-_{x+1,\sigma} - \psi^+_{x,\sigma}\psi^-_{x-1,\sigma} + 2\psi^+_{x,\sigma}\psi^-_{x,\sigma} \right) \right] \ , \\ N_0 &= \frac{1}{2S+1} \sum_{\sigma=\pm S} \sum_{x\in\Lambda} \left[\psi^+_{x,\sigma}\psi^-_{x,\sigma} \right] \ . \end{aligned} \tag{2.1}$$

In the above formulae μ_0 is the *chemical potential* and it is fixed by the density (we shall work in the *grand canonical ensemble*). The Hamiltonian (2.1) is also called the *tight binding* Hamiltonian.

Another class of models are the *continuum models*, in which the fermions are on the continuum and in such a case x assumes values on the segment $[-L/2, L/2]$. One imagines in this case that the positive charge of the ions is spreaded out in the metal (*jellium*). Then the corresponding Hamiltonian, again by neglecting

any form of interaction, is given simply by the *kinetic energy* operator

1.3
$$\begin{aligned} H_0 &= T_0 - \mu_0 N_0 \ , \\ T_0 &= \frac{1}{2S+1} \sum_{\sigma=\pm S} \int_{-L/2}^{L/2} \mathrm{d}x \, \psi^+_{x,\sigma} \frac{1}{2m} \partial_x^2 \psi^-_{x,\sigma} \ , \\ N_0 &= \frac{1}{2S+1} \sum_{\sigma=\pm S} \int_{-L/2}^{L/2} \mathrm{d}x \, \psi^+_{x,\sigma} \psi^-_{x,\sigma} \ . \end{aligned} \tag{2.2}$$

As in (2.2) μ_0 denotes the chemical potential.

p.2.2 **2.2.** *Interaction with the lattice.* We assume, as it is usual, that the Hamiltonian describing the interacting fermions is obtained by adding to the free Hamiltonian H_0 some other terms, according to the kind of interaction one wants to describe. In this way we get more realistic models with respect the ones considered in §2.1.

The conduction electrons interact through electric forces with the lattice of ions; in first approximation this interaction can be described in terms of a *pseudopotential*, which is assumed a regular periodic function which takes into account the lattice periodicity. In the continuum models one then adds to the Hamiltonian H_0 a term

1.6a
$$uP = u \int_{-L/2}^{L/2} \mathrm{d}x \, \varphi(x) \, \psi^+_{x,\sigma} \psi^-_{x,\sigma} \ , \tag{2.3}$$

with φ periodic with period T, *i.e.* $\varphi(x) = \varphi(x+T)$, and regular in its argument (what we mean exactly by "regular" will become clear later when we shall discuss in detail the model). It is well known that the presence of such a periodic potential leads to the formation of *energy bands*.

In lattice models the presence of the ion lattice is already described by the fact that one has $x \in \Lambda$; however to describe energy bands one can still add to the Hamiltonian H_0 a term

1.6b
$$uP = u \sum_{x \in \Lambda} \varphi(x) \, \psi^+_{x,\sigma} \psi^-_{x,\sigma} \ , \tag{2.4}$$

with $\varphi(x) = \varphi(x+T)$ for some integer $T > 1$.

For a long time solid state systems were considered as either *crystalline* (*i.e.* lattice periodic) or *amorphous*. The lattice periodicity was then described in terms of interactions with periodic pseudopotentials like (2.3) and (2.4). However in recent times several solid state systems with a *quasi-periodic* structure have been discovered (see for istance [AxG]). In some cases such materials have a basic structure and a periodic modulation superimposed on it, such that the periodicity of the modulation is *incommensurate* with the periodicity of the basic structure. Another possibility is that of structures composed by two periodic lattice subsystems, with mutually incommensurate periods.

In order to study the electronic properties of quasi-periodic systems, in case of lattice systems one can add to the Hamiltonian H_0 a term like (2.4), but in which one has $\varphi(x) = \varphi(x+T)$ with an *irrational* T, so that T is incommensurate with the period of the lattice (which is 1 in the units we have chosen); in the case of continuum systems one can write (2.3) with $\varphi(x)$ a quasi-periodic function, *i.e.* a function with two incommensurate intrinsic periods.

The lattice can be not exactly periodic or quasi-periodic, for the unavoidable presence of impurities: their presence can be modellized by the introduction of an additional term in the Hamiltonian describing the interaction with a white noise (for istance). Of course such possibilities are not incompatible, *i.e.* one can consider together both a stochastic and a periodic interaction.

p.2.3 **2.3.** *Interaction between the electrons.* The conduction electrons interact with each other: taking into account such interactions is essential for the understanding of many properties (superconductivity, magnetism, Mott

transition and so on). We can assume that the interaction between the fermions is given by a two-body potential. The interaction is assumed to have short range, as the Coulombian interaction should be screened in the metals. Then one can add to the Hamiltonian H_0 (or $H_0 + uP$), in the case of lattice systems, a term of the form

1.7
$$\lambda V = \lambda \frac{1}{(2S+1)^2} \sum_{\sigma,\sigma'=\pm S} \sum_{x,y\in\Lambda} v(x-y)\, \psi^+_{x,\sigma}\psi^+_{y,\sigma'}\psi^-_{y,\sigma'}\psi^+_{x,\sigma} \ , \tag{2.5}$$

with

1.8
$$|v(x-y)| \le v_0 e^{-\kappa|x-y|} \ , \tag{2.6}$$

for some positive constants κ and v_0. In some special cases, *e.g.* in the so called *Hubbard model*, $v(x-y) = \delta_{|x-y|,1}$.

In the continuum case one has

1.9
$$\lambda V = \lambda \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y\, v(x-y)\, \frac{1}{(2S+1)^2} \sum_{\sigma,\sigma'=\pm S} \psi^+_{x,\sigma}\psi^+_{y,\sigma'}\psi^-_{y,\sigma'}\psi^+_{x,\sigma} \ , \tag{2.7}$$

where v can be assumed to be a smooth function satisfying (2.6).

p.2.4 **2.4.** *Interaction with the phonons.* It is also important to consider the interaction with *phonons*, which are the quantized oscillations of the ion positions, *i.e.* of the lattice. One has to add to the Hamiltonian a term of the form

1.9a
$$H_B + \sum_{x\in\Lambda} \phi_x \left(\psi^+_{x,\sigma}\psi^-_{x,\sigma} - \frac{1}{2} \right) \ , \tag{2.8}$$

with

1.9b
$$H_B = -\frac{1}{\sigma_0^2} \sum_{x\in\Lambda} \frac{\partial^2}{\partial \phi_x^2} + \sum_{x\in\Lambda} \left(\phi_x^2 + b^2(\phi_x - \phi_{x+1})^2 \right) \ , \tag{2.9}$$

where ϕ_x is a boson quantum field, corresponding to a discretized vibrating string with linear density σ_0^2, optical frequency ω and maximum wave propagation speed c, so that $b = c\omega^{-1}$. One could take into account also acustic phonons.

p.2.5 **2.5.** *Spin-Hamiltonians.* Another class of models very related to the ones we are considering are the *spin-Hamiltonians*, like the *Heisenberg Hamiltonians*, where there is a 1/2-spin on each site of a lattice and the interaction is between nearest neighbours.

In dimension $d = 1$ a very general model is the XYZ model (which contains as limiting cases the XY model, the XXZ model and others) which is described by the Hamiltonian

1.12
$$H = \sum_{x=1}^{L-1} \left[J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1} + J_3 S^3_x S^3_{x+1} + h S^3_x \right] + U^1_L \ , \tag{2.10}$$

where $S^j_x = 2\sigma^j_x$, if σ^1_x, σ^2_x and σ^3_x are the Pauli matrices,

1.13
$$\sigma^1_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma^2_x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \qquad \sigma^3_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{2.11}$$

while U^1_L is a boundary interaction term. The Hamiltonian (2.10) can be written, [LSM], as a *fermion interacting spinless Hamiltonian*. In fact, it is easy to check that, if $\sigma^\pm_x = (\sigma^1_x \pm i\sigma^2_x)/2$, the operators

1.14
$$a^\pm_x \equiv \left[\prod_{y=1}^{x-1} (-\sigma^3_y) \right] \sigma^\pm_x \tag{2.12}$$

are a set of anticommuting operators and that we can write

1.15
$$\sigma_x^- = e^{-i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} a_x^- \ , \quad \sigma_x^+ = a_x^+ e^{i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} \ , \quad \sigma_x^3 = 2a_x^+ a_x^- - 1 \ . \tag{2.13}$$

Hence, if we normalize the interaction so that $J_1 + J_2 = 2$ and we introduce the *anysotropy*

1.15a
$$u = \frac{J_1 - J_2}{J_1 + J_2} \ , \tag{2.14}$$

we get

1.16
$$\begin{aligned} H = \sum_{x=1}^{L-1} \Big\{ -\frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - \frac{u}{2}[a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-] - \\ -J_3 \left(a_x^+ a_x^- - \frac{1}{2} \right) \left(a_{x+1}^+ a_{x+1}^- - \frac{1}{2} \right) \Big\} - h \sum_{x=1}^{L} \left(a_x^+ a_x^- - \frac{1}{2} \right) + U_L^2 \ , \end{aligned} \tag{2.15}$$

where U_L^2 is the boundary term in the new variables. We choose it so that the fermionic Hamiltonian (2.15) conicides with the Hamiltonian of a fermion system on the lattice with periodic boundary conditions, that is we put U_L^2 equal to the term in the first sum in the r.h.s. of (2.15) with $x = L$ and $a_{L+1}^{\pm} = a_1^{\pm}$ (in [LMS] this choice for the XY chain is called "c-cyclic"). Then the XYZ model can be considered as a fermionic model of the class we are discussing.

The XYZ Hamiltonian has a sort of anomalous potential of the form (generalizing it to the case of spinning fermions)

1.17
$$\xi B = \xi \sum_{x \in \Lambda} \left(\psi_{x,\sigma}^+ \psi_{x,-\sigma}^+ + \psi_{x,-\sigma}^- \psi_{x,\sigma}^- \right) \ . \tag{2.16}$$

Such a potential appears in mean field BCS theory in which the superconductivity phenomenon is approximately described in terms of an anomalous potential like (2.16). We shall consider the case of two one-dimensional interacting fermionic systems coupled by a Cooper interaction and we shall see that, in the analogous of the Bardeen approximation, one is led to consider an interacting fermion system with a term like (2.16)

p.2.6 **2.6.** *General interacting systems.* So in the following we can consider Hamiltonians which, in the most general case, could be of the form

1.20
$$H = H_0 + uP + \lambda V + \xi B + H_B \ . \tag{2.17}$$

Usually not all the possible interacting terms are considered together as the corresponding analysis would be very intricated. So we shall begin by considering a particular case, both for propedeutical and physical reasons: the analysis will be easier to perform (and still not so easy!) and in describing physical situations not all the interacting terms are expected to be at the same level at the same time.

p.2.7 **2.7.** *Other Hamiltonian models.* There are many other one-dimensional interacting fermionic models. One is the *Luttinger model*, [L] and [ML], which will play an important rôle in our analysis; there are many extensions of this model to spinning fermions, called the *Mattis model*, the *g-ological model*, the *Luther-Emery* model and so on. All such models are not true "fondamental" ones, in the sense that they are considered approximations, in some physical situations, of the models with Hamiltonians listed above; so we shall not discuss them here. We shall see that our methods make us to introduce such models in a natural way and to give a rigorous meaning to the intuition that such models are "close" to the one we are considering.

There are also many relativistic model, like the *Thirring model* or the *Yukawa$_2$ model*, which are closely related to the models with the Hamiltonians listed above, in some particular limit.

sec.3

3. Schwinger functions and physical observables

p.3.1 **3.1.** *Definition.* Fix $\beta > 0$. Setting $\mathbf{x} = (x, x_0)$, with $x \in \Lambda$ and $x_0 \in [-\beta/2, \beta/2)$, define $\psi^{\varepsilon}_{\mathbf{x},\sigma} = e^{x_0 H} \psi^{\varepsilon}_{x,\sigma} e^{-x_0 H}$.

If $\{t_1, \dots, t_s\}$ is a collection of time variables $t_i \in (-\beta/2, \beta/2)$, we shall denote by $\{\pi(1), \dots, \pi(s)\}$ the permutation of $\{1, \dots, s\}$ of parity p_π such that $t_{\pi(1)} > \dots > t_{\pi(s)}$.

At temperature $T = \beta^{-1}$ the finite-temperature imaginary-time correlation functions, or *Schwinger functions*, are defined by

3.1
$$S^{L,\beta}(\mathbf{x}_1, \varepsilon_1, \sigma_1; ...; \mathbf{x}_n, \varepsilon_n, \sigma_n) = (-1)^{p_\pi} \frac{\mathrm{Tr} e^{-\beta H} \psi^{\varepsilon_\pi(1)}_{\mathbf{x}_{\pi(1)}, \sigma_{\pi(1)}} \dots \psi^{\varepsilon_\pi(s)}_{\mathbf{x}_{\pi(s)}, \sigma_{\pi(s)}}}{\mathrm{Tr} e^{-\beta H}} \,. \tag{3.1}$$

In the spinless case we shall write simply $S^{L,\beta}(\mathbf{x}_1, \varepsilon_1; ...; \mathbf{x}_n, \varepsilon_n)$.

In the limit $\beta \to \infty$ the functions (3.1) define the zero temperature Schwinger functions: they describe the properties of the ground state of the system with Hamiltonian H given by (2.17) in the *grancanonical ensemble* with chemical potential μ_0.

p.3.2 **3.2.** *Physical relevance.* Most of the physical properties can be derived, at least in principle, by the knowledge of the Schwinger functions.

For istance by the two-point Schwinger function one can get information on the spectrum. If we consider the Fourier transform of the two-point Schwinger function, from the imaginary poles in k_0 one can compute the *spectral gap*; if $ik_{0,\alpha}, -ik_{0,\beta}$, with $k_{0,\alpha}, k_{0,\beta} > 0$, are such poles then it is well known that $\bar{\Delta} = \min_\alpha(k_\alpha) + \min_\beta(k_\beta)$, [BGL].

Another important quantity is the *occupation number*, defined as the average number of particle with "momentum" k. The momentum is the quantum number which allows us to classify the states of a "free" Hamiltonian, so the definition of occupation number depends on what we consider the free Hamiltonian. In a system with Hamiltonian H_0 (see (2.1) or (2.2)), the states are obtained considering Slater determinants of plane-waves, so that the occupation number is just given by

3.2
$$n_k \equiv \hat{S}(k; 0^-) \,, \tag{3.2}$$

if $\hat{S}(k, t)$ is the Fourier transform of the two-point Schwinger function with respect the only space variable (in such a case the Schwinger function is translationally invariant). On the other hand, if the Hamiltonian is $H_0 + uP$, with P given by (2.3) or (2.4), in the periodic case, the good quantum number is the *crystalline momentum* indicizing the *Bloch waves*, so that the good definition for the occupation number can be still written in the form (3.2), provided the "Fourier transform" has to be done with respect to the Bloch waves (instead of plane waves).

Other important physical quantities are the response functions; they measure the response of a physical observable to an infinitesimal external perturbation. For istance the density-density response measures the response of the system density to a perturbation proportional to the density of particles; it can be computed from the density-density correlation function (in terms of which the dielectric constant can be written, [Ma]), given, in the spinless case, by

3.3
$$S^{L,\beta}(\mathbf{x}, +; \mathbf{x}, -; \mathbf{0}, +; \mathbf{0}, -) - S^{L,\beta}(\mathbf{x}, +; \mathbf{0}, -)\, S^{L,\beta}(\mathbf{x}, +; \mathbf{0}, -) \,. \tag{3.3}$$

The magnetic response function measures the response of the spin to a magnetic perturbation, and the current-current response function measures the response of the current to an electric field.

p.3.3 **3.3.** *Schwiger functions for free systems.* Finally let us consider explicitly the Schwinger functions for free systems in which $H = H_0$. Suppose for instance H_0 to be given by (2.1).

The model described by H_0 is of course exactly solvable and all the Schwinger functions can be computed; they are obtained by the anticommutative *Wick rule* (for more details see §4 later) from the two-point Schwinger function.

The latter is given, if the fermions are on a lattice, by

1.4
$$S_0^{L,\beta}(\mathbf{x},-;\mathbf{y},+) \equiv g(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{k_0\in\mathcal{D}_\beta}\sum_{k\in\mathcal{D}_L}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-ik_0+E(k)}\,, \tag{3.4}$$

where $E(k)=1-\cos k-\mu_0$, with $|k|\le 2\pi$, is the *dispersion relation*, with the convection that $\mathbf{x}=(x,x_0)$, $\mathbf{y}=(y,y_0)$, $\mathbf{k}=(k,k_0)$, denoting by $\cdot$ the scalar product in $\mathbb{R}^2$, and defining

1.3*a*
$$\begin{aligned}\mathcal{D}_L &\equiv \{k=2\pi n/L,\quad n\in\mathbb{Z},\quad -[L/2]\le n\le[(L-1)/2]\}\,,\\ \mathcal{D}_\beta &\equiv \{k_0=2(n+1/2)\pi/\beta,\quad n\in\mathbb{Z},\quad -M\le n\le M-1\}\,,\end{aligned} \tag{3.5}$$

where M is a suitable cut-off to be removed at the end (see below).

If the fermions are on the continuum the dispersion relation becomes $E(k)=(k^2/2m)-\mu_0$ and the two-point Schwinger function is still given by (3.4), with the new definition of $E(k)$ and with $\mathcal{D}_L$ defined as

1.3*b*
$$\mathcal{D}_L \equiv \{k=2\pi n/L,\quad n\in\mathbb{Z},\quad -N\le n\le N-1\}\,, \tag{3.6}$$

with N a suitable cut-off (to be removed as M). Of course at the end we will be interested in removing the cut-offs M and N: we work with M and N finite, so that we are able to interpret the Schwinger functions as integrals on finite-dimensional Grassman algebras (see next section), and we find results which will be uniform in M and N, so that we can take the limits $M\to\infty$ and $N\to\infty$.

We can see that $g(\mathbf{x}-\mathbf{y})$ is the Fourier transform of a function singular for $k_0=0, k=p_F$, where p_F is the *Fermi momentum*, defined by the condition $E(p_F)=0$. In general, when adding to H_0 the interaction between particles, there is no reason for the Fourier transform of the Schwinger function to be singular for $k=p_F$; it could be singular at some interaction-dependent value. In order to take into account this fact it is useful to write

1.6
$$\mu_0 = \mu+\nu\,, \tag{3.7}$$

where ν is a counterterm which will be eventually suitably chosen in order to fix the position of the singularity at some interaction-indipendent point.

The Schwinger functions (3.1) can be expressed as functional integrals. In next section we shall review the basic concepts which allows us to introduce a functional integral representation in a fermionic theory, then in §5 and in §6 we shall discuss the notion of effective potential, hence in §7 we shall come back to the problem of studying the Schwinger functions.

*sec.*4
4. Fermionic functional integrals

*p.*4.1 **4.1.** *Grassman integrals and truncated expectations.* The Schwinger functions we shall be interested in are written as *Grassman integrals* (see the classical [Be] or any modern textbook like [NO]; see also §7). One introduces a finite dimensional *Grassman algebra*, which is a set of anticommuting *Grassman variables* $\psi\equiv\{\psi^+_\alpha,\psi^-_\alpha\}$, with α an index belonging to some finite set A. This means that

4.1
$$\psi^\sigma_\alpha,\psi^{\sigma'}_{\alpha'}+\psi^\sigma_\alpha,\psi^{\sigma'}_{\alpha'}=0\,,\qquad \forall\alpha,\alpha'\in A\,,\quad \forall\sigma,\sigma'=\pm\,; \tag{4.1}$$

in particular $(\psi^\sigma_\alpha)^2=0$ $\forall\alpha\in A$ and $\forall\sigma=\pm$.

Note that here and henceforth we use the same symbols to denote both the fermionic fields and the Grassman variables: this can be a little misleading, but it is the convention usually followed in quantum field theory, so we shall adopt it. Note also that confusion should not be made between the label $\sigma = \pm$ in (4.1) and the spin label σ used in the previous sections.

Let us introduce another set of Grassman variables $\{\mathrm{d}\psi^+_\alpha, \mathrm{d}\psi^-_\alpha\}$, $a \in A$, anticommuting with $\psi^+_\alpha, \psi^-_\alpha$, and an operation (*Grassman integration*) defined by

4.2
$$\int \psi^\sigma_\alpha \mathrm{d}\psi^\sigma_\alpha = 1 \,, \qquad \int \mathrm{d}\psi^\sigma_\alpha = 0 \,, \qquad a \in A \,, \quad \sigma = \pm 1 \,. \tag{4.2}$$

If $F(\psi)$ is any analytic function of the $\psi^+_\alpha, \psi^-_\alpha$, $\alpha \in A$, the operation

4.3
$$\int \prod_{\alpha \in A} \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\alpha F(\psi) \tag{4.3}$$

is simply defined by iteratively applying (4.2) and taking into account the anticommutation rules (4.1). It is easy to check that for all $a \in A$ and $C \in \mathbb{C}$

4.4
$$\frac{\int \mathrm{d}\psi^+_\alpha d\psi^-_\alpha e^{-\psi^+_\alpha C \psi^-_\alpha} \psi^-_\alpha \psi^+_\alpha}{\int \mathrm{d}\psi^+_\alpha d\psi^-_\alpha e^{-\psi^+_\alpha C \psi^-_\alpha}} = C^{-1} \,; \tag{4.4}$$

in fact $e^{-\psi^+_\alpha C \psi^-_\alpha} = 1 - \psi^+_\alpha C \psi^-_\alpha$ and by (4.2)

4.5
$$\int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\alpha e^{-\psi^+_\alpha C \psi^-_\alpha} = C \,, \tag{4.5}$$

while

4.6
$$\int \mathrm{d}\psi^+_\alpha d\psi^-_\alpha e^{-\psi^+_\alpha C \psi^-_\alpha} \psi^-_\alpha \psi^+_\alpha = 1 \,. \tag{4.6}$$

In the following we shall need also more complicate expressions involving more than a pair of Grassman variables, like

4.7
$$\frac{\int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\alpha \mathrm{d}\psi^+_\beta \mathrm{d}\psi^-_\beta \, e^{-\sum_{ij=\alpha}^{\beta} \psi^+_i M_{ij} \psi^-_j} \psi^-_{\alpha'} \psi^+_{\beta'}}{\int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\beta \mathrm{d}\psi^+_\beta \mathrm{d}\psi^-_\beta \, e^{-\sum_{ij=\alpha}^{\beta} \psi^+_i M_{ij} \psi^-_j}} = [M^{-1}]_{\alpha'\beta'} \,, \tag{4.7}$$

with $M \in \mathrm{GL}(2, \mathbb{C})$, for $\alpha \neq \beta \in A$ and $\alpha', \beta' \in \{\alpha, \beta\}$. Again (4.7) can be easily verified by using (4.2) and the anticommutation rules (4.1), which allow us to write

4.8
$$\int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\beta \mathrm{d}\psi^+_\beta \mathrm{d}\psi^-_\beta \, e^{-\sum_{ij=\alpha}^{\beta} \psi^+_i M_{ij} \psi^-_j} = M_{11} M_{22} - M_{12} M_{21} \equiv \det M \tag{4.8}$$

and

4.9
$$\int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\alpha \mathrm{d}\psi^+_\beta \mathrm{d}\psi^-_\beta \, e^{-\sum_{ij=\alpha}^{\beta} \psi^+_i M_{ij} \psi^-_j} \psi^-_{\alpha'} \psi^+_{\beta'} = M'_{\alpha'\beta'} \,, \tag{4.9}$$

if $M'_{\alpha'\beta'}$ is the minor complementary to the entry $M_{\alpha'\beta'}$ (*i.e.* $M'_{\alpha\alpha} = M_{\beta\beta}$, $M'_{\beta\beta} = M_{\alpha\alpha}$, $M'_{\alpha\beta} = -M_{\beta\alpha}$ and $M'_{\beta\alpha} = -M_{\alpha\beta}$).

The above formulae (4.4) and (4.7) closely remind us the Gaussian integrals: note however that there is no need that C or M are real or positive defined (but of course they have to be invertible).

Pursuing further the analogy with Gaussian integrals, we can consider a "measure" (a similar expression is found replacing g with a matrix, see (4.26) below)

4.10
$$P(\mathrm{d}\psi) = \prod_{\alpha \in A} \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\alpha g_\alpha e^{-\sum_{\alpha \in A} \psi^+_\alpha g^{-1}_\alpha \psi^-_\alpha} \,; \tag{4.10}$$

by construction one has

4.11
$$\int P(\mathrm{d}\psi) = 1 \,, \qquad \int P(\mathrm{d}\psi)\, \psi^-_\alpha \psi^+_\beta = \delta_{\alpha,\beta} g_\alpha \,. \tag{4.11}$$

In general $P(\mathrm{d}\psi)$ will be called a *Gaussian fermionic integration measure* (or *Grassman integration measure* or, as we shall do in the following, integration *tout court*) with covariance g: for any analytic function F defined on the Grassman algebra we can write

4.12
$$\int P(\mathrm{d}\psi)\, F(\psi) = \mathcal{E}(F) \,. \tag{4.12}$$

However note that $P(\mathrm{d}\psi)$ is not at all a real measure, as it does not satisfy the necessary positivity conditions, so that the terminology is only formal and the use of the symbol $\mathcal{E}$ (which stands for expectation value) is meant only by analogy.

Given p functions $X_1, \dots, X_p$ defined on the Grassman algebra and p positive integer numbers $n_1, \dots, n_p$, the *truncated expectation* is defined as

4.13
$$\mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) = \frac{\partial^{n_1+\dots+n_p}}{\partial\lambda_1^{n_1} \dots \partial\lambda_p^{n_p}} \log \int P(\mathrm{d}\psi)\, e^{\lambda_1 X_1(\psi) + \dots + \lambda_p X_p(\psi)} \Big|_{\lambda=0} \,, \tag{4.13}$$

where $\lambda = \{\lambda_1, \dots, \lambda_p\}$. It is easy to check that $\mathcal{E}^T$ is a linear operation, that is, formally,

4.14
$$\mathcal{E}^T(c_1 X_1 + \dots + c_p X_p; n) = \sum_{n_1+\dots+n_p=n} \frac{n!}{n_1! \dots n_p!} c_1^{n_1} \dots c_p^{n_p} \mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) \,, \tag{4.14}$$

so that the following relations immediately follow:

4.15
$$\begin{aligned} &(1) \qquad \mathcal{E}^T(X; 1) = \mathcal{E}(X) \,, \\ &(2) \qquad \mathcal{E}^T(X; 0) = 0 \,, \\ &(3) \qquad \mathcal{E}^T(X, \dots, X; n_1, \dots, n_p) = \mathcal{E}^T(X; n_1 + \dots + n_p) \,. \end{aligned} \tag{4.15}$$

Moreover one has

4.15*a*
$$\mathcal{E}^T(X_1, \dots, X_1, \dots, X_p, \dots, X_p; 1, \dots, 1, \dots, 1, \dots, 1) = \mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) \,, \tag{4.16}$$

where, for any $j = 1, \dots, p$, in the l.h.s. the function X_j is repeated n_j times and 1 is repeated $n_1 + \dots + n_p$ times.

We define also

4.15*b*
$$\mathcal{E}^T(X_1, \dots, X_p) \equiv \mathcal{E}^T(X_1, \dots, X_p; 1, \dots, 1) \,. \tag{4.17}$$

By (4.16) we see that all truncated expecations can be expressed in terms of (4.17); it is easy to see that (4.17) is vanishing if $X_j = 0$ for at least one j; see Appendix A3.

The truncated expectation appears naturally considering the integration of an exponential; in fact as a particular case of (4.13) one has

4.16
$$\mathcal{E}^T(X; n) = \frac{\partial^n}{\partial\lambda^n} \log \int P(\mathrm{d}\psi)\, e^{\lambda X(\psi)} \Big|_{\lambda=0} \,, \tag{4.18}$$

so that

4.17
$$\begin{aligned} \log \int P(\mathrm{d}\psi)\, e^{X(\psi)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial\lambda^n} \log \int P(\mathrm{d}\psi)\, e^{\lambda X(\psi)} \Big|_{\lambda=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T(X; n) \,. \end{aligned} \tag{4.19}$$

The following properties, immediate consequence of (4.2) and very similar to the properties of Gaussian integrations, follow; see also Appendix A3.

(1) *Wick rule.* Given two sets of labels $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1 \dots, \beta_m\}$ in A, one has

4.18
$$\int P(\mathrm{d}\psi)\, \psi^-_{\alpha_1} ... \psi^-_{\alpha_n} \psi^+_{\beta_1}, \dots, \psi^+_{\beta_m} = \delta_{n,m} \sum_{\pi} (-1)^{p_\pi} \prod_{i=1}^{n} \delta_{\alpha_i, \beta_{\pi(j)}} g_{\alpha_i} \ , \tag{4.20}$$

where the sum is over all the permutations $\pi = \{\pi(1), \dots, \pi(n)\}$ of the indices $\{1, \dots, n\}$ with parity p_π with respect to the fundamental permutation.

(2) *Addition principle.* Given two integrations $P(\mathrm{d}\psi_1)$ and $P(\mathrm{d}\psi_2)$, with covariance g_1 and g_2 respectively, then, for any function F which can be written as sum over monomials of Grassman variables, *i.e.* $F = F(\psi)$, with $\psi = \psi_1 + \psi_2$, one has

4.19
$$\int P(\mathrm{d}\psi_1) \int P(\mathrm{d}\psi_2) F(\psi_1 + \psi_2) = \int P(\mathrm{d}\psi)\, F(\psi) \ , \tag{4.21}$$

where $P(\mathrm{d}\psi)$ has covariance $g \equiv g_1 + g_2$. It is sufficient to prove it for $F(\psi) = \psi^- \psi^+$, then one uses the anticommutation rules (4.1). One has

4.20
$$\begin{aligned} &\int P(\mathrm{d}\psi_1) \int P(d\psi_2) \left(\psi_1^- + \psi_2^-\right) \left(\psi_1^+ + \psi_2^+\right) \\ &\qquad = \int P(\mathrm{d}\psi_1)\, \psi_1^- \psi_1^+ \int P(d\psi_2) + \int P(d\psi_1) \int P(d\psi_2)\, \psi_2^- \psi_2^+ = g_1 + g_2 \ . \end{aligned} \tag{4.22}$$

where (4.11) has been used.

(3) *Invariance of exponentials.* From the definition of truncated expectations, it follows that, if ϕ is an "external field", *i.e.* a not integrated field, then

4.21
$$\int P(\mathrm{d}\psi)\, e^{X(\psi+\phi)} = \exp\left[\sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T \left(X(\cdot + \phi); n\right)\right] \equiv e^{X'(\phi)} \ , \tag{4.23}$$

which is a main technical point: (4.23) says that integrating an exponential one still gets an exponential, whose argument is expressed by the sum of truncated expectations.

(4) *Change of integration.* If $P_g(\mathrm{d}\psi)$ denotes the integration with covariance g, then, for any analytic function $F(\psi)$, one has

4.22
$$\frac{1}{\mathcal{N}_\nu} \int P_g(\mathrm{d}\psi)\, e^{-\nu \psi^+ \psi^-} F(\psi) = \int P_{\tilde{g}}(\mathrm{d}\psi)\, F(\psi) \ , \qquad \tilde{g}^{-1} = g^{-1} + \nu \ , \tag{4.24}$$

where

4.23
$$\mathcal{N}_\nu = \frac{g^{-1} + \nu}{g^{-1}} = 1 + g\nu = \int P_g(\mathrm{d}\psi)\, e^{-\nu \psi^+ \psi^-} \ . \tag{4.25}$$

The proof is very easy from the definitions. More generally one has that, if M is an invertible 2×2 matrix and $P_M(\mathrm{d}\psi)$ is given by

4.24
$$P_M(\mathrm{d}\psi) = \int \mathrm{d}\psi^+_\alpha \mathrm{d}\psi^-_\beta \mathrm{d}\psi^+_\beta \mathrm{d}\psi^-_\beta \ \det M\, e^{-\sum_{ij=\alpha}^{\beta} \psi_i^+ M^{-1}_{ij} \psi_j^-} \ , \tag{4.26}$$

then, for $\sigma \in \mathbb{C}$,

4.25
$$\frac{1}{\mathcal{N}_\sigma} \int P_M(\mathrm{d}\psi)\, e^{-\sigma \psi_1^+ \psi_2^- - \sigma \psi_2^+ \psi_1^-} F(\psi) = \int P_{\tilde{M}}(\mathrm{d}\psi)\, F(\psi) \ , \qquad \tilde{M}^{-1} = M^{-1} + \sigma \sigma^1_x \ , \tag{4.27}$$

where σ_x^1 is the Pauli matrix (see (2.11)) and

4.26 $$\mathcal{N}_\sigma = \det\left(\mathbb{1} + \sigma S_x^1 M\right) = \frac{\det\left(M^{-1} + \sigma S_x^1\right)}{\det M^{-1}} = \int P_M(\mathrm{d}\psi)\, e^{-\sigma\psi_1^+\psi_2^- - \sigma\psi_2^+\psi_1^-} \ . \tag{4.28}$$

Moreover if $P_M(\mathrm{d}\psi)$ is the integration measure defined by (4.26), one has

4.26a $$\frac{1}{\mathcal{N}_N}\int P_M(\mathrm{d}\psi)\, e^{-\sum_{i,j=\alpha}^{\beta}\psi_i^+ N_{ij}^{-1}\psi_j^-}\, F(\psi) = \int P_{\tilde{M}}(\mathrm{d}\psi)\, F(\psi)\ , \tag{4.29}$$

where

4.26b $$\tilde{M}^{-1} = M^{-1} + N^{-1} \tag{4.30}$$

and

4.26c $$\mathcal{N}_N = \det\left(\mathbb{1} + N^{-1}M\right) = \frac{\det\left(M^{-1} + N^{-1}\right)}{\det M^{-1}} = \int P_M(\mathrm{d}\psi)\, e^{-\sum_{i,j=\alpha}^{\beta}\psi_\alpha^+ N_{ij}^{-1}\psi_\beta^-} \ . \tag{4.31}$$

p.4.2 **4.2.** *Truncated expectations and Feynman diagrams.* For concreteness we consider a system which is a perturbation of that described by the Hamiltonian H_0 given by (2.1).

We introduce a finite set of Grassman variables $\{\hat{\psi}_{\mathbf{k}}^\pm\}$, one for each $\mathbf{k} \in \mathcal{D}_{L,\beta}$, $\mathcal{D}_{L,\beta} \equiv \mathcal{D}_L \times \mathcal{D}_\beta$, with $\mathcal{D}_L$ and $\mathcal{D}_\beta$ defined in (3.5). Let be

4.27 $$P(\mathrm{d}\psi) = \Big(\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}} (L\beta\hat{g}(\mathbf{k}))\,\hat{\psi}_{\mathbf{k}}^+\hat{\psi}_{\mathbf{k}}^-\Big)\exp\Big[-\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} (L\beta\hat{g}(\mathbf{k}))^{-1}\,\hat{\psi}_{\mathbf{k}}^+\hat{\psi}_{\mathbf{k}}^-\Big]\ , \tag{4.32}$$

with

4.28 $$\hat{g}(\mathbf{k}) = \frac{1}{-ik_0 + E(k)} = \frac{1}{-ik_0 + \cos p_F - \cos k}\ , \tag{4.33}$$

where (see (3.4))

4.28a $$E(k) = 1 - \mu_0 - \cos k \equiv \cos p_F - \cos k\ . \tag{4.34}$$

So we are in the situation of §4.1 with the set of indices $A = \mathcal{D}_{L,\beta}$.

We introduce the *Grassman fields* $\psi_{\mathbf{x}}^\pm$ defined by

4.29 $$\psi_{\mathbf{x}}^\pm = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \hat{\psi}_{\mathbf{k}}^\pm e^{\pm i\mathbf{k}\cdot\mathbf{x}}\ , \tag{4.35}$$

where $\mathbf{k} = (k, k_0)$ and $\mathbf{k}\cdot\mathbf{x} = k_0x_0 + kx$, and such that

4.30 $$\int P(\mathrm{d}\psi)\psi_{\mathbf{x}}^-\psi_{\mathbf{y}}^+ = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\hat{g}(\mathbf{k}) \equiv g(\mathbf{x}-\mathbf{y})\ , \tag{4.36}$$

Of course the properties for the Grassman variables seen in §4.1 extend trivially to the Grassman fields.

In order to compute the truncated expectations there are two possible main ways.

One is the representation in terms of the ordinary *connected Feynman diagrams* defined in the following way. For a given set of indices P, define

4.31 $$\tilde{\psi}(P) = \prod_{f\in P}\psi_{\mathbf{x}(f)}^{\sigma(f)}\ , \tag{4.37}$$

with $\sigma(f) \in \{\pm\}$ and $\mathbf{x}(f) = (x(f), x_0(f)) \in \Lambda \times [-\beta, \beta]$, and call $|P|$ the number of elements in P. Then, given s sets of indices $P_1, \dots, P_s$, consider

4.32 $$\mathcal{E}^T\left(\tilde{\psi}(P_1), \dots, \tilde{\psi}(P_s)\right)\ , \tag{4.38}$$

for $s \geq 1$ (recall (4.17)).

First of all note that, by writing

4.32a
$$P_j = P_j^+ \cup P_j^+ \ , \qquad P_j^\pm = \{f \in P_j : \sigma(f) = \pm\} \ , \tag{4.39}$$

for each $j = 1, \dots, s$, one must have

4.32b
$$\sum_{j=1}^{s} |P_j^+| = \sum_{j=1}^{s} |P_j^-| \ , \tag{4.40}$$

because the truncated expectations can be written in terms of simple expectations (see Appendix A3) and the Wick rule (4.20) holds.

For any $\mathbf{x} = \mathbf{x}(f)$ and $\sigma = \sigma(f)$, we can represent each field $\psi_{\mathbf{x}}^\sigma$ as an oriented half-line emerging from a point $\mathbf{x}$ and carrying an arrow, pointing towards the point if $\sigma = -$ and opposite to the point if $\sigma = +$. We can enclose the points $\mathbf{x}(f)$ belonging to the set P_j, for some $j = 1, \dots, s$, in a box: in this way we obtain s disjoint boxes.

Then given n sets $P_1, \dots, P_s$, we associate to them a set of graphs Γ, called Feynman diagrams, obtained by joining pairwise the half-lines with consistent orientation (*i.e.* a half-line representing a field ψ^- with a half-line representing a field ψ^+ and *vice versa*) in such a way that the boxes are all connected; see Fig. 1. A line obtained by joining two half-lines will be denoted by ℓ and, if ℓ is a line contained in a diagram Γ, we shall write $\ell \in \Gamma$: the two half-lines are said to be *contracted* or to form a *contraction*.

To each line ℓ obtained joining the half-line representing $\psi_{\mathbf{x}(i)}^-$ with the half-line representing $\psi_{\mathbf{x}(j)}^+$ we associate a *propagator* $g_\ell \equiv g(\mathbf{x}(i)) - \mathbf{x}(j))$; as the line ℓ uniquely determines the points i and j, we shall write also $\mathbf{x}(i) - \mathbf{x}(j) = \mathbf{x}_\ell$.

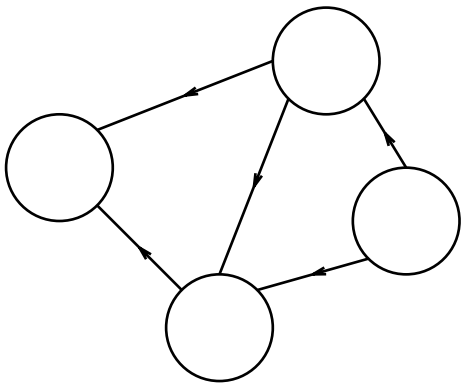


FIG. 1. A Feynman diagram Γ obtained by joining all the half-lines with consistent orientation emerging from the boxes encosing the sets P_1,...,P_s. The diagram Γ belongs to the set $\mathcal{G}_0$ in (4.42).

Then to each diagram Γ there corresponds a number, which will be called the *value* of the graph, given by the product of the propagators of the lines $\ell \in \Gamma$ (possibly up to a sign):

4.33
$$\mathrm{Val}(\Gamma) = (-1)^\pi \prod_{\ell \in \Gamma} g_\ell \ , \tag{4.41}$$

where π is a parity which depends on the way the lines are contracted between themselves. Then, if we denote by $\mathcal{G}_0$ the set of all Feynman diagrams which can be obtained by following the given prescription, one has

4.33*b*
$$\mathcal{E}^T\left(\tilde{\psi}(P_1),\dots,\tilde{\psi}(P_s)\right) = \sum_{\Gamma\in\mathcal{G}_0} \mathrm{Val}(\Gamma)\ . \tag{4.42}$$

As a consequence we see that all the sets $P_1,\dots,P_s$ have to be not empty if $s>1$, while one can have $P_1=\emptyset$ if $s=1$.

There is another possible (more compact) representation of the truncated expectations. Consider (4.38) and set $f=(j,i)$ for $f\in P_j$, with $i=1,\dots,|P_j|$, and $n=|P_1|+\dots+|P_s|$.

It is well known (see Appendix A3) that, up to a sign, if $s>1$,

4.34
$$\mathcal{E}^T\left(\tilde{\psi}(P_1),\dots,\tilde{\psi}(P_s)\right) = \sum_T \left(\prod_{\ell\in T} g_\ell\right) \int \mathrm{d}P_T(\mathbf{t}) \det G^T(\mathbf{t})\ , \tag{4.43}$$

where

(1) T is a set of lines forming an *anchored tree* between the clusters of points $P_1,\dots,P_s$, *i.e.* T is a set of lines which becomes a tree (see Appendix A1 for a formal definition of tree) if one identifies all the points in the same cluster,

(2) $\mathbf{t}$ is a set of parameters

4.35
$$\mathbf{t} = \{t_{j,j'}\in[0,1], \qquad 1\le j,j'\le s\}\ , \tag{4.44}$$

(3) $\mathrm{d}P_T(\mathbf{t})$ is a suitable (normalized) probability measure with support on a set of $\mathbf{t}$ such that $t_{j,j'}=\mathbf{u}_j\cdot\mathbf{u}_{j'}$, for some family of vectors $\mathbf{u}_j\in\mathbb{R}^s$ of unit norm, and

(4) $G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by

4.36
$$\left[G^T(\mathbf{t})\right]_{(j,i),(j',i')} = t_{j,j'} g(\mathbf{x}(j,i)-\mathbf{x}(j',i'))\ , \tag{4.45}$$

where $1\le j,j'\le s$ and $1\le i\le|P_j|$, $1\le i'\le|P_{j'}|$, such that the lines $\ell=\mathbf{x}(j,i)-\mathbf{x}(j',i')$ do not belong to T.

If $s=1$, the sum over T is empty, but we can still use the above equation, by interpreting the r.h.s. as

4.37
$$\begin{cases} 1\ , & \text{if } P_1 \text{ is empty}\ , \\ \det G(\mathbf{1})\ , & \text{otherwise}\ , \end{cases} \tag{4.46}$$

where $\mathbf{1}$ is obtained from (4.44) by setting $t_{j,j'}=1\ \forall j,j'$.

Note that, while in the first representation $\mathcal{E}^T$ was written as a sum over Feynman diagrams, in this second representation it is written as a sum over *trees* connecting the boxes. Fixing a tree T and expanding the determinant $\det G^T(\mathbf{t})$, one gets all the possible graphs which can be obtained by contracting the half-lines not belonging to T, *i.e.* one gets the Feynman diagrams and the representation (4.42) follows.

Of course the number of addends in the first representation (4.42) is much larger than in the second one, *i.e.* (4.43), where a large quantity of Feynman diagrams are grouped together.

It is important to stress the difference of the two representations of the truncated expectations, more precisely the difference between the number of addends appearing in the two representations. In the first one (4.42) a truncated expectation is written in terms of Feynman diagrams and the number of them can quite high: for istance, if $|P_i|=4$ in (4.36), they are $O(s!^2)$ (see Appendix A1), so while using such a representation it is difficult to verify the convergence of the perturbative series. In the other representation (4.43) we do not sum over the Feynman diagrams, but over the anchored trees (see Fig. 2), whose number is only $O(s!)$ (see Appendix A1). Of course there can be really a gain in expressing (4.38) by using (4.43) instead of (4.42) only if each summand of the two expressions admits the same bound, for instance a C^n bound for some constant C.

If the propagators are bounded by some constant C_0, $|g_\ell| \le C_0$, then one has $|\mathrm{Val}(\Gamma)| \le C_0^L$, where L is the number of lines in Γ (see (4.41)); as the number of Feynman diagrams in $\mathcal{G}_0$ is bounded by $O(s!^2)$ then we obtain a bound $s!^2 C^n$ from (4.42), for some constant C. On the other hand it is a remarkable inequality that the determinant in (4.43) can be still bounded by a constant to the power n (Gram-Hadamard inequality; see Appendix A3), so that a bound $s!C^n$ can be obtained for (4.38) by using the representation (4.43) instead of (4.42). Of course if one develops the determinant in (4.43) one obtains the expansion in Feynman diagrams (4.42): the dramatic improvement of the bound is due to the fact that one exploits cancellations among the Feynman diagrams (due to the Fermi statistics), which are lost if bounding each addend in (4.42) by its absolute value. More precisely one has

4.34*bis*

$$\begin{aligned} \left| \sum_T \left(\prod_{\ell \in T} g_\ell \right) \int \mathrm{d}P_T(\mathbf{t}) \det G^T(\mathbf{t}) \right| &\le \sum_T \left(\prod_{\ell \in T} |g_\ell| \right) C_1^{n-s+1} \\ &\le \sum_T C_0^{s-1} C_1^{n-s+1} \le s! \left(C \max\{C_0, C_1\} \right)^n \ . \end{aligned} \tag{4.47}$$

where C_1 is a constant (proportional to C_0) such that $|\det G^T(\mathbf{t})| \le C_1^{n-s+1}$ and $s!C^n$ takes into account the number of anchored trees which one has to sum over in (4.43); see Appendix A3.

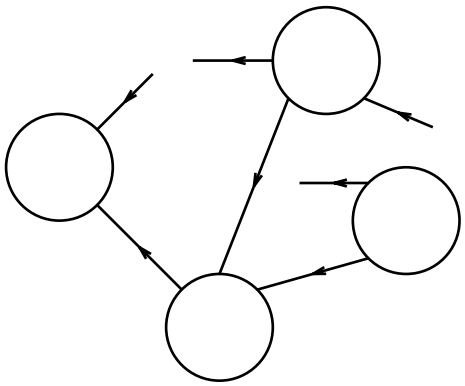


FIG. 2. A term contributing to the truncated expectation (4.38) according to the expansion (4.43). The lines connecting the sets $P_1,...,P_s$ form the anchored tree T. The other lines are left uncontracted, as the determinant in (4.47) takes into account all the possible ways to contract them.

We shall see that, as anticipated above, this will allow us to pass from a factorial $s!^2$ to a factorial $s!$ in the estimates , and that this will be enough in order to obtain convergence as a factor $1/s!$ arises from the perturbative expansion (see (5.22) and the comments around (5.40)); see the end of §5.4.

*sec.*5

5. The multiscale decomposition and power counting

*p.*5.1

5.1. *Tree expansion.* It is possible to write the functional integral introduced in §4 as sum over trees following two possible routes. [Note that the trees involved in the construction below have not to be confused with the (anchored) trees introduced in the previous section: they are called both trees because, as graphs, they have the same structure.]

The first route consists in looking at the Feynman diagrams and to realize that it is convenient to associate to each of them a set of boxes, called *clusters*, establishing a hierarchical order between the sizes of the momenta of the lines of the propagators. The reason for doing this is the following one: if the momenta of the lines in some box are larger than the momenta of the lines outside the box, one has a possibly "dangerous" contribution, while this is not the case in the opposite situation: it is natural that such two different contributions have to be treated in a different way. This argument will become clearer below and in §7. Note that such reasoning was followed by Bogolubov, Hepp and Zimmermann (see [B], [H] and [Z]). We shall see that the set of clusters associated to any graph can be very conveniently represented in terms of *trees*.

The other way for introducing trees follows the ideas of Wilson on the Renormalization Group, see [W]; one wants to implement the idea that, integrating the "irrelevant" degress of freedom of a theory, one gets an "effective theory" much simpler than the preceeding one and such that all the important physical informations are encoded in it. We will follow this route.

For concreteness we consider the discrete case, in which the free Hamiltonian is given by (2.1) (anyway the following discussion can be easily adapted to the continuum case). So, if we denote by $\mathbf{k} = (k, k_0)$ the momentum (see (3.5)), we have that k is defined modulo 2π. Let $\|\cdot\|_{\mathbb{T}}$ denote the distance on the one-dimensional torus $\mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z}$, *i.e.*

5.0
$$\|k\|_{\mathbb{T}} = \min_{n\in\mathbb{Z}} |k - 2\pi n| \ . \tag{5.1}$$

Fix $p_F = 2\pi n_F/L$, with $n_F \in \mathbb{N}$, such that $1 - \cos p_F = \mu_0$.

We introduce a smooth C^∞ function $\chi(\mathbf{k}')$ such that, if

5.1
$$|\mathbf{k}'| = \sqrt{k_0^2 + v_0 \|k'\|_{\mathbb{T}}^2} \ , \qquad v_0 = \left.\frac{\mathrm{d}E}{\mathrm{d}k}\right|_{k=p_F} = \sin p_F \ , \tag{5.2}$$

with $E(k)$ defined in (4.34), then

5.2
$$\chi(\mathbf{k}') = \begin{cases} 1 \ , & \text{if } |\mathbf{k}'| \le t_0 = a_0/\gamma \ , \\ 0 \ , & \text{if } |\mathbf{k}'| \ge a_0 \ , \end{cases} \tag{5.3}$$

where $a_0 = \min\{p_F/2, \pi - p_F/2\}$ and $\gamma > 1$; see Fig. 3.

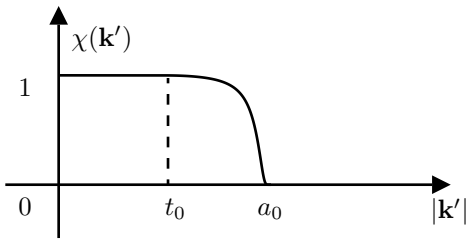


FIG. 3. The function $\chi(\mathbf{k}')$.

We can write in (4.33)

5.3
$$\begin{aligned} \hat g(\mathbf{k}) &= \hat g^{(\mathrm{u.v.})}(\mathbf{k}) + \hat g^{(\mathrm{i.r.})}(\mathbf{k}) \ , \\ \hat g^{(\mathrm{u.v.})}(\mathbf{k}) &\equiv \frac{1 - \chi(k_0, k + p_F) - \chi(k_0, k - p_F)}{-ik_0 + \cos p_F - \cos k} \ , \\ \hat g^{(\mathrm{i.r.})}(\mathbf{k}) &\equiv \frac{\chi(k_0, k + p_F) + \chi(k_0, k - p_F)}{-ik_0 + \cos p_F - \cos k} \ . \end{aligned} \tag{5.4}$$

We introduce, for any $\mathbf{k} \in \mathcal{D}_{L,\beta}$, two Grassman variables, $\psi^{(\mathrm{u.v.})}_{\mathbf{k}}$ and $\psi^{(\mathrm{i.r.})}_{\mathbf{k}}$, with propagators, respectively, $\hat{g}^{(\mathrm{u.v.})}(\mathbf{k})$ and $\hat{g}^{(\mathrm{i.r.})}(\mathbf{k})$; given a potential $\mathcal{V}(\psi)$, by the *addition principle*, we can write

5.4
$$\int P(\mathrm{d}\psi)\, e^{\mathcal{V}(\psi)} = \int P(\mathrm{d}\psi^{(\mathrm{i.r.})}) \int P(\mathrm{d}\psi^{(\mathrm{u.v.})})\, e^{\mathcal{V}(\psi^{(\mathrm{u.v.})}+\psi^{(\mathrm{i.r.})})}\ , \tag{5.5}$$

and, by using the *invariance of exponentials* property, we have

5.5
$$\int P(\mathrm{d}\psi^{(\mathrm{u.v.})})\, e^{\mathcal{V}(\psi^{(\mathrm{u.v.})}+\psi^{(\mathrm{i.r.})})} = \exp\left[\sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T_{\mathrm{u.v.}}\left(\mathcal{V}(\cdot + \psi^{(\mathrm{i.r.})}; n\right)\right] \equiv e^{\mathcal{V}^{(0)}(\psi^{(\mathrm{i.r.})})}\ . \tag{5.6}$$

We shall see better later why it can be of interest to consider an expression like (5.5).

It is convenient to represent the expansion for

5.6
$$\mathcal{V}^{(0)}(\psi^{(\mathrm{i.r.})}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T_{\mathrm{u.v.}}(\mathcal{V}(\cdot + \psi^{(\mathrm{i.r.})}); n) \tag{5.7}$$

as in Fig. 4.

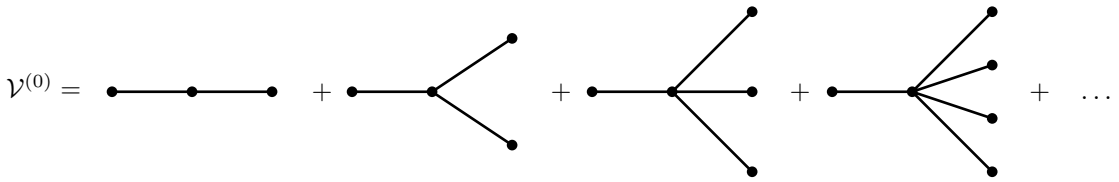


FIG. 4. Graphic representation of the expansion (5.7). We can associate some labels to the points: a label h=0 to the leftmost point, a label h=1 to the middle point and a label h=2 to all the rightmost points (endpoints).

One can say that we have "integrated out the high energy degrees of freedom", obtaining an "effective" theory describing fermions with momenta close to the Fermi surface. As $g^{(\mathrm{i.r.})}(\mathbf{k})$ is singular in two different points ($k = \pm p_F$, at $k_0 = 0$), it is natural to write

5.7
$$\hat{g}^{(\mathrm{i.r.})}(\mathbf{k}) = \frac{\chi(k_0, k + p_F)}{-ik_0 + \cos p_F - \cos k} + \frac{\chi(k_0, k - p_F)}{-ik_0 + \cos p_F - \cos k} \equiv \sum_{\omega \pm 1} \hat{g}^{(\mathrm{i.r.})}_{\omega}(\mathbf{k}) \tag{5.8}$$

and correspondingly we write

5.8
$$\int P(\mathrm{d}\psi^{(\mathrm{i.r.})}) = \prod_{\omega=\pm 1} \int P(\mathrm{d}\psi^{(\mathrm{i.r.})}_{\omega})\ ; \tag{5.9}$$

the fields $\psi^{(\mathrm{i.r.})\pm}_{\omega}$ are called *quasi-particle* Grassman fields: the label ω is sometimes called the *branch label*. Moreover we decompose each propagator $\hat{g}^{(\mathrm{i.r.})}_{\omega}(\mathbf{k})$ as an infinite sum of propagators

5.9
$$\hat{g}^{(\mathrm{i.r.})}_{\omega}(\mathbf{k}) = \sum_{h=-\infty}^{0} \frac{f_h(k + \omega p_F, k_0)}{-ik_0 + \cos p_F - \cos k} \equiv \sum_{h=-\infty}^{0} \hat{g}^{(h)}_{\omega}(\mathbf{k})\ , \tag{5.10}$$

where

5.10
$$f_h(\mathbf{k}') \equiv \chi(\gamma^{-h}\mathbf{k}') - \chi(\gamma^{-h+1}\mathbf{k}') \tag{5.11}$$

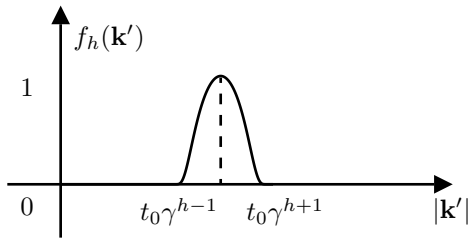


FIG. 5. The function $f_h(\mathbf{k}')$.

is such that $f_h(\mathbf{k}') = 0$ both for $|\mathbf{k}'| \le t_0\gamma^{h-1}$ and $|\mathbf{k}'| \ge t_0\gamma^{h+1}$, while $f_h(\mathbf{k}') = 1$ for $|\mathbf{k}'| = t_0\gamma^h$; see Fig. 5.

Note that in fact the series in (5.10) is a finite sum, if L, β are finite (that is only a finite number of terms can be really different from zero). In fact if L and β are fixed, one has $|k_0| \ge 2\pi/\beta$: so that $f_h(\mathbf{k}') = 0$ for any $h < h_\beta$, with

5.10a
$$h_\beta = \min\left\{h : t_0\gamma^{h+1} > \pi/\beta\right\} \; ; \tag{5.12}$$

note that $h_\beta = O(\log\beta)$.

Therefore, as far as β remains finite, one has a natural infrared cut-off h_β: of course we are interested in bounds uniform in such a cut-off, *i.e.* we want to consider the possibility of removing such a cut-off.

Using again the addition principle and the invariance of exponential property, calling $\psi^{(\le -1)}_\omega$ and $\psi^{(0)}_\omega$ the Grassman fields with propagators $\hat g^{(\le -1)}_\omega(\mathbf{k})$, if

5.11
$$\hat g^{(\le -1)}_\omega(\mathbf{k}) \equiv \sum_{h=-\infty}^{-1} \hat g^{(h)}_\omega(\mathbf{k}) \; , \tag{5.13}$$

and $\hat g^{(0)}_\omega(\mathbf{k})$, respectively, and writing

5.12
$$\int P(\mathrm{d}\psi^{(h)}) = \prod_{\omega=\pm 1} \int P(\mathrm{d}\psi^{(h)}_\omega) \; , \tag{5.14}$$

we obtain

5.13
$$\begin{aligned} \int P(\mathrm{d}\psi)\, e^{\mathcal{V}^{(0)}(\psi)} &= \int P(\mathrm{d}\psi^{(\le -1)}) \int P(\mathrm{d}\psi^{(0)})\, e^{\mathcal{V}^{(0)}(\psi^{(\le -1)} + \psi^{(0)})} \\ &\equiv \int P(\mathrm{d}\psi^{(\le -1)})\, e^{\mathcal{V}^{(-1)}(\psi^{(\le -1)})} \; , \end{aligned} \tag{5.15}$$

where

5.14
$$\begin{aligned} \mathcal{V}^{(-1)}(\psi^{(\le -1)}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T_0 \left(\mathcal{V}^{(0)}(\cdot + \psi^{(\le -1)}); n \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T_0 \left(\sum_{n'=0}^{\infty} \frac{1}{n'!} \mathcal{E}^T_{\text{u.v.}}(\mathcal{V}; n'); n \right) \; . \end{aligned} \tag{5.16}$$

A graphical representation of (5.16) is in Fig. 6, where the circles represent $\mathcal{V}^{(0)}$.

Writing the circles as in the second line of Fig. 6 we get immediately Fig. 7.

So $\mathcal{V}^{(-1)}$ is represented by a graph consisting in a set of lines and points arranged on the plane (x, y) in the following way. A line enters a point v_0 and $s \ge 1$ lines connect v_0 to other s points $v_1, \dots, v_s$: for each point v_j, with $j = 1, \dots, s$, there are $s'_j \ge 1$ exiting lines leading to s'_j points $v_{j1}, \dots, v_{js'_j}$, which we call endpoints. The endpoints (with the lines entering them) represent a graphic representation of $\mathcal{V}$, while the subgraphs consisting of a point v_j (with the line entering it) and of all the lines and points following v_j are

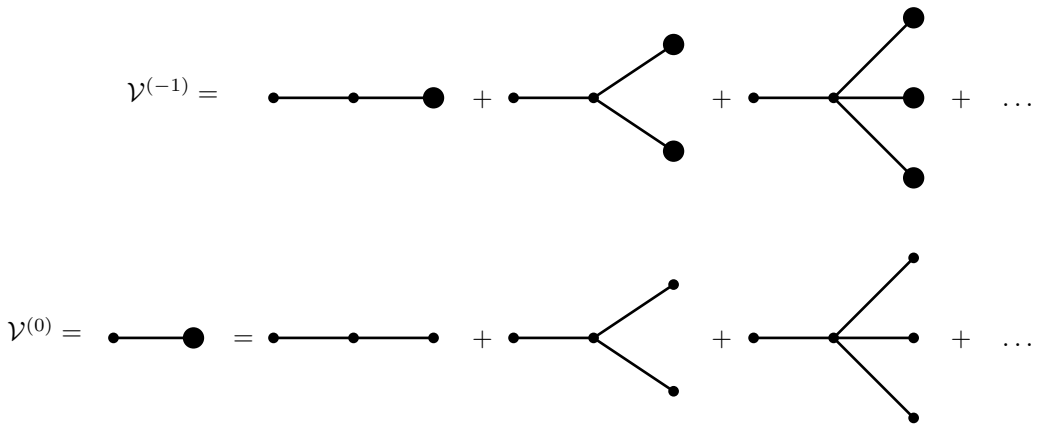


FIG. 6. Graphic representation of the expansion (5.16). The first line represents $\mathcal{V}^{(-1)}$ in terms of $\mathcal{V}^{(0)}$, while the second line defines a unique graph representations for all the contributions to $\mathcal{V}^{(0)}$ (and it is the same as in Fig. 4).

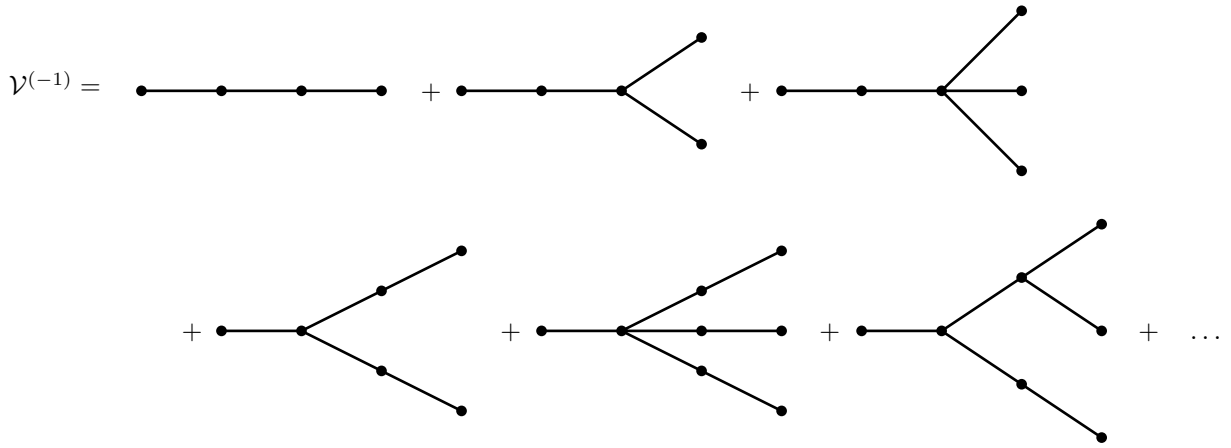


FIG. 7. Graphic representation of $\mathcal{V}^{(-1)}$ in terms of V: each term representing $\mathcal{V}^{(0)}$ in the first line of Fig. 6 is expanded by using the second line of Fig. 6. One should imagine that the leftmost node lays on a vertical line $h{=}{-1}$, the nodes immediately following it on a vertical line $h{=}0$, the endpoints on a vertical line $h{=}2$, while all the other nodes on a vertical line $h{=}1$, as it will be in Fig. 8 below.

graphic representations of $\mathcal{V}^{(0)}$: note that the circles are in fact expanded into such subgraphs. In conclusion one obtains a graph with a tree structure (see Appendix A1 for an introduction to tree graphs).

In order to have an aesthetically goodlooking picture we can draw all the points v_j, $j = 1, \dots, s$, on the same vertical line r_1 and all the points $v_{jj'}$, $j = 1, \dots, s$ and $j' = 1, \dots, s'_j$, on the same vertical line r_2. By introducing a coordinate system (x, y) we can denote by $x = 1$ and $x = 2$ the two lines r_1 and r_2, respectively; the point v_0 is on the line $x = 0$, while the root is on the line $x = -1$.

Now we can iterate further the above procedure, by integrating all the fields $\psi^{(\text{u.v.})}, \psi^{(0)}, \psi^{(-1)}, \dots, \psi^{(h+1)}$, so obtaining a contribution to $\mathcal{V}^{(h)}$, which is defined as

5.15
$$e^{\mathcal{V}^{(h)}(\psi^{(\le h)})} = \int P(\mathrm{d}\psi^{(h+1)}) \dots \int P(\mathrm{d}\psi^{(0)}) \int P(\mathrm{d}\psi^{(\text{u.v.})})\, e^{V(\psi^{(\le h)} + \psi^{(h+1)} + \dots + \psi^{(\text{u.v.})})}\ ; \tag{5.17}$$

the function $\mathcal{V}^{(h)}(\psi^{(\le h)})$ is the *effective potential* on scale h.

We can introduce also a scale label $h = 1$ to denote the ultraviolet scale, $\psi^{(1)} = \psi^{(\text{u.v.})}$, so that $\psi^{(\le 1)} \equiv \psi$ and $\mathcal{V}^{(1)}(\psi^{(\le 1)}) = \mathcal{V}(\psi)$.

By using iteratively the invariance of exponential property we see that $\mathcal{V}^{(h)}$ can be expressed in terms of $\mathcal{V}^{(h+1)}$ as

5.15a
$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T_{h+1} \left(\mathcal{V}^{(h+1)}(\cdot + \psi^{(\le h)}); n \right) , \tag{5.18}$$

where $\mathcal{V}^{(h+1)}$ in turn can be expressed in tersms of $\mathcal{V}^{(h+2)}$ as (5.18) with h replaced with $h+1$, and so on until $\mathcal{V}^{(h)}$ is expressed in terms of $\mathcal{V}^{(1)} \equiv \mathcal{V}$.

At each step of the iterative procedure a circle representing $\mathcal{V}^{(h')}$, for some $h < h' < 1$, is transformed into a point v on a vertical line $x = h' + 1$ (we use the coordinate system introduced above) with $s_v \ge 1$ exiting lines leading to s_v circles representing $\mathcal{V}^{(h'+1)}$ and so on. At the end only points are left (*i.e.* no circles remain): the ones on the line $x = 2$ are called endpoints.

By resuming the above discussion, we see that we can introduce a graph representation of $\mathcal{V}^{(h)}$ in terms of *labeled trees*.

We refer to Appendix A1 for a sistematic discussion on trees: here we confine ourselves to the basic notions, in order to make selfconsistent the following analysis.

On the plane (x, y) one draws the vertical lines $x = h, h+1, \dots, 0, 1, 2$ and one considers all the possible planar graphs obtained as follows, [GN].

One draws an orizontal line (a *branch* or a *line*) starting from a point r on the line $x = h$, the *root*, and leading to a point v_0 with coordinate $x = h_{v_0} > h$, the *first nontrivial vertex*. Such a point is the branching point of $s_{v_0} \ge 2$ lines (also branches or lines) forming a angles $\vartheta_j \in (-\pi/2, \pi/2)$, $j = 1, \dots, s_{v_0}$, with the x-axis and ending into points each of which is located on some vertical line $x = h_{v_0} + 1, h_{v_0} + 2, \dots$ (and it becomes another branching point). One proceeds in such a way until n points on the line $x = 2$ are reached, the *endpoints*. All the branching points between the root and the endpoints will be called the *nontrivial vertices*. The trivial vertices will be the points located at the intersections of the lines connecting two nontrivial vertices with the vertical lines. The integer n denoting the number of endpoints will be called the *order* of the tree. We associate to the endpoints a number 1 to n, ordered up to down. See Fig. 8.

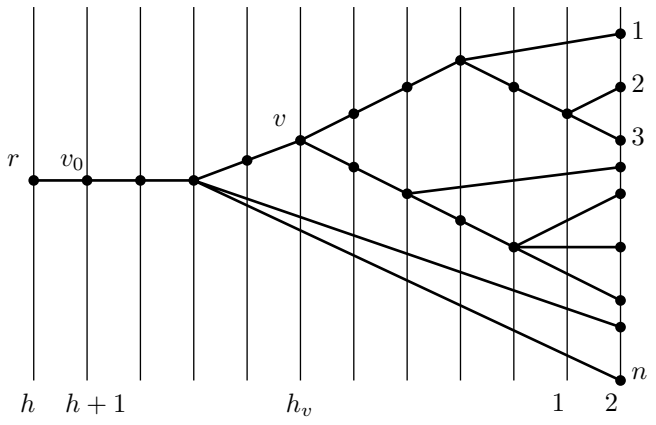


FIG. 8. A tree appearing in the graphic representation of $\mathcal{V}^{(h)}$. Such a tree is obtained by iterating the graph representations of the previous Figs. All the endpoints are on the verticale line corresponding to the line h=2.

If the tree has only one line connecting the root to a vertex on the line $x = 2$, we say that the tree is *trivial* and we shall write $\tau = \tau_0$. Note that in such a case the root has scale $h = 1$.

The graph so obtained is a *tree graph*: it consists of a set of lines connecting a partially ordered set of points (the vertices). The partial ordering of the vertices will be denoted by the symbol $\preceq$: if $v \prec w$ are two vertices, then $h_v < h_w$. Of course the lines are ordered as well: note that there is a correspondence one-to-one between vertices and lines, as a line uniquely identifies the vertex which it enters.

Note that to each vertex v an integer h_v is associated by construction: it is called the *scale label*. In particular we can associate the scale label h to r. We can associate with the unlabeled trees also some other labels: the values of such labels will depend on the particular problem we are studying.

Therefore we shall consider also the *labeled trees* (to be called simply trees in the following): we shall denote by the same symbol τ the labeled trees (in the following we shall deal only with labeled trees) and by $\mathcal{T}_{h,n}$ the set of all labeled trees with n endpoints (*i.e.* of order n) and with a scale label h associated to the root.

It is then easy to see that the number of unlabeled trees with n endpoints is bounded by 4^n; see Appendix A1.

If we include also the endpoints into the set of vertices, we have that the vertices can be either trivial vertices or nontrivial vertices (which include also the endpoints). We shall denote by $V(\tau)$ the set of vertices of a tree τ and by $V_{\rm f}(\tau)$ the set of vertices in $V(\tau)$ which are endpoints. By construction $h_v = 2$ for any $v \in V_{\rm f}(\tau)$, while $h < h_v < 2$ for any $v \in V(\tau) \setminus V_{\rm f}(\tau)$.

To each endpoint there corresponds one of the contributions to the interaction part of the Hamiltonian. With respect to the Hamiltonian (2.17), it is more convenient to consider a Hamiltonian containing some extra term having the same form of the terms defining the free Hamiltonian H_0 times some parameter: physically this is interpreted by saying that the interaction changes the "free" values of the parameters, *i.e.* the values of the parameters of the Hamiltonian describing the free system. By using the decomposition in (2.1) and (2.2) for H_0, we shall consider Hamiltonians of the form

5.16

$$\begin{aligned} H &= H_0 + \mathcal{V} \equiv H_0 + \alpha V_1 + \nu V_2 + u V_3 + \lambda V_4 + \xi V_5 \ , \\ V_1 &= T_0 \ , \\ V_2 &= N_0 \ , \\ V_3 &= P \ , \\ V_4 &= V \ , \\ V_5 &= B \ . \end{aligned} \tag{5.19}$$

Then with each endpoint v of scale $h_v = 2$ we associate one of the five contributions to $\mathcal{V}$: so we can associate to v a label $i \equiv i_v \in \{1, \dots, 5\}$ uniquely identifying the contribution V_i to $\mathcal{V}$ in (5.19): we shall say that the endpoint is
(1) of type α if $i = 1$,
(2) of type ν if $i = 2$,
(3) of type u if $i = 3$,
(4) of type λ if $i = 4$,
(5) of type ξ if $i = 5$.
We can also introduce a label r_v for $v \in V_{\rm f}(\tau)$ such that $r_v = \alpha$ if $i_v = 1$ and so on.

If n is the number of endpoints, $n = |V_{\rm f}(\tau)|$, we shall write $n = n_1 + \ldots + n_5$, where n_i is the number of endpoints $v \in V_{\rm f}(\tau)$ with $i_v = i$.

Moreover with such an endpoint v we associate also a set $\{\mathbf{x}_v\}$ of space-time points, which are the integration variables corresponding to the particular interaction contribution V_i: in particular $\{\mathbf{x}_v\}$ contains one point for any $i \neq 4$ and two points for $i = 4$.

Given a vertex v, which is not an endpoint, $\{\mathbf{x}_v\}$ will denote the family of all space-time points associated with the endpoints following v, *i.e.* with the endpoints $w \in V_{\rm f}(\tau)$ such that $v \prec w$.

We introduce a *field label* f to distinguish the fields appearing in the terms associated with the endpoints: the set of field labels associated with the endpoint v will be called I_v. Then $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field with label f. For instance, for $v \in V_{\rm f}(\tau)$ with $i_v = 4$, then $\{\mathbf{x}_v\} = \{\mathbf{x}, \mathbf{y}\}$ and $I_v = \{f_1, f_2\}$, if $\mathbf{x}(f_1) = \mathbf{x}$ and $\mathbf{x}(f_2) = \mathbf{y}$. We shall write also $\mathbf{x}(I_v) = \{\mathbf{x}(f) \ : \ f \in I_v\}$.

Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the end points following the vertex v.

p.5.2 **5.2.** *Clusters.* It is clear that, if $h \le 0$, the effective potential (if $\tilde{E}_h$ are normalization factors for any $h \le 2$) can be written in the following way:

5.17
$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + L\beta\tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) \ , \tag{5.20}$$

where $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ is defined iteratively as follows.

If τ is the trivial tree τ_0, then $h = 1$ and $\mathcal{V}^{(1)}(\tau_0, \psi^{(\le 1)})$ is given by one of the contributions to $\mathcal{V}(\psi)$, listed in (5.19).

If τ is not trivial and v_0 is the first vertex of τ and $\tau_1, \dots, \tau_s$ (with $s = s_{v_0}$) are the subtrees of τ with root v_0, then

5.18
$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \frac{1}{s!}\mathcal{E}^T_{h+1}\left(\mathcal{V}^{(h+1)}(\tau_1, \psi^{(\le h+1)}), \dots, \mathcal{V}^{(h+1)}(\tau_s, \psi^{(\le h+1)})\right) \ . \tag{5.21}$$

In general for each $v \in V(\tau)$ we denote by s_v the number of lines exiting from v ($s_v = 0$ if $v \in V_{\rm f}(\tau)$), so that, by iterating (5.21), one obtains

5.18a
$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = &\left(\prod_{v \in V(\tau)} \frac{1}{s_v!}\right) \\ &\mathcal{E}^T_{h+1}\left(\mathcal{E}^T_{h+2}\left(\mathcal{E}^T_{h+3} \dots \mathcal{E}^T_{-2}\left(\mathcal{E}^T_{-1}\left(\mathcal{E}^T_{0}\left(\mathcal{V}(\tau_0, \psi^{(\le 1)}), \dots\right), \dots\right), \dots\right), \dots\right), \dots\right) \ , \end{aligned} \tag{5.22}$$

where τ_0 is the trivial tree. The truncated expectations in (5.21) are meant to be computed starting from the endpoints towards the root.

The epression above can look a little intricated at first sight: the better way to understand it is to expecially work out some examples (for instance for low values of h like $h = 0, -1, -2, \dots$) and try to generalize them to any value of $h \le 0$.

Once a vertex v is reached, one has to consider an expression of the kind

5.18c
$$\frac{1}{s_v!}\mathcal{E}^T_{h_v}\left(\tilde{\psi}^{(\le h_v)}(P_{v_1}), \dots \tilde{\psi}^{(\le h_v)}(P_{v_{s_v}})\right) \ , \tag{5.23}$$

where s_v is the number of lines exiting from v and P_{v_j}, with $j = 1, \dots, s_v$, is a set of indices such that

5.18d
$$\tilde{\psi}^{(\le h_v)}(P_{v_j}) = \prod_{f \in P_{v_j}} \psi^{(\le h_v)\sigma(f)}_{\mathbf{x}(f),\omega(f)} \ , \qquad j = 1, \dots, s_v \ , \tag{5.24}$$

is a product of $|P_{v_j}|$ fields on scale $\le h_v$. This can be proven by induction on the scale h_v; see Appendix A6.

Therefore the effect of the truncated expectation $\mathcal{E}^T_{h_v}$ is to contract the fields on scale h_v appearing in the products (5.24) in all the possible ways.

If one uses the expansion (4.42) one obtains a sum over all the possible Feynman diagrams which can be obtained by contracting the half-lines emerging from the sets $P_{v_1}, \dots, P_{v_{s_v}}$. This means that, when the vertex v is reached moving along the tree τ, we construct a "diagram" formed by lines ℓ on scales $h_\ell \ge h_v$.

To any vertex $w \succ v$ there corresponds a subdiagram Γ_w such that all the lines on scale h_w form a connected set if all the subdiagrams Γ_{w_j}, $j = 1, \dots, w_{s_w}$, corresponding to the vertices immediately following w, are thought as contracted into points (this simply follows from the very definition of truncated expectation). We call P_v the set of labels corresponding to the fields associated to the external lines of Γ_v and set $n_v^e = |P_v|$. Then in (5.23) we have

5.18*m*

$$P_v = \bigcup_{j=1}^{s_v} Q_{v_j} \ , \tag{5.25}$$

if Q_{v_j} is the collection of the labels of the fields associated to the external fields of Γ_{v_j} which are not contracted on scale h_v (so that they become external fields of Γ_v). All such fields in turn either will be contracted on some scale $h' < h_v$ or will be the fields on scale $\le h$ whose product contributes to the effective potential $\mathcal{V}^{(h)}(\psi^{(\le h)})$.

We define *cluster* on scale h a set of endpoints which are contracted by lines on scale $h' \ge h$ such that there is at least one line on scale h. By extension we can consider also the endpoints as (trivial) clusters on scale $h = 2$.

An example of Feynman diagram, with the tree and the cluster structure associated to it, is given in Fig. 9.

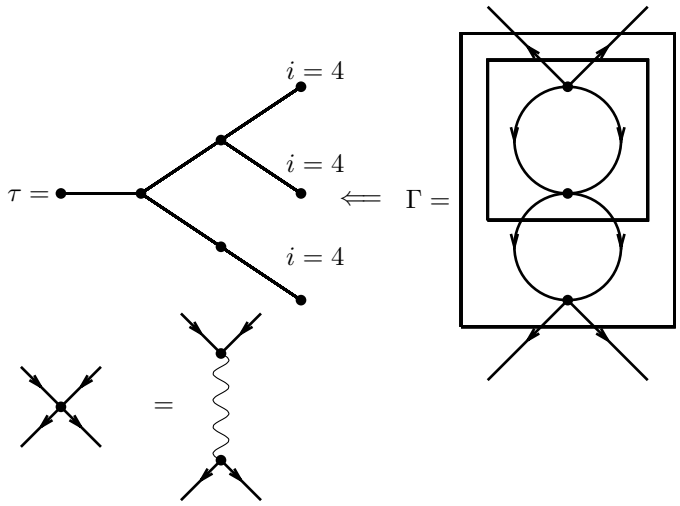


FIG. 9. An example of Feynman graph Γ with its clusters. The cluster structure uniquely identifies a tree τ. All the endpoints are supposed to be of type λ (*i.e.* i_v=4 $\forall v \in V_f(\tau)$; the graph elements corresponding to the endpoints are as will be shown in Fig. 11 below. It is customary to draw the graph elements representing λV by not explicitly drawing the ondulated line (representing the two-body potential), so that the two coordinates $\mathbf{x}$ and $\mathbf{y}$ appear as the were superimposed to each other.

We stress once more that we can choose between two possible expansions: either we really expand each truncated expectation into a sum over Feynman diagrams, as in (4.42), or, we use (4.43), so that we obtain a cluster structure in which one specifies the half-lines emerging from all clusters, but not the way in which the contractions are formed.

In the following we first use the expansion of the truncated expectations into Feynman diagrams, so obtaining bounds for each Feynman diagram: as they are based only on dimensional arguments, such bounds are called *dimensional bounds*. Then we shall show that, if one adopt the expansion (4.43) for the truncated expecation, then one obtains bounds for classes of Feynman diagrams: as they use the Gram-Hadamard

inequality for determinants, such bounds are sometimes called *determinat bounds*. Note that also the dimensional bounds are based on the cluster structure underlying the Feynman diagrams, simply no use is made of the Gram-Hadamard inequality for grouping together classes of diagrams, and on each Feynman diagram a bound is given.

The tree structure underlying (5.22) provides an arrangement of endpoints into a hierarchy of clusters contained into each other. With each vertex v we can associate the cluster G_v formed by the endpoints following v. Then by construction, there will be an inclusion relation by clusters such that $G_v \supset G_w$ if $v \prec w$.

So, given a tree, we can represent it as a set of clusters and *vice versa*; see Fig. 10, where only the clusters associated to nontrivial vertices are drawn.

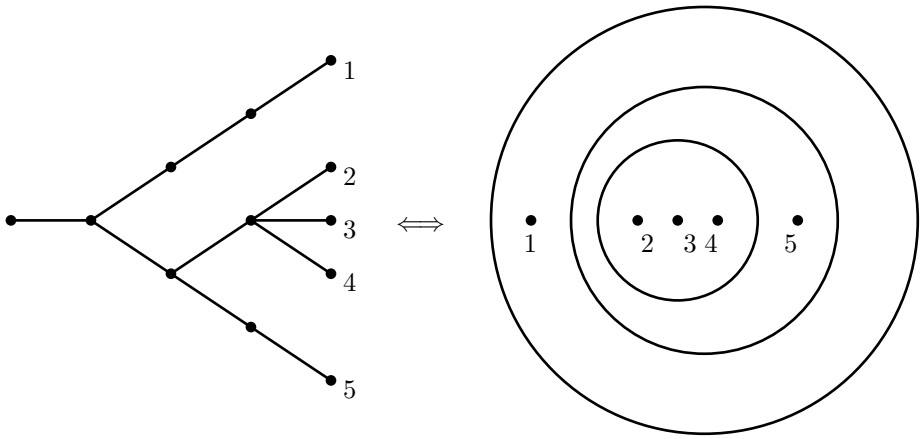


FIG. 10. A tree of order 5 and the corresponding clusters. Only the clusters corresponding to the nontrivial vertices are explicitly taken in consideration.

As we said above, given a cluster G_v, if all the maximal subclusters $G_{v_1}, \dots, G_{v_{s_v}}$ contained inside G_v are thought as points, then the set of points so obtained is connected: so it is possible to single out a set of $s_v - 1$ lines connecting them. Such a set will be called an *anchored tree*: it realizes a minimal connection between the maximal subclusters of G_v.

For each cluster G_v the set P_v determines the external lines of any diagram Γ_v which can be obtained by contracting the fields corresponding to the labels $f \in P_w$, with $v \preceq w$; by extension we shall say that such external lines are the external lines of the cluster G_v.

Each truncated expectation like (5.23) sees the clusters $P_{v_1}, \dots, P_{v_{s_v}}$ as points: by this we mean that its action is independet on the internal structures of the subclusters $G_{v_1}, \dots, G_{v_{s_v}}$ and depends only on the external lines of such clusters.

The crucial property is that, once a structure of clusters has been fixed, there will be a lot of diagrams compatible with it: to have a diagram instead of another will depend on the way the lines external to the clusters are contracted between themselves (see also the discussion at the end of §4.2).

Note that for each trivial vertex in τ the truncated expectation acts a simple expectation (see (1) of (4.15). Moreover if on one hand the truncated expecation requires the subclusters $G_{v_1}, \dots, G_{v_{s_v}}$ to be connected, on the other hand it does not forbid the external lines of the same cluster to be contracted between themselves (selfcontractions).

The final expression for the effective potential obtained through (5.22) is called the *nonrenormalized expansion* for reasons which will become clear later (once a "renormalized expansion" will have been introduced). As we shall see the procedure described here will be too naïve to produce a meaningful description of the

physics underlying the model we are studying: a more careful analysis will be necessary in order to correctly describe the model.

As said just at the beginning of §5.1, even without using the Gram-Hadamard inequality, the introduction of the clusters turns out to be a useful device in order to identify which propagators in a (class of) Feynman diagram are really dangerous. Given a Feynman diagram Γ, suppose to consider a (connected) subdiagram Γ' formed by some points and by the lines connecting them: we shall see later (see §5.4) that bad estimates can arise from such a subdiagram only if the number of external lines (*i.e.* of lines emerging from the the vertices internal to Γ' but not belonging to Γ') is equal to 2 or 4, for the class of models we are considering. A more careful analysis would show that such a contribution can really give problems only if all lines internal to Γ' have a momentum of size larger than the size of the momenta of the external lines. So if the subdiagram is a cluster such a property of the subdiagrams is automatically taken into account and, in terms of clusters, we can say that only clusters with 2 or 4 external lines can be source of problems: such an argument will be given a more rigorous formulation in §5.4 below.

p.5.3 **5.3.** *Values of Feynman diagrams.* Suppose (for simplicity and for concreteness) that each endpoint is of type λ: then the Feynman diagrams with p external lines are all the possible diagrams obtained by connecting all the clusters and leaving p uncontracted lines.

Expanding the truncated expectation in (5.21) by using the Feynman diagram expansion (see Fig. 1), one obtains a representation of $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ as sum over Feynman diagrams of quantities which are given by the product of fields times suitable coefficients called the *values* of the Feynman diagrams.

As we said before, for the moment we are supposing that all the truncated expecations are written in terms of Feynman diagrams: then we shall obtain some bounds on the values of the Feynman diagrams. A final bound on the kernels of the effective potentials can be obtained simply bu multiplyng the bound holding for a generic Feynman diagram times the number of Feynman diagrams.

In §6 we shall prove that the same dimensional arguments can still be performed by directly studying (5.22) and making use also of the expansion (4.43) for the truncated expectations. The final expression for the effective potential will be called the *nonrenormalized expansion* for reasons which will become clear later (once a "renormalized expansion" will have been introduced). As we shall see the procedure described here will be too naïve to produce a meaningful description of the physics underlying the model we are studying: a more careful analysis will be necessary in order to correctly describe the model.

Now let us come back to the bounds on Feynman diagrams. The value to be assigned to any Feynman diagram is obtained in the following way (for istance in momentum space).

Given a line ℓ of a Feynman diagram, there will be a cluster G_v on scale h_v such that ℓ is contained in G_v but it is outside any other cluster internal to G_v; moreover the momentum of the propagator corresponding to such a line will be of the form $\mathbf{k} = \mathbf{k}' + \omega p_F$, for some values of $\mathbf{k}'$ of size γ^{h_v} (otherwise, by the support properties of the χ functions, the value of the corresponding diagram is vanishing) and of $\omega = \pm 1$. Then with the line ℓ the following labels will be associated: $\mathbf{k}_\ell = \mathbf{k}$, $\omega_\ell = \omega$ and $h_\ell = h_v$.

One associates with each contracted line ℓ the propagator

5.19a
$$g_\ell \equiv g^{(h_\ell)}_{\omega_\ell}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}_\ell\cdot(\mathbf{x}-\mathbf{y})}\, \hat g^{(h_\ell)}_{\omega_\ell}(\mathbf{k}_\ell)\ , \tag{5.26}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the points connected by the line ℓ: here again for concretensess purposes we are supposing that the model described with free Hamiltonian H_0 given by (2.1) is considered.

Each line has a momentum according to the usual momentum conservation rules, the independent momenta are integrated and with the lines which are non contracted the *external fields* $\psi^{(\le h)}$ are associated, if h is the scale of the root of the tree. Then the coefficient by which the product of external fields is multiplied is the value of the Feynman diagram.

Note that Feynman diagrams associated with a set of clusters naturally appear: we have seen that if one looks at standard Feynman diagrams, one is naturally led to introduce clusters to identify the subgraphs

responsible of divergences (which are the subdiagrams such that their internal lines momenta are larger than the momenta of their external lines).

p.5.4 **5.4.** *Power counting.* It is quite easy to estimate the above Feynman diagrams. First note that each propagator $g_\omega^{(h)}(\mathbf{x})$ is finite and, for any integer N, it is bounded by

5.20
$$\left|g_\omega^{(h)}(\mathbf{x})\right| \le \gamma^h \frac{C_N}{1+\left(\gamma^h|\mathbf{x}|\right)^N}\ , \tag{5.27}$$

as it is easy to derive by using (5.10), see Appendix A4. We perform the estimates in the coordinate space; at this level the estimates could be performed also in the momentum space and no conceptual difference would arise, but we shall see that in the nonperturbative estimates it is convenient to work in the coordinate space.

Note that, given a Feynman diagram Γ, there is a tree which can be associated to it, uniquely determined by the cluster strucure of Γ: let us call it τ.

Then, as all the clusters have to be connected, by the very definition of the truncated expectation (see §4.2), the integrations, up to a constant C^n, produce a factor

5.21
$$\prod_{v\notin V_{\rm f}(\tau)} \gamma^{-2h_v(s_v-1)}\ , \tag{5.28}$$

if s_v is the number of subtrees coming out from v and $v\notin V_{\rm f}(\tau)$ stands for $v\in V(\tau)\setminus V_{\rm f}(\tau)$; see Appendix A4.

Moreover, for any cluster G_v, $v\notin V_{\rm f}(\tau)$, by using (5.27) we get, up to a constant C^n, a factor

5.22
$$\gamma^{h_v n_v^0}\ , \tag{5.29}$$

if n_v^0 is the number of propagators internal to a cluster G_v but not to any smaller one.

So the bound for the value of a generic Feynman diagram Γ is given by

5.23
$$\int \mathrm{d}\mathbf{x}(I_{v_0})\,|\mathrm{Val}(\Gamma)| \le C^n \prod_{v\notin V_{\rm f}(\tau)} \gamma^{h_v\left(n_v^0-2(s_v-1)\right)}\ , \tag{5.30}$$

where τ is the tree associated to Γ, $I_{v_0}=\{1,\dots,n+n_4\}$ (if n is the number of endpoints and n_4 is the number of endpoints v with $i_v=4$) and

5.23a
$$\int \mathrm{d}\mathbf{x}(I_{v_0}) = \prod_{f\in I_{v_0}} \sum_{x(f)\in\Lambda} \int_{-\beta/2}^{\beta/2} \mathrm{d}x_0(f). \tag{5.31}$$

For simplicity we shall consider only the case in which for $v\in V_{\rm f}(\tau)$ one has either $i_v=4$ or $i_v=2$, see (5.19). Let $m_{4,v}$ be the number of endpoints contained in the cluster v to which is associated a label $i=4$ and let $m_{2,v}$ be the the number of endpoints contained in the cluster G_v to which is associated a label $i=2$. Moreover let n_v^e be the number of fields external to the cluster G_v.

Then the following relations can be easily checked to hold, if v' is the vertex preceeding v on the tree:

5.24
$$\sum_{v\notin V_{\rm f}(\tau)} (h_v-h)\,(s_v-1) = \sum_{v\notin V_{\rm f}(\tau)} (h_v-h_{v'})\,(m_{4,v}+m_{2,v}-1) \tag{5.32}$$

and

5.25
$$\sum_{v\notin V_{\rm f}(\tau)} (h_v-h)\,n_v^0 = \sum_{v\notin V_{\rm f}(\tau)} (h_v-h_{v'})\left(2m_{4,v}+m_{2,v}-\frac{n_v^e}{2}\right)\ . \tag{5.33}$$

Note that $h_v - h_{v'} = 1$ by construction.

Inserting the above two equalities into (5.30), one gets

5.26
$$\int \mathrm{d}\mathbf{x}(I_{v_0}) \, |\mathrm{Val}(\Gamma)| \leq C^n \gamma^{-h\left(n^e_{v_0}/2-2+m_{2,v_0}\right)} \prod_{v \notin V_{\mathrm{f}}(\tau)} \gamma^{-(h_v-h_{v'})(n^e_v/2-2)} \gamma^{-(h_v-h_{v'})m_{2,v}} \, , \tag{5.34}$$

where v_0 is the node immediately following the root.

Note that in (8.41) it is more convenient to redecompose

8.26c
$$h m_{2,v_0} + \sum_{v \in V(\tau)} (h_v - h_{v'}) \, m_{2,v} = \sum_{v \in V_{\mathrm{f}}(\tau)} h_{v'} m_{2,v} \, , \tag{5.35}$$

where, for an endpoint v, one has

8.26d
$$m_{2,v} = \begin{cases} 1 \, , & v \text{ is of type } \nu \, , \\ 0 \, , & \text{otherwise} \ . \end{cases} \tag{5.36}$$

The identity (5.35) can be easily verified analogously to (5.32) and (5.33). In this way we obtain a factor $\gamma^{-h_{v'}}$ for each endpoint v with $i_v = 2$ (see (5.19)). As all endpoints are on scale $h = 2$, this means that we have a factor γ^{-1} for each endpoint of type ν: we prefer to maintain the writing $h_{v'}$ for reasons that will become clear in the following (when the renormalization procedure will have been introduced).

Given a cluster G_v we denote by P_v the sets of of labels f such that $\mathbf{x}(f)$ is an endpoint contained in G_v, so that $n^e_v = |P_v|$, and $\psi^{(\leq h)\sigma_{\mathrm{f}}}_{\mathbf{x}(f),\omega(f)}$ is the field associated to a line external to G_v. By defining

5.26a
$$D(P_v) = \frac{n^e_v}{2} - 2 \, , \tag{5.37}$$

we can rewrite the bound (5.34) as

5.26b
$$\int \mathrm{d}\mathbf{x}(I_{v_0}) \, |\mathrm{Val}(\Gamma)| \leq C^n \gamma^{-hD(P_{v_0})} \left(\prod_{v \notin V_{\mathrm{f}}(\tau)} \gamma^{-(h_v-h_{v'})D(P_v)} \right) \left(\prod_{v \in V_{\mathrm{f}}(\tau)} \gamma^{-h_{v'} m_{2,v}} \right) \, . \tag{5.38}$$

The above estimate is of course finite (contrary to the power counting of the all theory for propagators which are singular), but the problems come out if one wants to perform the sum over the scales of a tree. If $n^e_v \geq 6$, then $(n^e_v/2 - 2) \geq 1$ and so, by using that $h_v - h_{v'} > 0$,

5.27
$$\sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\{P_v\}} \left(\prod_{v \notin V_{\mathrm{f}}(\tau)} \gamma^{-(h_v-h_{v'})(n^e_v/2-2)} \right) \leq \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\{P_v\}} \left(\prod_{v \in V_{\mathrm{f}}(\tau)} \gamma^{-(h_v-h_{v'})} \right) \leq C^n \, , \tag{5.39}$$

for some (different) constant C, as it is proven in Appendix A6.1.

Note that (5.37) could suggest that each time we have an endpoint $v \in V_{\mathrm{f}}(\tau)$ of type ν, we gain an extra unit contributing to $D(P_w)$ for all $w \preceq v$, so that one could think that no problems arise for $n^e_v = 4$ when $m_{2,v} \geq 1$ and for $n^e_v = 2$ when $m_{2,v} \geq 2$. Nevertheless this is not true as all such gains are payed by an extra bad factor γ^{-hn_2} in front of the product in (5.37).

Then we identify immediately the following problem. If $n^e_v \leq 4$ the above sum cannot be performed; then the clusters with 2 or 4 external lines have to be *renormalized*. At this level this simply means that there is something to do if one wants to obtain something of meaningful: one will have to consider a different expansion.

The above problem manifests itself at a *perturbative* level, as the effect of the bounds for single diagrams, if the sum over the trees is performed. However there is also a *nonperturbative problem*; even if $n^e_v \geq 6$, we cannot conclude from such bounds that the theory has a meaning. The reason is the following one.

As we see from (5.22) we have a factor $1/s_v!$ for each vertex $v \in V(\tau)$. If we expand the truncated expectation $\mathcal{E}^T_{h_v}$ in terms of Feynman diagrams we obtain $O(s_v!^2)$ terms (see Appendix A1). Then the overall combinatorial factor is proportional to

5.27a
$$\prod_{v\in V(\tau)} \frac{s_v!^2}{s_v!} = \prod_{v\in V(\tau)} s_v! \,, \tag{5.40}$$

where n is the number of endpoints in τ. This means that for any vertex there are too many diagrams and the factor $1/s_v!$ arising from the expansion into product of truncated expecations (5.22) is not enough to try to compensate the number of Feynman diagrams. So the sum over n cannot be performed.

We shall consider a particular model for introducing all the Renormalization Group formalism, rather than developing it *in abstracto*. By following [M1] we choose the Holstein-Hubbard model in which, with respect to the adiabatic Holstein model, there is also a quartic term in the interaction, so that we dispose in such a way of a model which presents most of the interesting features of one-dimensional fermionic systems: by simply putting $\lambda = 0$ (*i.e.* by neglecting the two-body interaction), we recover the adiabatic Holstein model. The problem arising by the bound (5.40) is then solved through the use of the Gram-Hadamard inequality which allow us to obtain $s_v!$ terms for each $v \in V(\tau)$ instead of $s_v!^2$ terms. The technical details are deferred to next section and to Appendix A3.

After treating the Holstein-Hubbard model we shall show (see §12 below) how similar methods can produce many results in a number of fermionic models.

p.5.5 **5.5.** *Comparison with Wilson's method.* In the previous sections we have seen how to define a sequence of effective potentials $\mathcal{V}^{(0)}, \mathcal{V}^{(-1)}, \dots, \mathcal{V}^{(h)}$ integrating the fields $\psi^{(1)}, \psi^{(0)}, \dots, \psi^{(h+1)}$. It is interesting to remark the similarity of this approch with the Renormalization Group of Wilson, [W]. Calling $\psi^{(\geq\Lambda)}$ and $\psi^{(\leq\Lambda)}$ fields with momentum $\mathbf{k} = (k, k_0)$ with $|\mathbf{k}'|$ bigger or lower than some prefixed scale Λ, in the approach of Wilson, one computes (see for istance [MCD])

5.19
$$\int P(\mathrm{d}\psi^{(\geq\Lambda)})\, e^{\mathcal{V}(\psi)} \equiv e^{\mathcal{V}^{(\Lambda)}(\psi^{(\leq\Lambda)})} \,. \tag{5.41}$$

Comparing $\mathcal{V}^{(\Lambda)}$ and $\mathcal{V}^{(\Lambda+\mathrm{d}\Lambda)}$, for $|\mathrm{d}\Lambda| \ll 1$, one gets in the limit $\mathrm{d}\Lambda \to 0$, some differential equation for the running coupling constants which will be introduced in §8 below. One can see that this is what we do in the limit $\gamma \to 1$ and considering a sharp partition of unity through ϑ-functions instead of the χ-functions introduced through (5.3). The reason why we do not do this will become clear in the following: essentially it is that one has to perform derivatives and the derivative of a ϑ-function is a δ-function, so that this causes some spurious technical difficulties. We think that it is possible to extend our formalism closer to Wilson's original formulation, but there is essentially no simplification in doing this; therefore we will not discuss further such a point here.

sec.6
6. Nonperturbative estimates for the nonrenormalized expansion

p.6.1 **6.1.** *Kernels of the effective potentials.* By the analysis of the previous section we have that $\mathcal{V}^{(h)}(\psi^{(\leq h)})$, the effective potential on scale h, can be written as

6.1
$$\begin{aligned} \mathcal{V}^{(h)}(\psi^{(\leq h)}) &= \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) \,, \\ \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) &= \int \mathrm{d}\mathbf{x}(I_{v_0}) \sum_{P_{v_0}\subset I_{v_0}} \tilde{\psi}^{(\leq h)}(P_{v_0}) \mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(I_{v_0})\right) \,, \end{aligned} \tag{6.1}$$

where $\mathcal{T}_{h,n}$ is the set of labelled trees of order n contributing to $\mathcal{V}^{(h)}(\psi^{(\le h)})$ and

6.2
$$\int \mathrm{d}\mathbf{x}(I_{v_0}) = \prod_{f\in I_{v_0}} \sum_{x(f)\in\Lambda} \int_{-\beta/2}^{\beta/2} \mathrm{d}x_0(f)\ , \tag{6.2}$$

with $I_{v_0} = \{1,\dots,n+n_4\}$, if n is the number of endpoints and n_2 is the number of endpoints v with $i_v = 4$.

By using (5.21) and (6.1) we obtain for the kernel $\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}_{v_0})$ the following recursive relation:

6.3
$$\begin{aligned}\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}(I_{v_0})) = \sum_{P_{v_1},\dots,P_{v_{s_{v_0}}}} \left(\prod_{j=1}^{s_{v_0}} \mathcal{W}^{(h+1)}\left(\tau_j, P_{v_j}, \mathbf{x}(I_{v_j})\right)\right) \\ \frac{1}{s_{v_0}!}\mathcal{E}^T_{h+1}\left(\tilde\psi^{(h+1)}(P_{v_1}\setminus Q_{v_1}),\dots \tilde\psi^{(h+1)}(P_{v_{s_{v_0}}}\setminus Q_{v_{s_{v_0}}})\right)\ ,\end{aligned} \tag{6.3}$$

where

6.3a
$$Q_{v_j} = P_{v_0}\cap P_{v_j}\ , \qquad j = 1,\dots,s_{v_0}\ . \tag{6.4}$$

Then (6.3) can be iterated leading to

6.4
$$\begin{aligned}&\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}(I_{v_0}))\\ &\quad= \sum_{\{P_v\}_{v\in V(\tau)}} \left(\prod_{v\notin V_{\rm f}(\tau)} \mathcal{E}^T_{h_v}\left(\tilde\psi^{(h_v)}(P_{v_1}\setminus Q_{v_1}),\dots \tilde\psi^{(h+1)}(P_{v_{s_v}}\setminus Q_{v_{s_v}})\right)\right)\left(\prod_{v\in V_{\rm f}(\tau)} r_v\right)\ ,\end{aligned} \tag{6.5}$$

where s_v is the number of lines exiting from the vertex v (whose value is fixed by the tree τ), while r_v is the constant appearing in (5.19) associated to the endpoint v ($r_v = \lambda$ if v is of type λ and so on; see (5.19)).

The sum

6.5
$$\sum_{\{P_v\}_{v\in V(\tau)}} \tag{6.6}$$

in (6.5) is over all the possible choices of the sets P_v corresponding to the vertices of τ, except P_{v_0} which is fixed. The sets Q_v are uniquely determined by the sets $\{P_v\}$ by taking into account that for any $v\in V(\tau)$ one has

6.6
$$Q_v \subset P_v\ , \qquad P_v = \bigcup_{j=1}^{s_v} Q_{v_j}\ , \tag{6.7}$$

so that for any $v\in V(\tau)$ and for any v_j immediately following v one has

6.6a
$$Q_{v_j} = P_v\cap P_{v_j}\ , \qquad j = 1,\dots,s_v\ , \tag{6.8}$$

which extends (6.4) to any vertex in $V(\tau)$.

Then we can write (6.1) as

6.7
$$\begin{aligned}\mathcal{V}^{(h)}(\psi^{(\le h)}) &= \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau,\psi^{(\le h)})\ ,\\ \mathcal{V}^{(h)}(\tau,\psi^{(\le h)}) &= \sum_{P_{v_0}\subset I_{v_0}} \int \mathrm{d}\mathbf{x}(P_{v_0})\,\tilde\psi^{(\le h)}(P_{v_0})\,\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}(P_{v_0}))\ ,\end{aligned} \tag{6.9}$$

where

6.8
$$\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}(P_{v_0})) = \int \mathrm{d}\mathbf{x}(I_{v_0}\setminus P_{v_0})\,\mathcal{W}^{(h)}(\tau, P_{v_0}, \mathbf{x}(I_{v_0}))\ . \tag{6.10}$$

The kernel $\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)$ depends only on the variables

6.9
$$\mathbf{x}(P_{v_0}) = \{\mathbf{x}(f)\}_{f\in P_{v_0}} \,, \tag{6.11}$$

and, as we are going to prove, they satisfy the bound

6.10
$$\int \mathrm{d}\mathbf{x}(P_{v_0}) \left|\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)\right| \le \beta L \gamma^{-hD(P_{v_0})} \sum_{\{P_v\}} \left(\prod_{v\notin V_{\rm f}(\tau)} \gamma^{-(h_v-h_{v'})D(P_v)}\right) (C\varepsilon)^n \,, \tag{6.12}$$

where $\varepsilon = \max\{|\alpha|, |\nu|, |u|, |\lambda|, |\xi|\}$, C is a suitable constant depending on N (through the bound (5.27) holding for the propagators) and $D(P_v)$ is defined in (5.37).

The sums over the sets $\{P_v\}$ in (6.14) and over $\tau \in \mathcal{T}_{h,n}$ in order to recover the complete kernels of the effective potential require $D(P_v) > 0$. However this is not true when $n_v^e \le 4$: it will become possible only after that the renormalization procedure has been applied (so that $D(P_v)$ will be modified into $D(P_v) + z_v$, with z_v such that $D(P_v) + z_v > 0$).

p.6.2 **6.2.** *Proof of (6.12).* First note that (6.12) involves the integrations of all the endpoints. For all the endpoints $V \in V_{\rm f}(\tau)$ with $i_v = 4$ we can use the potential $v(x-y)\,\delta(x_0-y_0)$ in order to integrate one of the two variables $\{\mathbf{x}, \mathbf{y}\}$: so we are left with n integrations.

Recall that, by (4.47),

6.11
$$\left|\mathcal{E}^T\left(\tilde\psi(P_1), \dots, \tilde\psi(P_s)\right)\right| \le \sum_T \left(\prod_{\ell\in T} |g_\ell|\right) C_1^{n-s+1} \,, \tag{6.13}$$

if $n = |P_1| + \dots + |P_s|$ and C_1 is a constant proportional to the bound C_0 on the propagator $\mathcal{E}(\psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{y},\sigma'})$. If $\mathcal{E} = \mathcal{E}_h$ one has $C_0 = C_N\gamma^h$, see (5.27).

Then, by introducing (4.43) into (6.5) and using (6.13), we can bound

6.12
$$\begin{aligned}&\left|\mathcal{E}^T_{h_v}\left(\tilde\psi^{(h_v)}(P_{v_1}\setminus Q_{v_1}), \dots \tilde\psi^{(h+1)}(P_{v_{s_v}}\setminus Q_{v_{s_v}})\right)\right| \\ &\qquad \le \sum_T \left(\prod_{\ell\in T} g_\ell\right) (CC_N)^{\sum_{j=1}^{s_v}|P_{v_j}|-|P_v|}\, \gamma^{h_v\left(\sum_{j=1}^{s_v}|P_{v_j}|-|P_v|\right)} \,,\end{aligned} \tag{6.14}$$

as also the propagators g_ℓ, $\ell \in T$, are on scale h_v and we used that the number of lines internal to G_v which are contracted on scale h_v is given by

6.12a
$$\sum_{j=1}^{s_v} |P_{v_j}| - |P_v| \,. \tag{6.15}$$

Then for each anchored tree T contributing to the sum we can use the $s_v - 1$ propagators g_ℓ, with $\ell \in T$, in order to perform $s_v - 1$ integrations: this gives a factor

6.13
$$\gamma^{-2h_v(s_v-1)} \,, \tag{6.16}$$

as it can be easily proved by using the compact support properties of the propagators (compare with (5.28)); see Appendix A4.

As the number of integration variables is n (see the initial comments of this section) and

6.14
$$\sum_{v\notin V_{\rm f}(\tau)} (s_v - 1) = |V_{\rm f}(\tau)| - 1 = n - 1 \,, \tag{6.17}$$

we see that, at the end, all the integrations can be performed, up to one, corresponding to a single endpoint of the tree: such an integration gives the factor (βL) in (6.12).

Moreover we have

6.15
$$\prod_{v\in V_{\rm f}(\tau)} |r_v| \le \varepsilon^n \ , \tag{6.18}$$

by the definition of ε after (6.12) and by the fact that $|V_{\rm f}(\tau)| \le n$.

Noting that

6.17
$$\sum_{j=1}^{s_v} |P_{v_j}| - |P_v| = n_v^0 \ , \tag{6.19}$$

where n_v^0 is defined after (5.29), then we obtain, for the left hand side of (6.12), a bound

6.18
$$\beta L \left(\prod_{v\notin V_{\rm f}(\tau)} \gamma^{h_v\left(n_v^0-2(s_v-1)\right)} \right) (C\varepsilon)^n \ , \tag{6.20}$$

for some constant C, depending on N, so that, by using the relations (5.32) and (5.33) and the definition (5.37), then (6.12) immediately follows (simply reason as in §5.4 about the Feynman diagrams).

sec.7

7. Schwinger functions as Grassman integrals

p.7.1 **7.1.** *Perturbation theory and euclidean formalism.* The Schwinger functions have been introduced in §3.1. The standard perturbation theory allows us to express them in terms of Feynman graphs.

By using the representation

7.1
$$e^{-tH} = \lim_{n\to\infty} \left[e^{-tH_0/n} \left(1 - \frac{t\mathcal{V}}{n} \right) \right]^n \ , \tag{7.1}$$

where H_0 is defined in (2.1) in the discrete case and in (2.2) in the continuum case, while (for instance, see (2.17) and (5.19))

7.2
$$\mathcal{V} = uP + \lambda V + \nu H_0 \ , \tag{7.2}$$

one finds for the numerator of (3.1) the following representation.

By introducing $p_1 + \ldots + p_{s+1}$ variables t'_j such that one has $t'_j \ge t'_{j+1}$ for any $1 \le j < p_1 + \ldots + p_{s+1}$ and the values $t'_{p_1}, \ldots, t'_{p_1+\ldots+p_s}$ are fixed to be $t_1, \ldots, t_s$, respectively, we define $\mathbf{t} = \{t'_j\}$ and set

7.3
$$\mathcal{V}(t) = e^{H_0 t} \mathcal{V} e^{-tH_0} \ . \tag{7.3}$$

Then the numerator of (3.1) becomes

7.4
$$\sum \pm \int \mathrm{d}\mathbf{t}\, \mathrm{Tr}\, e^{-\beta H_0} \mathcal{V}(t_1) \ldots \mathcal{V}(t'_{p_1-1})\, \psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1} \mathcal{V}(t'_{p_1+1}) \ldots \psi^{\varepsilon_s}_{\mathbf{x}_s,\sigma_s} \ldots \mathcal{V}(t'_{p_1+\ldots+p_{s+1}}) \ , \tag{7.4}$$

where the sum is over the integers $p_1 + \ldots + p_{s+1}$, the integral is over all the variables t'_j, with the constraints described above, and the sign $\pm$ is $+$ if the number of the $\mathcal{V}$ factors is even and $-$ otherwise.

By taking into account that each term contributing to $\mathcal{V}$ in (7.2) is an integral on space variables and that H_0 is quadratic in the field operators, the terms in (7.4) can be expressed as integrals of sums of products of propagators

7.5
$$g(\mathbf{x}-\mathbf{y}) = \begin{cases} \mathrm{Tr}\, e^{-\beta H_0} \psi^-_{\mathbf{x}} \psi^+_{\mathbf{y}} / \mathrm{Tr}\, e^{-\beta H_0} \ , & \text{if } x_0 > y_0 \ , \\ -\mathrm{Tr}\, e^{-\beta H_0} \psi^+_{\mathbf{y}} \psi^-_{\mathbf{x}} / \mathrm{Tr}\, e^{-\beta H_0} \ , & \text{if } x_0 \le y_0 \ , \end{cases} \tag{7.5}$$

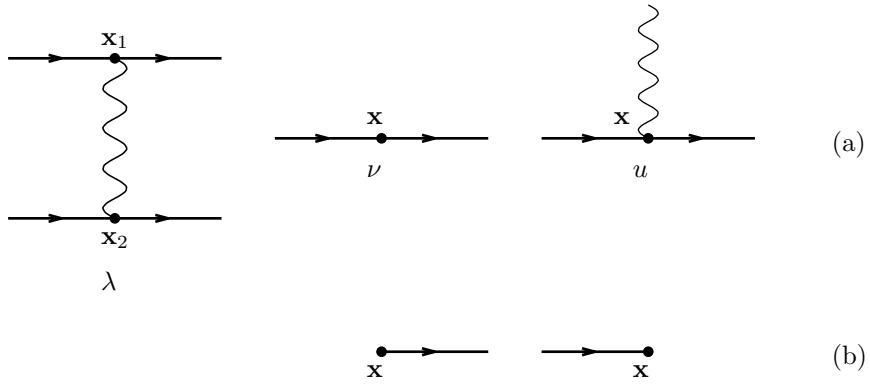


FIG. 11. The graph elements for the model described by the Hamiltonian with interaction given by (7.2). Note that the ondulated lines appearing in two of the graph elements of the form (a) have a different meaning: for the graph element associated to endpoints of type λ it represents the potential $v(x-y)\delta(x_0-y_0)$, while for the graph element associated to endpoints of type λ it represents the potential $\varphi(x)\delta(x_0)$.

Then each term can be graphically represented in terms of Feynman diagrams, which are obtained by contracting in all the possible ways the *graph elements* represented in Fig. 11.

One has s elements of the last forms (b) in Fig.11 and n elements of one of the remaining forms (a). The lines are then contracted as described in §4.2.

It is a remarkable result, [AGD], that all the non-connected graphs cancel exactly the denominator of (3.1), which of course can be dealt with as the numerator and gives a formula analogous to (7.4), with the only difference that only $\mathcal{V}$ factors appear and only graph elements of the form (a) in Fig. 11.

This explains why only connected graphs have to be considered.

The Schwinger functions can be expresses also in terms of fermionic functional integrations introduced in §4. The expansion (7.4) in terms of fermionic fields can be shown, [BG2], to be equivalent to the expansion in terms of Grassman variables given by

7.6
$$S(\mathbf{x}_1,\varepsilon_1,\sigma_1,\dots,\mathbf{x}_s,\varepsilon_s,\sigma_s) = \frac{\partial^n}{\partial\phi^{\varepsilon_1}_{\mathbf{x}_1}\dots\partial\phi^{\varepsilon_s}_{\mathbf{x}_s}} \log\int P(\mathrm{d}\psi)\,\exp\left[\mathcal{V}(\psi)+\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}\mathrm{d}x_0\,\left(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x},\sigma}+\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x},\sigma}\right)\right], \tag{7.6}$$

where the derivatives are meant as (formal) functional derivatives. The equivalence is formally an identity: it is enough to interpret the propagator (7.5) as an expectation value of the product of two Grassman fields (see (4.36)).

Therefore one finds for the Schwinger functions a graphical expression analogous to that of the effective potentials: the only difference is that the interaction is slightly changed by allowing an interaction with a fictitious "external field". Without considering the multiscale decomposition of the propagators and the renormalization effects the relation between the effective potential (obtained by integrating all the scales) and the Schwinger functions would be easily to derive (see for instance [BG]; see also §11 later): however the multiscale decomposition and the renormalization, mostly the change introduced into the "free measure", makes such a relation not so obvious and the explicit representation of the Schwinger functions in terms of truncated expectations becomes a little involved: this will be carried out later in §11, by starting from (7.6).

For instance, in the case of the two-point Schwinger functions, one has to compute

7.7
$$S(\mathbf{x},-,\sigma,\mathbf{y},+,\sigma') \equiv S(\mathbf{x},\mathbf{y}) = \frac{\int P(\mathrm{d}\psi)\, e^{\mathcal{V}(\psi)} \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{y},\sigma'}}{\int P(\mathrm{d}\psi)\, e^{\mathcal{V}(\psi)}}$$
$$= \frac{\partial^n}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} \log \int P(\mathrm{d}\psi)\, \exp\left[\mathcal{V}(\psi) + \sum_{x\in\Lambda} \int_{-\beta/2}^{\beta/2} \mathrm{d}x_0 \left(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x},\sigma} + \psi^+_{\mathbf{x}}\phi^-_{\mathbf{x},\sigma}\right)\right], \tag{7.7}$$

where the interaction $\mathcal{V}(\psi)$ is as above. We note since now that the second expression in (7.7) - as well as (7.6) in the general case of any s-point Schwinger functions - is more convenient for practical purposes as it allows to follow the same strategy adopted for the effective potentials (simply with a different "interaction Hamiltonian") consisting in integrating the scales in a hyerarchical way.

p.7.2 **7.2.** *Feynman graphs and origin of divergences.* The expansion given above for the effective potentials and the one hinted for the Schwinger functions (which, as anticipated, will be carried out in detail in §11) are finite sums with finite coefficients if L, β are finite; however in general there is no hope that the above series are still convergent in the limits $L, \beta \to \infty$. The reasons are a lot and quite easy to understand. If $\lambda = 0$ (in the limit $L, \beta \to \infty$) the Fourier transform of the Schwinger function is singular at $k_0 = 0$, $|k| = p_F$; even in the most favorable case, in which the interacting Schwinger function has the same kind of singularity of the free one (such systems are generally called *Fermi liquids*), there is no reason for which the singularity of the free and the interacting Schwinger functions have to be located at the same point, *i.e.* $k_0 = 0$, $|k| = p_F$, but in general will be in some other point $k_0 = 0$, $|k| = p_F + O(\lambda)$.

This phenomenon is quite general (there is the remarkable exception of the Luttinger model in which, as we shall see, the singularity is λ-independent, due the the relativistic invariance of the model) and not limited to the case $d = 1$. By the way in more than one dimension the situation is even more complicated as, in absence of rotation invariance, the singularity is shifted by an angle-dependent quantity, see [FTS1]. It is quite clear that this produces problems in a naïve expansion for the correlation function. Assume that the interacting Schwinger function is simply

4.40
$$\frac{1}{-ik_0 + \cos k - \mu - \nu(\lambda)}\,, \tag{7.8}$$

which has the same nature of singularity as in the $\lambda = 0$ case, but at the point $\cos^{-1}(\mu + \nu(\lambda))$; an expansion in powers of λ needs a preliminary expansion

4.41
$$\left[\frac{1}{-ik_0 + \cos k - \mu}\right] \sum_{n=0}^{\infty} \left(\frac{\nu(\lambda)}{-ik_0 + \cos k - \mu}\right)^n\,, \tag{7.9}$$

which of course has no meaning for k close to p_F. This is one of the reasons for which we expect that the expansion in terms of Feynman diagrams cannot be well defined and why it is not the right expansion to consider. It is also very easy to isolate some of the diagrams reflecting the above shift of the singularity; for instance the diagram represented in Fig. 12.

Note that the divergences occur when the momenta of the propagators of the lines in the boxes are larger than the momenta of the lines external to the boxes; in fact in the other case the small momentum of the external lines is compensated by the momentum of the lines internal to the boxes, and no accumulation of propagators with small momenta is present.

Then by the above simple example we learn that we have to divide the integration domains for each Feynman diagram to single out the true dangerous contributions; in other words we have to factorize the product of propagators for each Feynman diagram according to the relative size of the momenta of the propagators associated to the lines. Such a "factorization" is essential in a consistent theory of renormalization; if one

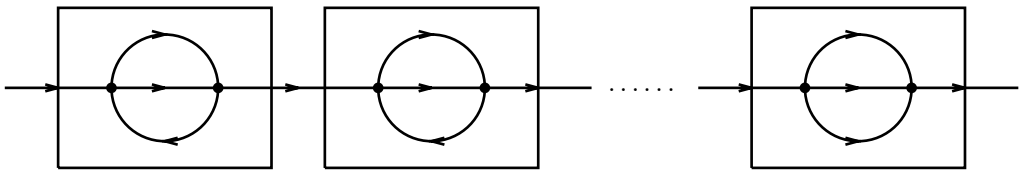


FIG. 12. A chain of clusters with two external lines. As said in Fig. 9, the graph elements with four external lines are drawn by not expicitly representing the two-body potential as an ondulated line, but simply gluing together the two points $\mathbf{x}$ and $\mathbf{y}$.

does not do it, one finds well known problems like the *overlapping divergences problem*, [R], or the *renormalon problem*, [R], due to the fact that one "subtracts too much". We have seen in the §5 that diagrams including the above factorization are naturally generated in a Wilsonian Renormalization Group framework: mathematically the key notion is that of clusters.

The one explained above is the simpler source of problem in the expansion. A more serious one is the change of the exponent of the singularity (*anomalous dimension*), leading to logarithmic divergencies; for instance one can think to the function $x^{-(1+\varepsilon)}$ and its expansion $x^{-1}\sum_{n=0}^{\infty}[\varepsilon \log x]^n/n!$: each addend has a $O(\log|x|^n)$ behaviour. Even more serious is the change of the nature of the singularity, for istance in the case in which there is a *gap generation*, so that for instance the Fourier transform of the Schwinger function is not singular at all: this is what is believed to happen in superconductivity or in $d = 1$ when there is the formation of charge or spin density wave. From such considerations it is clear the necessity of different expansions, which will be described in the next section. From a mathematical point of view it is remarkable that one is attempting at constructing perturbatively, by a suitable expansion in the perturbative parameters, quantities which are not analytic in such parameters (so that a power expansion fails).

*sec.*8

8. The Holstein-Hubbard model: a paradigmatic example

*p.*8.1 **8.1.** *The model.* To fix ideas we study a system of interacting fermions on a lattice subject to a quasi-periodic potential, following the analysis in [M1]. In the physical literature such systems are studied in connection with the so called *quasi-crystals*, see for instance [VMG] and [CS]. Such a case contains all the relevant features (anomalous dimension, dynamical Bogolubov transformations, small divisors problem); we shall see that the results for all the models listed in §13 can be obtained through suitable changes and adaptations of the arguments we explain here in details.

The Hamiltonian of the *Holstein-Hubbard model* is given by

8.1
$$H = H_0 + uP + \lambda V + \nu N_0 \,, \tag{8.1}$$

where H_0 and N_0 are given by (2.1), P by (2.3) and V by (2.5), with $S = 0$ and with $\varphi(x)$ a periodic function with period incommensurate with the lattice step (which is assumed to be 1, see §2.1).

As there is no dependence on the spin we can write simply $\psi^{\pm}_{\mathbf{k},\sigma}$ as $\psi^{\pm}_{\mathbf{k}}$ in (8.1), so that the Hamiltonian

becomes

8.1a
$$\begin{aligned} H = & \sum_{x\in\Lambda}\left[\frac{1}{2}\left(-\psi^+_x\psi^-_{x+1}-\psi^+_x\psi^-_{x-1}+2\psi^+_x\psi^-_x\right)\right]-\mu_0\sum_{x\in\Lambda}\psi^+_x\psi^-_x+u\sum_{x\in\Lambda}\varphi(x)\,\psi^+_x\psi^-_x \\ & +\lambda\sum_{x,y\in\Lambda}v(x-y)\,\psi^+_x\psi^+_y\psi^-_y\psi^-_x+\nu_0\sum_{x\in\Lambda}\psi^+_x\psi^-_x\,. \end{aligned} \tag{8.2}$$

Let us fix $p_F=\bar m p$, with $\bar m\in\mathbb{N}$ and $p=\pi/T$, if T is the period of the potential φ (*i.e.* $\varphi(x+T)=\varphi(x)$ for any x; see (2.4)). Suppose also the function φ to be analytic (in a strip around the real axis).

In the following we assume also the functions φ in (2.4) and v in (2.5) to be even in their arguments: this is not essential, but parity considerations simplify a few aspects of the following analysis.

By the definition of p we can write $\varphi(x)=\bar\varphi(2px)$ with $\bar\varphi$ is a 2π-periodic function and p/π is an irrational number; moreover the Fourier transform of $\bar\varphi$ is exponentially decreasing (*i.e.* $\bar\varphi$ is supposed to be analytic in a strip around the real axis). In order to perform a rigorous analysis one cannot assume that p/π is a generic irrational number, but it has to belong to a class of numbers called *Diophantine* characterized by the following arithmetic properties: there exist two constants C_0 and τ such that, for any integers k,n,

8.2
$$|2np+2k\pi|\ge C_0|n|^{-\tau}\qquad\forall(n,k)\in\mathbb{Z}^2\setminus\{(0,0)\}\,; \tag{8.3}$$

the Diophantine vectors (p,π) are of full measure for $\tau>1$, [Sch]. Note that we can write (8.3) as

8.2a
$$\|2np\|_{\mathbb{T}}\ge C_0|n|^{-\tau}\qquad\forall n\in\mathbb{Z}\setminus\{0\}\,; \tag{8.4}$$

which is satisfied by a full measure set of p's in the real axis.

We can apply the iterative procedure seen in §5.1 by introducing the quasi-particle fields $\psi^{(h)\pm}_\omega$; after integrating the ultraviolet scale and denoting

8.3
$$\begin{aligned} \hat v(\mathbf{k}) &= \delta_{k_0,0}\sum_{x\in\Lambda}v(x)\,e^{-ikx}\,, \\ \hat\varphi_m &= \sum_{x\in\Lambda}\varphi(x)\,e^{-2impx}\,, \end{aligned} \tag{8.5}$$

where, by the analyticity assumption,

8.3a
$$|\hat\varphi_m|\le F_0e^{-\kappa|m|}\qquad\forall m\in\mathbb{Z}\,, \tag{8.6}$$

for suitable positive constants F_0,κ, we obtain

8.3b
$$\begin{aligned} \mathcal{V}^{(0)}(\psi^{(\le0)}) = & \lambda\frac{1}{(L\beta)^4}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_4\in\mathcal{D}_{L,\beta}}\psi^{(\le0)+}_{\mathbf{k}_1}\psi^{(\le0)-}_{\mathbf{k}_2}\psi^{(\le0)+}_{\mathbf{k}_3}\psi^{(\le0)-}_{\mathbf{k}_4}\,\hat v(\mathbf{k}_1-\mathbf{k}_2)\,\delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-\mathbf{k}_4) \\ & +\frac{1}{(L\beta)^4}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_4\in\mathcal{D}_{L,\beta}}\psi^{(\le0)+}_{\mathbf{k}_1}\psi^{(\le0)-}_{\mathbf{k}_2}\psi^{(\le0)+}_{\mathbf{k}_3}\psi^{(\le0)-}_{\mathbf{k}_4}\,W(\mathbf{k}_1,\dots,\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-\mathbf{k}_4) \\ & +\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}(\nu+F(\mathbf{k}))\,\psi^{(\le0)+}_{\mathbf{k}}\psi^{(\le0)-}_{\mathbf{k}} \\ & +u\sum_{m=1}^\infty\hat\varphi_m\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\left(\psi^{(\le0)+}_{\mathbf{k}}\psi^{(\le0)-}_{\mathbf{k}+2m\mathbf{p}}+\psi^{(\le0)+}_{\mathbf{k}}\psi^{(\le0)-}_{\mathbf{k}-2m\mathbf{p}}\right) \\ & +\sum_{n=2}^\infty\sum_{m=1}^\infty\frac{1}{(L\beta)^n}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\mathcal{D}_{L,\beta}}\psi^{(\le0)\sigma_1}_{\mathbf{k}_1}\dots\psi^{(\le0)\sigma_n}_{\mathbf{k}_n}\,\hat W^{(0)}_{n,m}(\mathbf{k}_1,\dots,\mathbf{k}_n)\,\delta\Big(\sum_{i=1}^n\sigma_i\mathbf{k}_i+2m\mathbf{p}\Big)\,, \end{aligned} \tag{8.7}$$

where $\sigma_i = \pm$, $|F(\mathbf{k})| \le C|\lambda|$, $|W(\mathbf{k}_1, \dots, \mathbf{k}_4)| \le C|\lambda|^2$ and the kernels $\hat W^{(0)}_{n,m} \equiv \hat W^{(0)}_{n,m}(\mathbf{k}_1, \dots, \mathbf{k}_n)$ satisfy the conditions:
(1) $\hat W^{(0)}_{n,m} = \hat W^{(0)}_{n,-m}$, if φ and v are even functions;
(2) $|\hat W^{(0)}_{n,m}| \le C^n \varepsilon^{\max(2, n/2-1)}$ if $\varepsilon = \max\{|\lambda|, |u|, |\nu|\}$; moreover $\mathbf{p} = (p, 0)$ and the delta-function $\delta(\mathbf{k}) = L\beta\delta_{k_0,0}\delta_{k,0}$ is defined modulo 2π in k.

Such conditions are easily verified: it is enough to express $\mathcal{V}^{(0)}$ in terms of Feynman diagrams by using the rules given in §3 and to check that the parity properties of the interaction imply the condition (1), while the condition (2) follows from the fact that in order to have a cluster on scale $h = 0$ with n external lines one needs at least $N \ge 2$ points such that $N \ge 2 + (n-3)/2$.

p.8.2 **8.2.** *Effective potentials.* We decompose the fields and their propagators as in §5.1. For each field $\psi^{(\le h)\sigma}_{\mathbf{k},\omega}$ we write

8.3c $$\mathbf{k} = \mathbf{k}' + \omega \mathbf{p}_F \ , \tag{8.8}$$

where $\mathbf{p}_F = (p_F, 0)$, so that $\mathbf{k}' = (k', k_0)$ measures the distance from the Fermi surface (if the field is on scale h then $|\mathbf{k}'| \approx \gamma^h$; see (5.2) for notations).

Then by integrating iteratively the fields as shown in §5.1, one obtains the effective potentials $\mathcal{V}^{(h)}$, which can be written as $\mathcal{V}^{(0)}$ in (8.7). More precisely one can write

8.3d $$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(L\beta)^n} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathcal{D}_{L,\beta}} \psi^{(\le h)\sigma_1}_{\mathbf{k}_1} \dots \psi^{(\le h)\sigma_n}_{\mathbf{k}_n} \hat W^{(h)}_{n,m}(\mathbf{k}_1, \dots, \mathbf{k}_n)\, \delta\big(\sum_{i=1}^{n} \sigma_i \mathbf{k}_i + 2m\mathbf{p}\big) \ . \tag{8.9}$$

We shall call $\hat W^{(h)}_{n,m}(\mathbf{k}_1, \dots, \mathbf{k}_n)$ the *value of the cluster* with external lines $\psi^{(\le h)\sigma_1}_{\mathbf{k}_1}, \dots, \psi^{(\le h)\sigma_n}_{\mathbf{k}_n}$.

If h is not the scale of the root then the cluster is a subcluster of a bigger cluster and some of its external lines ℓ can be contracted on scales $h_\ell < h$.

The power counting argument of the previous section tells us that we have to renormalize all the clusters with two and four external lines. More precisely the bound (6.13) and the definition (5.37) show that we need at least a gain $\gamma^{2(h_v - h_{v'})}$ when $|P_v| = 2$ and a gain $\gamma^{h_v - h_{v'}}$ when $|P_v| = 4$.

However in this case there are infinite many kinds of clusters with two and four external lines (depending on the value of m) and renormalizing all of them would be clearly a problem.

So we shall try to *improve the power counting*: this is a typical phenomenon arising in many fermionic systems studied by RG methods (also in $d = 2$ one has to improve the power counting with a similar trick). The idea in this case is the following one.

LEMMA 1. *Assume that in*

8.3e $$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(L\beta)^n} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathcal{D}_{L,\beta}} \psi^{(\le 0)\sigma_1}_{\mathbf{k}_1} \dots \psi^{(\le 0)\sigma_n}_{\mathbf{k}_n} \hat W^{(0)}_{n,m}(\mathbf{k}_1, \dots, \mathbf{k}_n)\, \delta\big(\sum_{i=1}^{n} \sigma_i \mathbf{k}_i + 2m\mathbf{p}\big) \tag{8.10}$$

one has

8.4 $$\sum_{i=1}^{n} \sigma_i \omega_i p_F + 2mp \bmod 2\pi \neq 0 \ . \tag{8.11}$$

Then the contribution (8.9) to $\mathcal{V}^{(h)}$ is vanishing unless one has

8.5 $$|m| \ge C_1 \left[\frac{\gamma^{-h/\tau}}{n^{1/\tau}} \right] - \bar m n \ , \tag{8.12}$$

if C_1 is a suitable positive constant.

Proof. Consider a contribution to $\mathcal{V}^{(h)}$ as in (8.9), arising from a cluster with n external lines (on scale $\le h$): by the momentum conservation one has

8.5a $$\sum_{i=1}^{n} \sigma_i k_i + 2mp = 0 \ , \tag{8.13}$$

so that

8.5b
$$\sum_{i=1}^{n} \sigma_i k'_i = - \left(\sum_{i=1}^{n} \sigma_i \omega_i p_F + 2mp \right) . \tag{8.14}$$

Using the compact support property of the propagators corresponding to the Grassman fields $\psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i}$ (see (5.2) and (5.11)) and the Diophantine condition (8.4), we can bound

8.6
$$n a_0 \gamma^h \ge \Big\| \sum_{i=1}^{n} \sigma_i k'_i \Big\|_{\mathbb{T}} \ge \Big\| \sum_{i=1}^{n} \sigma_i \omega_i p_F + 2mp \Big\|_{\mathbb{T}} \ge C_0 \left(n\bar{m} + |m| \right)^{-\tau} , \tag{8.15}$$

from which (8.12) follows with $C_1 = (C_0/a_0)^{1/\tau}$. ■

Using a terminology coming from Classical Mechanics (introduced by Eliasson, [E2]), the clusters with two or four external lines for which the above condition (8.11) is not verified are called *resonances* or *resonant clusters*.

Let us denote by N_v the integer number such that, if $\mathbf{k}_i = \mathbf{k}'_i + \omega_i \mathbf{p}_F$ are the momenta of the n^e_v lines entering or exiting the cluster G_v, one has

8.7
$$\sum_{i=1}^{n^e_v} \sigma_i k_i = \sum_{i=1}^{n^e_v} \sigma_i \left(k'_i + \omega_i p_F \right) = 2 N_v p . \tag{8.16}$$

We can define inductively

8.7a
$$N_v = \begin{cases} N_{v_1} + \ldots + N_{v_{s_v}} , & \text{if } v \in V(\tau) \setminus V_{\rm f}(\tau) \text{ and } w' = v \ \forall w \in \{v_1, \ldots, v_{s_v}\} , \\ m_v , & \text{if } v \in V_{\rm f}(\tau) . \end{cases} \tag{8.17}$$

The above equation (8.12) says that, up to the case of resonances, in order to have a cluster with scale h_v one needs N_v to be greater than $\gamma^{-h_{v'}/\tau}$, a big number if h_v is very negative: but it is clear that the larger N_v the smaller the value associated with the cluster. This is obvious if the cluster contains only endpoints, as $|\hat{\varphi}_n| \le F_0 e^{-\kappa |n|}$. The general case will be discussed below.

p.8.3 **8.3.** *Renormalization.* The above lemma says that the clusters such that (8.11) is not satisfied, *i.e.*

8.8
$$\sum_{i=1}^{n} \sigma_i \omega_i p_F + 2mp \bmod 2\pi = 0 , \tag{8.18}$$

are in some sense special as N_v can be small without limit and in such cases there is no power counting improvement exploiting the fact that N_v has to be large and the exponential decay of the factors $|\hat{\varphi}_n|$. Note that (8.18) can be a source of problem only in a few particular cases, depending on G_v and N_v, as only the clusters with two and four external lines have to be renormalized in order to improve the power counting.

As we said we call such contributions resonances. In Classical Mechanics the resonances have only two external lines, ([G1]; see also [GM1]): if $\lambda = 0$ the model is technically similar to the perturbative series for invariant tori.

The renormalization operator $\mathcal{R} = \mathbb{1} - \mathcal{L}$ is a linear operator defined in the following way. [The definitions below should have to be slightly modified for L, β finite, anyway we prefer to ignore such a technical aspect in order not to overwhelm the notations; see [BeM] for a technically more satisfactory discussion.]

• If $n > 4$ then

8.9
$$\mathcal{L} \Bigg\{ \frac{1}{(L\beta)^n} \sum_{\mathbf{k}'_1, \ldots, \mathbf{k}'_n \in \mathcal{D}_{L,\beta}} \left(\prod_{i=1}^{n} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \right) \hat{W}^{(h)}_{n,m} \left(\mathbf{k}'_1 + \omega_1 \mathbf{p}_F, \ldots, \mathbf{k}'_n + \omega_n \mathbf{p}_F \right) \delta\Big(\sum_{i=1}^{n} \sigma_i (\mathbf{k}'_i + \omega_i \mathbf{p}_F) + 2m\mathbf{p} \Big) \Bigg\} = 0 . \tag{8.19}$$

• If $n = 4$ then

$$\begin{aligned}
&\mathcal{L}\Bigg\{ \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \Big(\prod_{i=1}^{4} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \Big) \\
&\qquad \hat{W}^{(h)}_{4,m}\left(\mathbf{k}'_1+\omega_1\mathbf{p}_F,\dots,\mathbf{k}'_4+\omega_4\mathbf{p}_F\right) \delta\big(\sum_{i=1}^{4}\sigma_i(\mathbf{k}'_i+\omega_i\mathbf{p}_F)+2m\mathbf{p}\big)\Bigg\} \\
&= \delta_{(\sigma_1\omega_1+\sigma_2\omega_2+\sigma_3\omega_3+\sigma_4\omega_4)\mathbf{p}_F+2m\mathbf{p},\mathbf{0}} \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \Big(\prod_{i=1}^{4} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \Big) \\
&\qquad \hat{W}^{(h)}_{4,m}\left(\omega_1\mathbf{p}_F,\dots,\omega_4\mathbf{p}_F\right) \delta\big(\sum_{i=1}^{4}\sigma_i\mathbf{k}'_i\big) \ .
\end{aligned} \tag{8.20}$$

8.10

• If $n = 2$ then

$$\begin{aligned}
&\mathcal{L}\Bigg\{ \frac{1}{(L\beta)^2} \sum_{\mathbf{k}'_1,\mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} \Big(\prod_{i=1}^{2} \psi^{(\le 0)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \Big) \\
&\qquad \hat{W}^{(h)}_{2,m}\left(\mathbf{k}'_1+\omega_1\mathbf{p}_F,\mathbf{k}'_2+\omega_2\mathbf{p}_F\right) \delta\big(\sum_{i=1}^{2}\sigma_i(\mathbf{k}'_i+\omega_i\mathbf{p}_F)+2m\mathbf{p}\big)\Bigg\} \\
&= \delta_{(\omega_1-\omega_2)\mathbf{p}_F,\mathbf{0}} \frac{1}{(L\beta)} \sum_{\mathbf{k}'_1,\mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} \Big(\prod_{i=1}^{2} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \Big) \\
&\qquad \Big[W^{(h)}_{2,m}\left(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F\right) + \omega_1 E(k'+\omega_1 p_F)\partial_k \hat{W}^{(h)}_{2,m}\left(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F\right) + k_0 \partial_{k_0} W^{(h)}_{2,m}\left(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F\right) \Big] \ , \\
&+ \delta_{(\omega_1+\omega_2)\mathbf{p}_F,\mathbf{0}} \frac{1}{(L\beta)} \sum_{\mathbf{k}'_1,\mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} \Big(\prod_{i=1}^{2} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i \mathbf{p}_F,\omega_i} \Big) \ \hat{W}^{(h)}_{2,m}\left(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F\right) \ ,
\end{aligned} \tag{8.21}$$

8.11

where

$$E(k'+\omega p_F) = \cos p_F - \cos k = v_0 \omega \sin k' + (1-\cos k')\cos p_F \ , \qquad v_0 = \sin p_F \ , \tag{8.22}$$

8.12

the delta function is always defined modulo 2π in k and and the symbols $\partial_k, \partial_{k_0}$ denote discrete derivatives; see Appendix A2.

Note that the action of the localization operator is nontrivial (*i.e.* different from zero) only for the resonant clusters, *i.e.* for the clusters with two or four external lines such that

$$\sum_{i=1}^{n} \sigma_i \omega_i p_F + 2mp = -\sum_{i=1}^{n} \sigma_i k'_i = 0 \bmod 2\pi \ , \qquad n = 2, 4 \, . \tag{8.23}$$

8.12*a*

By setting

$$\begin{aligned}
\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=2}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(L\beta)^n} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_n \in \mathcal{D}_{L,\beta}} \psi^{(\le h)\sigma_1}_{\mathbf{k}_1} \dots \psi^{(\le h)\sigma_n}_{\mathbf{k}_n} \\
\mathcal{L}\hat{W}^{(0)}_{n,m}(\mathbf{k}_1,\dots,\mathbf{k}_n)\, \delta\big(\sum_{i=1}^{n}\sigma_i\mathbf{k}_i + 2m\mathbf{p}\big) \ ,
\end{aligned} \tag{8.24}$$

8.12*b*

we can write (8.19), (8.20) and (8.21) as

8.12c
$$\begin{aligned}
&\mathcal{L}\hat{W}^{(h)}_{2,m}(\mathbf{k}'_1+\omega_1\mathbf{p}_F,\mathbf{k}'_2+\omega_2\mathbf{p}_F)=\delta_{(\omega_1-\omega_2)\mathbf{p}_F,\mathbf{0}}\Big[\hat{W}^{(h)}_{2,m}(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F)\\
&\qquad+\omega_1 E(k'+\omega_1 p_F)\partial_k\hat{W}^{(h)}_{2,m}(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F)+k_0\partial_{k_0}\hat{W}^{(h)}_{2,m}(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F)\Big]\\
&\qquad+\delta_{(\omega_1+\omega_2)\mathbf{p}_F,\mathbf{0}}\,\hat{W}^{(h)}_{2,m}(\omega_1\mathbf{p}_F,\omega_2\mathbf{p}_F)\ ,\\
&\mathcal{L}\hat{W}^{(h)}_{4,m}(\mathbf{k}'_1+\omega_1\mathbf{p}_F,\dots,\mathbf{k}'_4+\omega_4\mathbf{p}_F)=\delta_{(\sigma_1\omega_1+\sigma_2\omega_2+\sigma_3\omega_3+\sigma_4\omega_4)\mathbf{p}_F+2m\mathbf{p},\mathbf{0}}\hat{W}^{(h)}_{4,m}(\omega_1\mathbf{p}_F,\dots,\omega_4\mathbf{p}_F)\ ,\\
&\mathcal{L}W^{(h)}_{n,m}(\mathbf{k}'_1+\omega_1\mathbf{p}_F,\dots,\mathbf{k}'_n+\omega_n\mathbf{p}_F)=0\ ,\qquad n\ge 6\ .
\end{aligned}\tag{8.25}$$

Note that the r.h.s of (8.20) and (8.21) are vanishing unless (8.18) is verified. The *localization operator* $\mathcal{L}$ is aimed to characterize the resonances, *i.e.* the terms such that $\sum_{i=1}^n\sigma_i\mathbf{k}'_i=0$, with $n=2,4$ (see (8.23)). One can wonder why, for $n=2$, we localize the term with $\omega_1=\omega_2$ at the second order while for the term with $\omega_1=-\omega_2$ only a first order localization is performed: the reason is that the marginal (according to a naïve power counting) terms of the form $k_0\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},-\omega}$ are indeed irrelevant; as we shall see such terms contain a factor $\sigma_h\gamma^{-h}$ and this will improve the power counting, see below.

We can write then $\mathcal{L}\mathcal{V}^{(h)}$ in the following more compact way:

8.12d
$$\mathcal{L}\mathcal{V}^{(h)}(\psi)=\gamma^h n_h F^{(\le h)}_\nu+\gamma^h s_h F^{(\le h)}_\sigma+z_h F^{(\le h)}_\zeta+a_h F^{(\le h)}_\alpha+l_h F^{(\le h)}_\lambda\ ,\tag{8.26}$$

where

8.13
$$\begin{aligned}
F^{(\le h)}_\nu&=\sum_{\omega=\pm1}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\ ,\\
F^{(\le h)}_\sigma&=\sum_{\omega=\pm1}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le h)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\ ,\\
F^{(\le h)}_\alpha&=\sum_{\omega=\pm1}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}E(k'+\omega p_F)\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\ ,\\
F^{(\le h)}_\zeta&=\sum_{\omega=\pm1}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}(-ik_0)\,\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\ ,\\
F^{(\le h)}_\lambda&=\frac{1}{(L\beta)^4}\sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4\in\mathcal{D}_{L,\beta}}\psi^{(\le h)+}_{\mathbf{k}'_1+\mathbf{p}_F,1}\psi^{(\le h)+}_{\mathbf{k}'_1-\mathbf{p}_F,-1}\psi^{(\le 0)-}_{\mathbf{k}'_3+\mathbf{p}_F,1}\psi^{(\le 0)-}_{\mathbf{k}'_4-\mathbf{p}_F,-1}\,\delta\Big(\sum_{i=1}^4\sigma_i\mathbf{k}_i\Big)\ ,
\end{aligned}\tag{8.27}$$

and, for $h=0$,

8.14
$$\begin{cases}
n_0=\nu+O(\lambda^2)\ ,\\
s_0=u\hat{\varphi}_{\bar{m}}+O(u\lambda^2)\ ,\\
a_0=O(\lambda^2)\ ,\\
z_0=O(\lambda^2)\ ,\\
l_0=\lambda\,(\hat{v}(0)-\hat{v}(2p_F))+O(\lambda^2)\ .
\end{cases}\tag{8.28}$$

We call n_h,s_h,a_h,z_h,l_h the *running coupling constants*. As a matter of fact we shall see that the renormalization performed until now will be not enough in order to solve all problems, so that we shall be forced to introduce other running coupling constants and modify the ones defined in (8.23). So the "final" running coupling constants will not be exactly the ones defined so far: this is the reason why we denote them by latin characters, while the final ones will be denoted by greek characters; see (8.64) below.

Let us recall that $V_{\rm f}(\tau)$ denotes the vertices of τ which are endpoints (see §5.1). We can define $V^*_{\rm f}(\tau)\subset V_{\rm f}(\tau)$ the subset of endpoints which no running coupling constants are associated with. Such endpoints will

be all endpoints on scale $h = 2$ associated with the (nonlocalized) contributions to uP, *i.e.* $v \in V_{\rm f}^*(\tau)$ if $h_v = 2$ and $N_v = m_v \neq 0$.

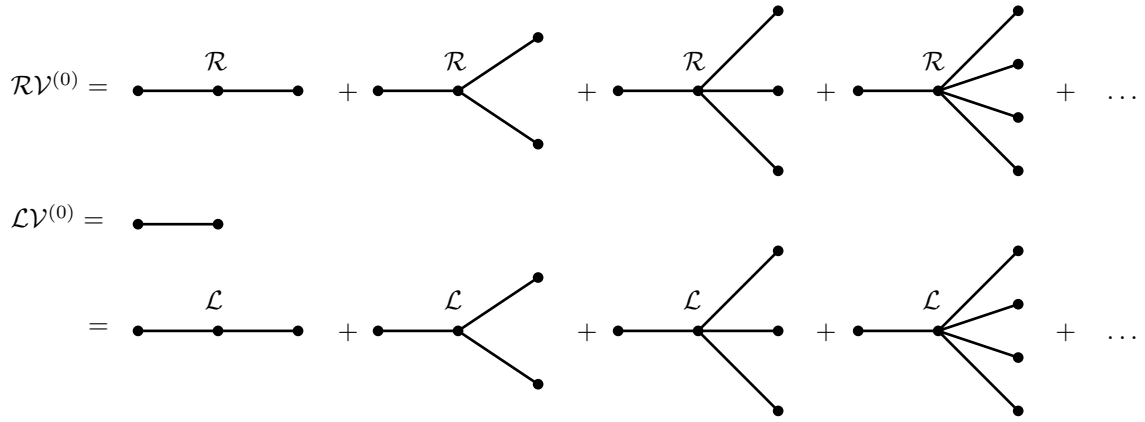
p.8.4 **8.4.** *Renormalized trees.* The iterative integration is done then in the following way.

$$\begin{aligned}\int P(\mathrm{d}\psi)\, e^{\mathcal{V}(\psi)} &= \int P(\mathrm{d}\psi^{(\mathrm{u.v.})}) \int P(\mathrm{d}\psi^{(\mathrm{i.r.})})\, e^{V(\psi^{(\mathrm{u.v.})}+\psi^{(\mathrm{i.r.})})} \\ &= \int P(\mathrm{d}\psi^{(\mathrm{i.r.})})\, e^{V^{(0)}\psi^{(\mathrm{i.r.})})} \\ &= \int P(\mathrm{d}\psi^{(<0)}) \int P(\mathrm{d}\psi^{(0)})\, e^{\mathcal{L}V^{(0)}(\psi^{(\le 0)})+\mathcal{R}V^{(0)}(\psi^{(\le 0)})} \\ &= \int P(\mathrm{d}\psi^{(<-1)}) \int P(\mathrm{d}\psi^{(-1)})\, e^{\mathcal{L}V^{(-1)}(\psi^{(\le -1)})+\mathcal{R}V^{(0)}(\psi^{(\le -1)})} \text{ and so on.}\end{aligned} \tag{8.29}$$

8.15

Of course one can represent this operations in terms of a new kind of trees, which will be called *renormalized trees*, and which can be obtained in the following way.

One writes $\mathcal{V}^{(0)}$ as in Fig. 13: there can be endpoint on scale $h < 2$, representing contributions arising from $\mathcal{L}\mathcal{V}^{(1)}$.



0FIG. 13. Splitting of the effective potential $\mathcal{V}^{(0)}$ as sum of two contributions: the renormalized part $\mathcal{R}\mathcal{V}^{(0)}$ and the localized part $\mathcal{L}\mathcal{V}^{(0)}$.

Then one writes $\mathcal{V}^{(-1)}$ as in Fig. 6, by using the representation in Fig. 13 for $\mathcal{V}^{(0)}$, so obtaining the expansion given in Fig. 14 for $\mathcal{L}\mathcal{V}^{(-1)}$. The same expansion holds also for $\mathcal{R}\mathcal{V}^{(-1)}$, the only difference being that an $\mathcal{R}$ operator is associated also to the first node (compare $\mathcal{L}\mathcal{V}^{(0)}$ and $\mathcal{R}\mathcal{V}^{(0)}$ in Fig. 13).

In conclusion the renormalized trees are given by the same trees as in the previous sections, with the following differences. See Fig. 15.

(1) With each vertex $v \notin V_{\rm f}(\tau)$ an $\mathcal{R}$ operation is associated, up to the first vertex v_0 which can have associated either an $\mathcal{R}$ operation or an $\mathcal{L}$ operation.

(2) There are endpoints v with scale h_v (before each endpoint was at scale $h_v = 2$). If $h_v < 2$ with the endpoint v a contribution $\mathcal{L}V^{(h)}$ is associated, while if $h_v = 2$ either a contribution $\mathcal{L}V^{(0)}$ or a contribution $\mathcal{R}V^{(0)}$ are associated with v. If v is an endpoint and $h_v \le -1$ than $h_v = h_{v'} + 1$ is v' is the nontrivial vertex immediately preceding v. The running coupling constants corresponding to the endpoint v will be denoted by r_v: one has $r_v = \nu_h$ if $h = h_{v'}$ and the contribution $F_\nu^{(\le h)}$ to $\mathcal{L}\mathcal{V}^{(h)}(\psi)$ is considered, and so on.

Of course we can write a Feynman diagram expansion, in which each cluster value is written as $W^{(h)} = (\mathbb{1}-\mathcal{L})W^{(h)} + \mathcal{L}W^{(h)}$ (see (8.23)). We shall see that the bound for $(\mathbb{1}-\mathcal{L})W^{(h)}$ has an extra factor $\gamma^{z_v(h_v - h_{v'})}$

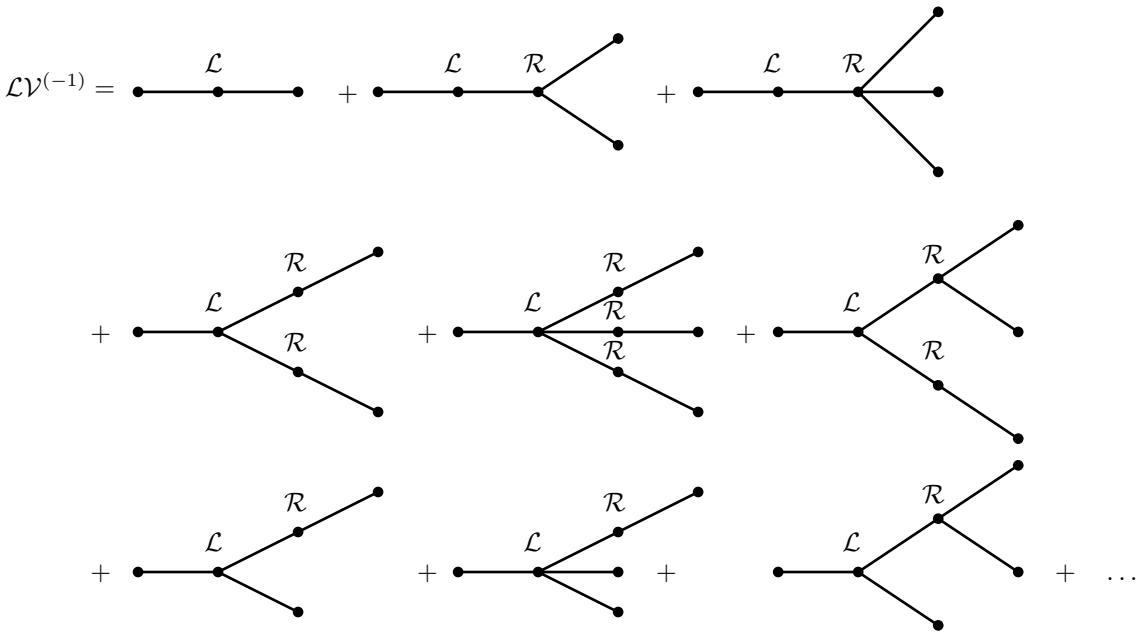


FIG. 14 Graphic representation of the localized effective potential $\mathcal{V}^{(-1)}$.

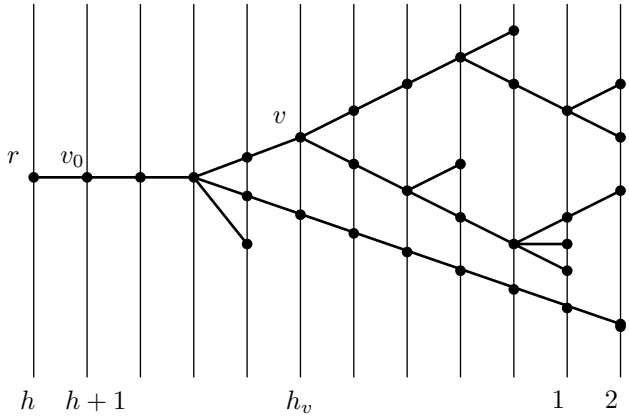


FIG. 15. A renormalized tree appearing in the graphic representation of $\mathcal{RV}^{(h)}$ or $\mathcal{LV}^{(h)}$.

for each $v \in V(\tau)$, with respect to the bound for $W^{(h)}$, for a suitable integer z_v. It will turn out to be $z_v = 1, 2$ for the clusters on which $\mathcal{R}$ acts: such a factor is just what we need in order to perform the sum over the trees, as it converts the exponent in (5.34) from $n_v^e/2 + m_{2,v} - 2$ to $n_v^e/2 + m_{2,v} - 2 + z_v$. Therefore, by taking into account the analysis performed in §5.3 and the value of z_v, the factor $n_v^2/2 + m_{2,v} - 2 + z_v$ becomes positive.

In order to understand how the gain factor $\gamma^{z_v(h_v - h_{v'})}$ arises, we can consider explicitly an example.

Consider a resonant cluster with two external fields: if $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta associated to the external lines of the cluster, one has $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}' + \omega \mathbf{p}_F$, so that we can set $W_{2,0}^{(h)}(\mathbf{k}_1, \mathbf{k}_2) \equiv \Xi^{(h)}(\mathbf{k}')$.

We know from the previous analysis that for such a cluster a second order renormalization is required

if $\omega_1 = \omega_2$, while a first order renormalization is enough if $\omega_1 = -\omega_2$: this should produce a gain factor $\gamma^{z(h_v - h_{v'})}$ where $z = 1, 2$, respectively.

For simplicity we explicitly consider now the case of clusters with only two external lines with $\omega_1 = -\omega_2$: so a first order renormalization is enough in order to obtain a "first order gain" $\gamma^{h_v - h_{v'}}$. This means that, as far as the following heuristic discussion is concerned, we suppose that all the involved clusters on which the action of the renormalization operator is nontrivial are clusters with two external lines and with $\omega_1 = -\omega_2$, *i.e.* such that

8.15a

$$\begin{aligned} \mathcal{L} \Bigg\{ & \frac{1}{(L\beta)^2} \sum_{\mathbf{k}'_1, \mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} \left(\prod_{i=1}^{2} \psi^{(\le 0)\sigma_i}_{\mathbf{k}'_i + \omega_i \mathbf{p}_F, \omega_i} \right) \\ & \hat{W}^{(h)}_{2,m} (\mathbf{k}'_1 + \omega_1 \mathbf{p}_F, \mathbf{k}'_2 + \omega_2 \mathbf{p}_F)\, \delta\big(\sum_{i=1}^{2} \sigma_i (\mathbf{k}'_i + \omega_i \mathbf{p}_F) + 2m\mathbf{p} \big) \Bigg\} \\ & = \delta_{(\omega_1 + \omega_2) p_F, 0} \frac{1}{(L\beta)} \sum_{\mathbf{k}'_1, \mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} \left(\prod_{i=1}^{2} \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i + \omega_i \mathbf{p}_F, \omega_i} \right) \hat{W}^{(h)}_{2,m} (\omega_1 \mathbf{p}_F, \omega_2 \mathbf{p}_F) \ . \end{aligned} \tag{8.30}$$

Then, as the argument is a simply dimensional one, one can easily convince himself that, when needed, a second order normalization produces a "second order gain".

Of course, as we said, the clusters with two external lines and with $\omega_1 = \omega_2$ have to be renormalized twice according to the prescription given in the previous section: anyway the following discussion can be performed for a second order renormalization without any relevant change but from a notational point of view, so that, in order to not make uselessly cumbersome the analysis, we suppose to deal only with a first order renormalization.

Write for a the first order renormalization of the resonant cluster we are considering

8.16

$$\Xi^{(h)}(\mathbf{k}') = \Xi^{(h)}(\mathbf{0}) + \mathbf{k}' \cdot \int_0^1 \mathrm{d}t \partial_{\mathbf{k}'} \Xi^{(h)}(t\mathbf{k}') \ , \tag{8.31}$$

with $\partial_{\mathbf{k}'} = (\partial_{k'}, \partial_{k_0})$: by (8.30) the first term in the right hand side of (8.31) would take into account the localized contribution to the effective potential, while the second term would represent the renormalized contribution.

Recall that $\hat{W}^{(h)}$ is the integral of a product of propagators g_ℓ with scales $\ge h_v$ the derivative in (8.31) produces an extra dimensional factor γ^{-h_v}, while the "zero" $\mathbf{k}'$ produces an extra factor $\gamma^{h_{v'}}$ (by the compact support of the propagator).

There is a technical point that should be stressed. Of course it is possible that there are many clusters inside each other to be renormalized. Suppose that $G_{v_1}, \dots, G_{v_m}$ are clusters to be renormalized, with $v_1 \prec v_2 \prec \dots \prec v_m$: so $G_{v_m} \subset \dots \subset G_{v_1}$. Start by renormalizing G_{v_1}, *i.e.* the most external one: then a derivative can be applied on all the propagators corresponding to the lines inside G_{v_1}. In particular it can be applied to the propagator of a line inside G_{v_m}. Next we renormalize v_2: again the derivative can be applied on all the propagators corresponding to the lines inside G_{v_2}. And so on: after m renormalization steps all the clusters $G_{v_1}, \dots, G_{v_m}$ have been renormalized, but among all the contributions which have been obtained, also terms like $\partial^m_{\mathbf{k}'} g_\ell$, with $\ell \in G_{v_m}$, have been obtained: this, in addition to the rigth dimensional factor, contributes to the bound with a factor $O(m!^\alpha)$, $\alpha \ge 1$ (one would have $\alpha = 1$ if the support function was analytic, and it is even worse for the choise done in §5.1). Therefore the graph value in general can be only bounded $O(n!^\alpha)$.

There are several ways to solve this problem. In the case of exponential (analytic) cut-off function [BGPS] or Gevray class function (nonanalytic and with compact support, but with Fourier transform bounded by $e^{-(\kappa n)^{1/s}}$), [R], one can still bound these extra $O(m!^\alpha)$ terms; see for instance [BGPS] and [DR1].

Another way to see that there is no problem is to show simply that all the propagators are at most derived twice (see [BM1]), essentially by exploiting the (simple) idea that once a gain has been obtained corresponding to some resonance there is no need more to renormalize it (a fact already used in [GS]): to be more precise, the argument is the following. Note since now that in such a way no assumptions on the cut-off function are necessary, except the smoothness one (and in fact one can weaken also such a hypothesis, see [BGGM]).

The argument is very simple is we consider a first order renormalization, as the one we are discussing here, and it can be trivially extended. Consider the cluster G_{v_1} and assume that the derivative is applied just on a propagator inside G_{v_n} and outside $G_{v_{n+1}}$, for some $1 < n < m$; in this way we get a factor $\gamma^{(h_{v'_1}-h_{v_n})}$, which we can rewrite as

8.17
$$\gamma^{(h_{v'_1}-h_{v_n})} = \gamma^{(h_{v'_1}-h_{v_1})}\gamma^{(h_{v_2}-h_{v'_2})}\dots\gamma^{(h_{v'_n}-h_{v_n})} , \tag{8.32}$$

so that each cluster has the factor $\gamma^{h_v-h_{v'}}$ and there is no need of other normalization; and in fact all the other renormalizations are vanishing as

8.18
$$\partial_{\mathbf{k}'}\mathcal{R}\Xi^{(h)}(\mathbf{k}') = \partial_{\mathbf{k}'}\Xi^{(h)}(\mathbf{k}') , \tag{8.33}$$

which means that there are no renormalizations acting on the clusters encluded between G_{v_1} and G_{v_n}.

The above analysis is performed in Fourier space and skips the problem of implementing the Gram-Hadamard inequality in order to control the number of terms arising from the perturbative expansion. On the other hand, as we shall see better in Appendix A3, the Gram-Hadamard inequality is applied in the coordinate space. The renormalization procedure gives rise to factors $\mathbf{k}'$ (see (8.31)) which, in the coordinate space, correspond to derivatives of fields, hence to derivatives of propagators once such lines are contracted. This creates a series of intricacies and technical problems, for the discussion and solution of which we refer to the original papers: see [BGPS], [M1] and [BeM].

p.8.5 **8.5.** *Renormalized bounds.* Proceding as in §6 we get for the renormalized expansion

8.19
$$\begin{aligned}\int d\mathbf{x}(P_{v_0}) \left|\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)\right| \le C^n \gamma^{-h\left[D(P_{v_0})+z_{v_0}(N_{v_0},P_{v_0})\right]} \\ \left(\prod_{v\notin V_{\rm f}(\tau)} \gamma^{-[D(P_v)+z_v(N_v,P_v)](h_v-h_{v'})}\right)\left(\prod_{v\in V_{\rm f}(\tau)\setminus V^*_{\rm f}(\tau)} |r_v|\right) \\ \left(\prod_{v\in V_{\rm f}(\tau)} \gamma^{-h_{v'}m_{2,v}}\right)\left(\prod_{v\in V^*_{\rm f}(\tau)} |\hat\varphi_{m_v}|\right) ,\end{aligned} \tag{8.34}$$

where $V^*_{\rm f}(\tau)$ is the set of endpoints such that no running coupling constant is associated to them (see the end of §8.3), and $m_{2,v}$ is defined in (5.36), while
(1) $z_v(N_v,P_v) = 1$ if G_v has four external lines ($|P_v| = 4$) and it is a resonance, *i.e.* $\sum_{i=1}^4 \sigma_i\omega_i\mathbf{p}_F + 2N_v\mathbf{p} = \mathbf{0}$,
(2) $z_v(N_v,P_v) = 2$ if G_v has two external lines ($|P_v| = 2$) and it is a resonance, *i.e.* $(\omega_1-\omega_2)\mathbf{p}_F + 2N_v\mathbf{p} = \mathbf{0}$, such that $\omega_1 = \omega_2$,
(3) $z_v(N_v,P_v) = 1$ if G_v has two external lines ($|P_v| = 2$ and it is a resonance, *i.e.* $(\omega_1-\omega_2)\mathbf{p}_F + 2N_v\mathbf{p} = \mathbf{0}$, such that $\omega_1 = -\omega_2$.

Note that now the endpoints v can have also a scale $h_v < 2$, so that we cannot set $\gamma^{-h_{v'}} = \gamma^{-1}$ in the last line of (8.34).

The bound (8.34) is obtained by using the Gram-Hadamard inequality like in §6: the presence of the renormalization makes a little involved the construction, as also derived fields have to be considered in the space-time coordinates (in which the inequality can be applied). However the bounds (8.34) obtained for

the renormalized expansion is not yet sufficient for proving nondiverging bounds when the sum over trees is performed for a number of reasons.

(1) The factor $D(P_v) + z_v(N_v, P_v)$ can be still equal to -1 or 0, in correspondence of non-resonant clusters with two and four external lines; we have to extract from

8.20
$$\prod_{v \in V_f^*(\tau)} |\hat{\varphi}_{m_v}| \tag{8.35}$$

some good factor by using the lemma in §8.2.

(2) Also for resonances with two external lines such that $\omega_1 = -\omega_2$, by definition of $\mathcal{R}$, one can have $D(P_v) + z_v(N_v, P_v) = 0$, as $z_v(N_v, P_v) = 1$ in such a case.

(3) There are two relevant running coupling costants, namely $\gamma^h n_h$ and $\gamma^h s_h$. We deduce from the above discussion that it is necessary to put a factor γ^h in front of them (*i.e.* to assume that they are decreasing at least as γ^h) to have a renormalizable power counting: in fact each endpoint v with $m_{2,v} = 2$ carries a factor $\gamma^{-h_{v'}}$, which we choose to delete by putting a factor $\gamma^{h_{v'}}$ in front of the corresponding running coupling constant themselves: of course such an operation is meaningful only if after one can prove that n_h and s_h remain bounded. While there is a counterterm ν in the Hamiltonian (8.1) which can be fixed (hopefully) in order that this can be really done, this is not the case for s_h. We shall see in the next section that, while n_h is related to the shift of the singularity of the interacting two-point Schwinger function, s_h is due to the effect of the opening of a gap in the spectrum; because of such a term the propagator becomes essentially "of the form" $kf_h/(k^2 + \sigma_h^2)$, for some constant σ_h, so that its expansion in terms of σ_h gives an expression "of the form"

8.21
$$\frac{1}{k} \sum_{n=0}^{\infty} \left(-\frac{\sigma_h^2}{k^2} \right)^n , \tag{8.36}$$

which would be convergent only if $\sigma_h \simeq \gamma^h s_h$, with s_h bounded, since $k \simeq \gamma^h$. It is clear that by a Bogolubov transformation (see [ADG]) we can put the gap term in the fermionic integration: however, see below, as the true gap is not of order $O(u)$, but of order $O(u^{1-\lambda+\cdots})$, many Bogolubov transformations are necessary, one for each scale, as the gap itself has a nontrivial flow.

(4) Finally z_h, α_h are not bounded uniformly in h. In fact one can write the flow to second order as

8.22
$$\begin{aligned} l_{h-1} &= l_h , \\ a_{h-1} &= a_h + \beta_1 \lambda_h^2 , \\ z_{h-1} &= z_h + \beta_1 \lambda_h^2 , \end{aligned} \tag{8.37}$$

where β_1 is a constant, so that one obtains $a_h = z_h = O(\lambda^2 h)$.

We shall see that the arising logarithmic divergence is due to the "infinite" wave function renormalization: if $u = 0$ the large distance behaviour of the two-point Schwinger function is not $|\mathbf{x}|^{-1}$, but $|\mathbf{x}|^{-(1+\lambda^2+\ldots)}$; see §10, §12 and §13.

The above considerations show the necessity of a new, *anomalous* expansion.

p.8.6 **8.6.** *Anomalous integration.* The integration is performed iteratively: at each step $h \le 0$ the Grassman integration measure is changed by using the results in §2.1 and the fields are rescaled by a suitable factor. The change of the integration measure can be interpreted as a shift of some terms contributing to the effective potential into the integration measure.

Practically one proceeds by introducing a sequence of constants Z_h, with $h \le 0$, and $Z_0 = 1$, in the following way.

Define

8.23a
$$C_h(\mathbf{k}')^{-1} = \sum_{j=h_\beta}^{h} f_j(\mathbf{k}') \;, \tag{8.38}$$

where h_β is given by (5.12).

Once that the fields $\psi^{(0)}, \dots, \psi^{(h+1)}$ have been integrated we have

8.24
$$\int P_{Z_h}(\mathrm{d}\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \;, \tag{8.39}$$

where, up to a constant,

8.25
$$\begin{aligned} P_{Z_h}(\mathrm{d}\psi^{(\le h)}) = &\left(\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \prod_{\omega=\pm 1} \mathrm{d}\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \mathrm{d}\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \right) \\ &\exp\Big\{ -\frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} \sum_{\omega=\pm 1} C_h(\mathbf{k}') Z_h \Big[\Big(-ik_0 + (1-\cos k')\cos p_F + \omega v_0 \sin k' \Big) \\ &\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} + \sigma_h(\mathbf{k}')\, \psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega} \Big] \Big\} \;, \end{aligned} \tag{8.40}$$

where $\sigma_h(\mathbf{k}')$ is defined iteratively (see (8.43) below).

As before it is convenient to split $\mathcal{V}^{(h)}$ as sum of two summands $\mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, with $\mathcal{R} = \mathbb{1} - \mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is the operator defined in the previous section.

We write, if $\mathcal{N}_h$ is a constant

8.26
$$\int P_{Z_h}(\mathrm{d}\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} = \frac{1}{\mathcal{N}_h} \int \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(\le h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{\le h})} \;, \tag{8.41}$$

where

8.27
$$\begin{aligned} \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(\le h)}) = &\left(\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \prod_{\omega=\pm 1} \mathrm{d}\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \mathrm{d}\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \right) \\ &\exp\Big\{ -\frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} \sum_{\omega=\pm 1} C_h(\mathbf{k}') Z_{h-1}(\mathbf{k}') \Big[\Big(-ik_0 + (1-\cos k')\cos p_F + \omega v_0 \sin k' \Big) \\ &\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} + \sigma_{h-1}(\mathbf{k}')\, \psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega} \Big] \Big\} \;, \end{aligned} \tag{8.42}$$

with

8.28
$$\begin{aligned} Z_{h-1}(\mathbf{k}') &= Z_h \left(1 + C_h^{-1}(\mathbf{k}') z_h\right) \;, \\ Z_{h-1} &= Z_h \left(1 + z_h\right) \;, \\ Z_{h-1}(\mathbf{k}')\sigma_{h-1}(\mathbf{k}') &= \begin{cases} Z_h \left(\sigma_h(\mathbf{k}') + C_h^{-1}(\mathbf{k}') s_h\right) \;, & \text{if } h < 0 \;, \\ C_0^{-1}(\mathbf{k}') s_0 \;, & \text{if } h = 0 \;, \end{cases} \\ \tilde{\mathcal{V}}^{(h)} &= \mathcal{L}\tilde{\mathcal{V}}^{(h)} + (\mathbb{1} - \mathcal{L})\, \mathcal{V}^{(h)} \;, \\ \mathcal{L}\tilde{\mathcal{V}}^{(h)}(\psi) &= \gamma^h n_h F_\nu^{(\le h)} + (a_h - z_h)\, F_\alpha^{(\le h)} + l_h F_\lambda^{(\le h)} \;. \end{aligned} \tag{8.43}$$

Note that the functions $Z_h(\mathbf{k}')$ and $\sigma_h(\mathbf{k}')$ are defined iteratively for $h \le 0$ by (8.43) itself (for a better understanding of the integration procedure one can work out explicitly the first scales $h = 0, -1, \dots$). In particular one has

8.28a
$$\sigma_h(\mathbf{k}') = \sum_{h'=h}^{0} C_{h'}^{-1}(\mathbf{k}') s_{h'} \;, \tag{8.44}$$

so that, if $\mathbf{k}'$ is such that $C_h^{-1}(\mathbf{k}') \neq 0$ (*i.e.* $|\mathbf{k}'| \leq t_0\gamma^{h+1}$), one has

8.28b
$$\sigma_h(\mathbf{k}') = C_h^{-1}(\mathbf{k}')s_h + \sum_{h'=h+1}^{0} s_j \ , \tag{8.45}$$

as $C_{h'}^{-1}(\mathbf{k}') = 1$ for $h' \geq h+1$ for such $\mathbf{k}'$. Therefore $\sigma_h(\mathbf{k}')$ is a smooth function on $\mathbb{T}\times\mathbb{R}$. We define

8.28c
$$\sigma_h = \sum_{h'=1}^{0} s_{h'} \ . \tag{8.46}$$

The right hand side of (8.41) can be written as

8.29
$$\frac{1}{\mathcal{N}_h}\int P_{Z_{h-1}}(\mathrm{d}\psi^{(\leq h-1)})\int \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \ , \tag{8.47}$$

where $P_{Z_{h-1}}(\mathrm{d}\psi^{(\leq h-1)})$ is given by (8.42) with
(1) $Z_{h-1}(\mathbf{k}')$ replaced by Z_{h-1},
(2) $C_h(\mathbf{k}')$ replaced with $C_{h-1}(\mathbf{k}')$,
(3) $\psi^{(\leq h)}$ replaced with $\psi^{(\leq h-1)}$,
while $\tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})$ is given by (8.42) with
(1) $Z_{h-1}(\mathbf{k}')$ replaced by Z_{h-1},
(2) $C_h(\mathbf{k}')$ replaced with $\tilde{f}_h^{-1}(\mathbf{k}')$, if

8.30
$$\tilde{f}_h(\mathbf{k}') = Z_{h-1}\left(\frac{C_h^{-1}(\mathbf{k}')}{Z_{h-1}(\mathbf{k}')} - \frac{C_{h-1}^{-1}(\mathbf{k}')}{Z_{h-1}}\right) \ , \tag{8.48}$$

(3) $\psi^{(\leq h)}$ replaced with $\psi^{(h)}$.

This can be esily proven by using the addition principle and the change of integration for fermionic integrations discussed in §4.

Note also that $\tilde{f}_h(\mathbf{k}')$ is a compact support function, with support of width $O(\gamma^h)$ and far $O(\gamma^h)$ from the "singularity", *i.e.* from $\mathbf{p}_F$.

The Grassman integration $\tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})$ has propagator given by

8.31
$$\frac{g^{(h)}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} = \sum_{\omega,\omega'=\pm 1} e^{-i(\omega x-\omega' y)p_F}\,\frac{g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} \ , \tag{8.49}$$

with

8.32
$$\frac{g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} = \int \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})\,\psi^{(h)-}_{\mathbf{x},\omega}\psi^{(h)+}_{\mathbf{y},\omega'} \tag{8.50}$$

such that

8.32a
$$g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\tilde{f}_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \ , \tag{8.51}$$

where the 2×2 matrix $T_h(\mathbf{k}')$ has elements

8.33
$$\begin{cases} [T_h(\mathbf{k}')]_{1,1} = (-ik_0 + (1-\cos k')\cos p_F + v_0\sin k') \ , \\ [T_h(\mathbf{k}')]_{1,2} = [T_h(\mathbf{k}')]_{2,1} = \sigma_{h-1}(\mathbf{k}') \ , \\ [T_h(\mathbf{k}')]_{2,2} = (-ik_0 + (1-\cos k')\cos p_F - v_0\sin k') \ , \end{cases} \tag{8.52}$$

which is well defined on the support of $\tilde{f}_h(\mathbf{k}')$, so that, if we set

8.34
$$A_h(\mathbf{k}') = \det T_h(\mathbf{k}') = [-ik_0 + (1-\cos k')\cos p_F]^2 - (v_0 \sin k')^2 - [\sigma_{h-1}(\mathbf{k}')]^2 \ , \tag{8.53}$$

then

8.35
$$T_h^{-1}(\mathbf{k}') = \frac{1}{A_h(\mathbf{k}')} \begin{pmatrix} [\tau_h(\mathbf{k}')]_{1,1} & [\tau_h(\mathbf{k}')]_{1,2} \\ [\tau_h(\mathbf{k}')]_{2,1} & [\tau_h(\mathbf{k}')]_{2,2} \end{pmatrix} \ , \tag{8.54}$$

with

8.36
$$\begin{cases} [\tau_h(\mathbf{k}')]_{1,1} = [-ik_0 + (1-\cos k')\cos p_F - v_0 \sin k'] \ , \\ [\tau_h(\mathbf{k}')]_{1,2} = [\tau_h(\mathbf{k}')]_{2,1} = -\sigma_{h-1}(\mathbf{k}') \ , \\ [\tau_h(\mathbf{k}')]_{2,2} = [-ik_0 + (1-\cos k')\cos p_F + v_0 \sin k'] \ . \end{cases} \tag{8.55}$$

Note there exist two positive constants c_1, c_2 such that

8.37
$$c_1 \sigma_h \le \sigma_h(\mathbf{k}') \le c_2 \sigma_h \ , \qquad \sigma_h \equiv \sum_{h'=h}^{0} s_{h'} \ , \tag{8.56}$$

where the definition (8.46) has been used.

The large distance behaviour of the propagator (8.48) is given by the following lemma (which can be proven by reasoning as for proving (5.27) in Appendix A4).

LEMMA 2. *The propagator $g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ in (8.51) can be bounded as follows. For $\omega = \omega'$ one has*

8.38
$$g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) = g^{(h)}_{0;\omega}(\mathbf{x}-\mathbf{y}) + C^{(h)}_{1,\omega}(\mathbf{x}-\mathbf{y}) + C^{(h)}_{2,\omega}(\mathbf{x}-\mathbf{y}) \ , \tag{8.57}$$

where

8.39
$$g^{(h)}_{0;\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{\tilde{f}_h(\mathbf{k}')}{-ik_0 + \omega v_0 k'} \ , \tag{8.58}$$

$C^{(1)}_{1,\omega}(\mathbf{x}-\mathbf{y})$ is independent on $\sigma_h(\mathbf{k}')$ and $C^{(h)}_{1,\omega}$ and $C^{(h)}_{2,\omega}$ are such that, for any integer $N > 1$ and for $|x-y| \le L/2$, $|x_0 - y_0| \le \beta/2$, one has

8.40
$$\left| C^{(h)}_{1,\omega}(\mathbf{x}-\mathbf{y}) \right| \le \frac{\gamma^{2h} C_N}{1 + (\gamma^h(\mathbf{x}-\mathbf{y}))^N} \ , \qquad \left| C^{(h)}_{2,\omega}(\mathbf{x}-\mathbf{y}) \right| \le \left(\frac{|\sigma_h|}{\gamma^h} \right)^2 \frac{\gamma^h C_N}{1 + (\gamma^h(\mathbf{x}-\mathbf{y}))^N} \ , \tag{8.59}$$

for a suitable constant C_N. For $\omega = -\omega'$ one has

8.41
$$\left| g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) \right| \le \frac{|\sigma_h|}{\gamma^h} \frac{\gamma^h C_N}{1 + (\gamma^h(\mathbf{x}-\mathbf{y}))^N} \ , \tag{8.60}$$

where C_N can be chosen the same constant as in (8.59).

Note that $g^{(h)}_{0;\omega}(\mathbf{x}-\mathbf{y})$ coincides with the propagator "at scale γ^h" of the Luttinger model, [L] (this remark will be crucial for studying the Renormalization Group flow): it admits the bound

8.41a
$$\left| g^{(h)}_{0;\omega}(\mathbf{x}-\mathbf{y}) \right| \le \frac{\gamma^h C_N}{1 + (\gamma^h(\mathbf{x}-\mathbf{y}))^N} \ , \tag{8.61}$$

so that we see that, we respect to $g^{(h)}_{0;\omega}$, the propagtaors $C^{(h)}_{1,\omega}$, $C^{(h)}_{2,\omega}$ and $g^{(h)}_{\omega,-\omega}$ have some extra good factors, which are, respectively, γ^h, $(|\sigma_h|/\gamma^h)^2$ and $|\sigma_h|/\gamma^h$.

We *rescale* the fields by rewriting (8.47) as

8.42
$$\frac{1}{\mathcal{N}_h}\int P_{Z_{h-1}}(\mathrm{d}\psi^{(\le h-1)})\int \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}\ , \tag{8.62}$$

so that

8.43
$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(\le h)} + \delta_h F_\alpha^{(\le h)} + \lambda_h F_\lambda^{(\le h)}\ , \tag{8.63}$$

where, by definition,

8.44
$$\begin{aligned} \nu_h &= \frac{Z_h}{Z_{h-1}} n_h\ , \\ \delta_h &= \frac{Z_h}{Z_{h-1}}\,(a_h - z_h)\ , \\ \lambda_h &= \left(\frac{Z_h}{Z_{h-1}}\right)^2 l_h\ . \end{aligned} \tag{8.64}$$

We call the set $\vec{v}_h = (\nu_h, \delta_h, \lambda_h)$ the *running coupling constants*. They will be the true running coupling constants of the model and replace the ones defined through (8.28).

We perform the integration

8.45
$$\int \tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} = e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})+\tilde{E}_h}\ , \tag{8.65}$$

where $\tilde{E}_h$ is a suitable constant and

8.46
$$\begin{aligned} \mathcal{L}\mathcal{V}^{(h-1)}(\psi^{(\le h-1)}) = {}& \gamma^{h-1} n_{h-1} F_\nu^{(\le h-1)} + s_{h-1} F_\sigma^{(\le h-1)} \\ & + a_{h-1} F_\alpha^{(\le h-1)} + z_{h-1} F_\zeta^{(\le h-1)} + l_{h-1} F_\lambda^{(\le h-1)}\ . \end{aligned} \tag{8.66}$$

Note that the above procedure allows us to write the running coupling constants $\vec{v}_h$ in terms of $\vec{v}_{h'}$, $h' \ge h+1$,

8.47
$$\vec{v}_h = \vec{\beta}\,(\vec{v}_{h+1}, \dots, \vec{v}_0)\ ; \tag{8.67}$$

the function $\vec{\beta}(\vec{v}_{h+1}, \dots, \vec{v}_0)$ is called the *beta function*.

Recall that, if no renormalization is performed, the effective potential $\mathcal{V}^{(h)}(\psi)$ is a sum of terms of the form

8.48a
$$\frac{1}{(L\beta)^n}\sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_n\in\mathcal{D}_{L,\beta}}\left(\prod_{i=1}^n \psi^{(\le h)\sigma_i}_{\mathbf{k}'_i+\omega_i\mathbf{p}_F,\omega_i}\right)\hat{W}^{(h)}_{n,m}(\mathbf{k}_1,\dots,\mathbf{k}_n)\,\delta\Big(\sum_{i=1}^n \sigma_i(\mathbf{k}'_i+\omega_i\mathbf{p}_F)+2m\mathbf{p}\Big)\ ; \tag{8.68}$$

see (8.8). The renormalization procedure described above produced a new sequence of (renormalized) effective potentials which are the form (8.68) with the fields $\psi^{(\le h)}$ replaced with $\sqrt{Z_h}\psi^{(\le h)}$ and the kernels $\hat{W}^{(h)}_{n,m}(\mathbf{k}_1,\dots,\mathbf{k}_n)$ computed with the new rules: we shall call them the *renormalized values of the clusters*.

The effective potentials can be written as

8.48b
$$\begin{aligned} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) &= \sum_{n=1}^\infty \sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)})\ , \\ \mathcal{V}^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}) &= \int \mathrm{d}\mathbf{x}(I_{v_0}) \sum_{P_{v_0}\subset I_{v_0}} \sqrt{Z_h}^{|P_{v_0}|}\,\tilde{\psi}^{(\le h)}(P_{v_0})\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(I_{v_0})\right)\ . \end{aligned} \tag{8.69}$$

Here the kernels

8.48c
$$\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right) = \int \mathrm{d}\mathbf{x}(I_{v_0}\setminus P_{v_0})\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(I_{v_0})\right) \tag{8.70}$$

are the functions of which the renormalized values $\hat{W}^{(h)}_{n,m}(k_1,\dots,kk_n)$ mentioned above represent the Fourier transforms.

Define

8.48
$$h^* = \inf\{h \geq h_\beta : a_0\gamma^{h+1} \geq 2|\sigma_h|\} \; . \tag{8.71}$$

We shall prove in the following that the running coupling constants σ_h remain bounded from below (uniformly in β): as γ^{h+1} can be arbitrarily small for $\beta \to \infty$ and h small enough, the definition (8.71) of h^* makes sense.

By the previous lemma one immediately gets the following result.

LEMMA 3. *For $h > h^*$ and for any integer $N > 1$, it is possible to find a constant C_N such that*

8.49
$$\left| g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) \right| \leq \frac{C_N \gamma^h}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^N} \; , \tag{8.72}$$

for $|x-y| \leq L/2$ and $|x_0-y_0| \leq \beta/2$.

We shall see that, using the above lemmata and assuming that the running coupling constants are bounded (assumption which will be checked to hold *a posteriori*), the integrations $\tilde{P}_{Z_{h-1}}(\mathrm{d}\psi^{(\leq h)})$ are well defined for $0 \geq h > h^*$.

The integration of the scale from h^* to h_β can be performed "in a single step" as

8.50
$$\int P_{Z_{h^*}}(\mathrm{d}\psi^{(\leq h^*)}) e^{-\mathcal{V}^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)})} = \frac{1}{\mathcal{N}_{h^*}} \int \tilde{P}_{Z_{h^*-1}}(\mathrm{d}\psi^{(\leq h^*)}) e^{-\tilde{\mathcal{V}}^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)})} \; , \tag{8.73}$$

where the integration measure $\tilde{P}_{Z_{h^*-1}}(\mathrm{d}\psi^{(\leq h^*)})$ is defined by the propagator

8.51
$$\int \tilde{P}_{Z_{h^*-1}}(\mathrm{d}\psi^{(\leq h^*)})], \psi^{-(\leq h^*)}_{\mathbf{x}} \psi^{+(\leq h^*)}_{\mathbf{y}} \equiv \frac{g^{(\leq h^*)}(\mathbf{x}-\mathbf{y})}{Z_{h^*-1}} \; . \tag{8.74}$$

The integration in the r.h.s. in (8.73) is well defined, as it follows from the following bound.

LEMMA 4. *Assume that h^* is finite uniformly in L, β, so that $|\sigma_{h^*-1}|/\gamma^{h^*} \geq \bar{\kappa}$, for a suitable constant $\bar{\kappa}$. Then for any integer N it is possible to find a constant C_N such that one has*

8.52
$$\left| g^{(\leq h^*)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) \right| \leq \frac{C_N \gamma^{h^*}}{1+(\gamma^{h^*}|\mathbf{x}-\mathbf{y}|)^N} \; , \tag{8.75}$$

for $|x-y| \leq L/2$ and $|x_0-y_0| \leq \beta/2$.

Comparing the previous lemmata, we see that the propagator of the integration of all the scale between h^* and h_β has the same bound as the propagator of the integration of a single scale greater than h^*: this will be used to perform the integration of the scales $\leq h^*$ altogether.

In fact γ^{h^*} is a momentum scale and, roughly speaking, for momenta larger than γ^{h^*} the theory is "essentially" a massless theory (up to corrections $O(\sigma_h\gamma^{-h})$), while for momenta smaller than γ^{h^*}, it is a "massive" theory with mass $O(\gamma^{h^*})$.

By the lemma in §8.2 we see that it is possible to have quartic or bilinear contribution to $\mathcal{V}^{(h)}$, if $|h|$ is large enough, such that (8.18) with $n = 2, 4$ is *not* satisfied only with an extremely large $N_v \equiv m$, namely $|m| \geq C\gamma^{-h/\tau}$, for some constant C. In order to show that such terms are irrelevant, we shall have to use the fact that $|\hat{\varphi}_m| \leq Ce^{-\kappa|m|}$, for some constants C and κ; see below.

p.8.7 **8.7.** *Bounds for the renormalized expansion.* We want to prove the following result.

THEOREM 1. *Let $h > h^*$, with h^* defined by (8.71). If, for some constants c_1, one has*

8.100
$$\sup_{h'>h} |\vec{v}_{h'}| \equiv \varepsilon_h \quad \sup_{h'>h} \left| \frac{Z_{h'}}{Z_{h'-1}} \right| \leq e^{c_1\varepsilon_h^2}, \quad \sup_{h'>h} \left| \frac{\sigma_{h'}}{\sigma_{h'-1}} \right| \leq e^{c_1\varepsilon_h}, \tag{8.76}$$

and if there exists a constant $\bar\varepsilon$ (depending on c_1) such that $\varepsilon_h \le \bar\varepsilon$, then, for a suitable constant c_0, independent of c_1, as well as of u, L and β, one has

8.101
$$\sum_{\tau\in\mathcal{T}_{h,n}} [|n_h(\tau)| + |z_h(\tau)| + |a_h(\tau)| + |l_h(\tau)|] \le (c_0\varepsilon_h)^n \ , \tag{8.77}$$

8.102
$$\sum_{\tau\in\mathcal{T}_{h,n}} |s_h(\tau)| \le |\sigma_h|(c_0\varepsilon_h)^n \ , \tag{8.78}$$

8.103
$$\sum_{\tau\in\mathcal{T}_{h,n}} |\tilde E_{h+1}(\tau)| \le \gamma^{2h}(c_0\varepsilon_h)^n \ , \tag{8.79}$$

and, setting $D(P_v) = (n^e_v/2) - 2 + z_v(N_v, P_v) + \tilde z_v(N_v, P_v)/2$, with $\tilde z_v = 1$ if $|P_v| = 2$ and $\omega_1 = -\omega_2$ and 0 otherwise

8.104
$$\sum_{\tau\in\mathcal{T}_{h,n}} \int d\mathbf{x}(P_{v_0}) \left|\mathcal{R}\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)\right| \le L\beta\gamma^{-D(P_{v_0})h}(c_0\varepsilon_h)^n \ , \tag{8.80}$$

where (8.80) corresponds to trees such that a $\mathcal{R}$ operation is associated to the first vertex, (8.77) and (8.78) correspond to trees such that a $\mathcal{L}$ operation is associated to the first vertex, and (8.79) represents a constant (i.e. field-independent) contribution to the effective potential.

Let us recall the discussion in §8.5 and let us consider first the case of a cluster G_v with two external lines such that $\hat W^{(h)}_{2,m}(\mathbf{k}'_1+\omega_1\mathbf{p}_F, \mathbf{k}'_2+\omega_2\mathbf{p}_F)$ has $\omega_1 = -\omega_2$. Contributions to the effective potential corresponding to such clusters will be generated by at least a nondiagonal propagator.

Therefore, if for some term contributing to the effective potential on scale h, there is a nondiagonal propagator on scale h', then one has an extra term $|\sigma_{h'}\gamma^{-h'}|$ with respect to the bound holding in absence of nondiagonal propagators (compare (8.60) with (8.61)). If G_v is a cluster containing the line corresponding to such a propagator one has $h' \ge h_v$: the scale h' will be the scale of the minimal cluster containing the line such that the nondiagonal propagator is associated with it.

We shall prove in the following that

8.52*a*
$$\left|\frac{\sigma_h}{\sigma_{h-1}}\right| \le e^{c\varepsilon} \ , \tag{8.81}$$

for some constant c, so that

8.52*b*
$$\left|\frac{\sigma_{h_v}}{\sigma_{h^*}}\right| \le e^{c\varepsilon(h_v-h^*)} \ . \tag{8.82}$$

Then one has

8.53
$$|\sigma_{h_v}\gamma^{-h_v}| \le \left|\frac{\sigma_{h_v}}{\sigma_{h^*}}\right| \left|\frac{\sigma_{h^*}}{\gamma^{h_v}}\right| \le \gamma^{(h-h_v)(1-O(\varepsilon))} \le \gamma^{(h-h_v)/2} \ , \tag{8.83}$$

as $|\sigma_{h^*}| \le C\gamma^{h_*} \le C\gamma^h$ (for a suitable constant C), by (8.71).

Let us consider vertices $v_1, v_2, .., v_m$ ordered so that $v_1 \prec v_2 \prec \ldots \prec v_m$, with $v_{i-1} = v'_i$ for $i = 2, \ldots, m$ and v'_1 is the root. Suppose that G_{v_1} is a cluster containing a nondiagonal propagator on scale $h' = h_v$. Then one has

8.54
$$\left|\sigma_{h_{v_1}}\gamma^{-h_{v_1}}\right| \le \prod_{i=1}^m \gamma^{(h_{v_i}-h_{v'_i})/2} \ . \tag{8.84}$$

This means that we can associate a factor $\gamma^{(h_{v'}-h_v)/2}$ to each cluster containing a cluster G_v with two external lines such that $\omega_1 = -\omega_2$. Note also that as the values of such cluster G_v is marginal in a naïve power counting analysis (and $z_v = 1$), the corresponding gain factor is enough to ensure the convergence, as far the value of G_v is concerned.

It remains to improve the power counting of the nonresonant clusters with two or four external lines (*i.e.* with $|P_v| = 2$ or $|P_v| = 4$) which are such that

8.56
$$\sum_{f \in P_v} \sigma(f)\omega(f)p_F + 2N_v p \neq 0 \; . \tag{8.85}$$

We have a bound analogous to (8.34) namely, if $\rho_v \in \{\delta_{h_v}, \nu_{h_v}, \lambda_{h_v}\}$

8.54e
$$\begin{aligned} \int d\mathbf{x}(P_{v_0}) \left| \mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)\right| &\leq C^n \gamma^{-h\left[D(P_{v_0}) + z_{v_0}(N_{v_0}, P_{v_0} + \tilde{z}_{v_0}(N_{v_0}, P_{v_0}/2)\right]} \\ &\left(\prod_{v \notin V_{\rm f}(\tau)} \gamma^{-[D(P_v)+z_v(N_v,P_v)+\tilde{z}_v(N_v,P_v)/2](h_v - h_{v'})}\right) \left(\prod_{v \in V_{\rm f}(\tau)} |\rho_v|\right) \left(\prod_{v \in V_{\rm f}^*(\tau)} |\hat{\varphi}_{m_v}|\right) \; , \end{aligned} \tag{8.86}$$

and we can write in (8.86)

8.55
$$\begin{aligned} &\prod_{v \notin V_{\rm f}(\tau)} \gamma^{-[D(P_v)+z_v(N_v,P_v)](h_v - h_{v'})} \\ &\qquad \leq \prod_{v \notin V_{\rm f}(\tau)} \gamma^{-|P_v|/6} \left(\prod_{v \in V_4(\tau)} \gamma^{h_v - h_{v'}}\right) \left(\prod_{v \in V_{2'}(\tau)} \gamma^{h_v - h_{v'}}\right) \left(\prod_{v \in V_2(\tau)} \gamma^{2(h_v - h_{v'})}\right) \; , \end{aligned} \tag{8.87}$$

where the following notations have been used.
(1) $V_4(\tau)$ is the set of vertices $v \in V(\tau)$ with G_v nonresonant and $|P_v| = 4$.
(2) $V_{2'}(\tau)$ is the set of vertices $v \in V(\tau)$ with G_v nonresonant, $|P_v| = 2$ and a derivative acting on one of the external lines.
(3) $V_2(\tau)$ is the set of vertices $v \in V(\tau)$ with G_v nonresonant, $|P_v| = 2$ and no derivative acting on the external lines.

With any vertex $v \in V_{nt}(\tau)$ we associate a *depth* label D_v defined inductively as follows. If v is an endpoint then $D_v = 1$, otherwise

8.57
$$D_v = 1 + \max_{w \in V_{nt}(\tau)} \{D_w \; : \; w \succ v\} \; . \tag{8.88}$$

Note that $D_v \leq -h_{v'} + 2$, if v' is the nontrivial vertex immediately preceeding v.

We call B_D the set of $v \in V_{\rm f}^*(\tau)$ such that the nontrivial vertex immediately preceeding v has depth D.

Now consider the factors $\gamma^{h_v - h_{v'}}$ in (8.90) corresponding to the vertices $v \in V_4(\tau) \cup V_{2'}(\tau) \cup V_2(\tau)$.

For any pair v_1, v_2 of nontrivial vertices such that $v_2' = v_1$ we can consider all the trivial vertices following v_1 and preceeding v_2 together with v_2 and write

8.57a
$$\left(\prod_{\substack{v_1 \prec w \preceq v_2 \\ w \in V_4(\tau) \cup V_{2'}(\tau)}} \gamma^{h_w - h_{w'}}\right) \left(\prod_{\substack{v_1 \prec w \preceq v_2 \\ w \in V_2(\tau)}} \gamma^{2(h_w - h_{w'})}\right) \leq \gamma^{-h_{v_1}} \; , \tag{8.89}$$

so that we can bound (8.87) by

8.58
$$\begin{aligned} &\prod_{v \notin V_{\rm f}(\tau)} \gamma^{-[D(P_v)+z_v(N_v,P_v)+\tilde{z}_v(N_v,P_v)/2](h_v - h_{v'})} \\ &\qquad \leq \prod_{v \notin V_{\rm f}(\tau)} \gamma^{-|P_v|/12} \left(\prod_{v \in V_4(\tau) \cup V_{2'}(\tau) \cup V_2(\tau)} \gamma^{-2h_{v'}}\right) \; , \end{aligned} \tag{8.90}$$

simply using (8.89) for any path of trivial vertices between two nontrivial vertices: it is obvious that tin this way we exhaust all the vertices in $V(\tau)$.

One can prove inductively that

8.59
$$\prod_{v\in V_{\rm f}^*(\tau)} |\hat\varphi_{m_v}| \le \left(\prod_{v\in V_{\rm f}^*(\tau)} e^{-\kappa|m_v|/2}\right)\left(\prod_{v\in V_{\rm f}^*(\tau)} e^{-\kappa|N_v|/2^{D_v+1}}\right) ; \tag{8.91}$$

see Appendix A6 for a proof.

Using that $D_v \le -h_{v'}+2$ and defining the *indicator function* χ as

8.59a
$$\chi(\text{property}) = \begin{cases} 1\,, & \text{if property holds}\,,\\ 0\,, & \text{otherwise}\,. \end{cases} \tag{8.92}$$

one has

8.60
$$\begin{aligned}\prod_{v\notin V_{\rm f}(\tau)} \chi^*\left(|N_v| \ge C_1\big(\gamma^{-h_{v'}/\tau}|P_v|^{-1/\tau} - \bar m|P_v|\big)\right)\left(\prod_{v\notin V_{\rm f}(\tau)} |\hat\varphi_{m_v}|\right)\\ \le C^n\left(\prod_{v\notin V_{\rm f}(\tau)} e^{-\kappa|m_v|/2}\right)\left(\prod_{v\in V_4(\tau)\cup V_{2'}(\tau)\cup V_2(\tau)} e^{-\kappa C_2\gamma^{-h_{v'}/\tau}/2^{-h_{v'}+3}}\right)\,,\end{aligned} \tag{8.93}$$

where the star on χ means that the constraint represented by the indicator function is used only for the the vertices $v \in V_4(\tau)\cup V_{2'}(\tau)\cup V_2(\tau)$, for which $|P_v|\le 4$. At the end we obtain

8.61
$$\begin{aligned}\sum_{\{m_v\}_{v\in V_{\rm f}^*(\tau)}} \int d\mathbf{x}(P_{v_0})\left|\mathcal{W}^{(h)}\left(\tau, P_{v_0}, \mathbf{x}(P_{v_0})\right)\right| \le C^n\varepsilon^n\gamma^{-h[D(P_{v_0})+z_{v_0}+\tilde z_{v_0}/2]}\sum_{\{P_v\}}\left(\prod_{v\notin V_{\rm f}(\tau)}\gamma^{-|P_v|/12}\right)\\ \left(\prod_{v\in V_4(\tau)\cup V_{2'}(\tau)}\gamma^{-h_{v'}}e^{-\kappa C_2\gamma^{-h_v/\tau}/2^{-h_v+3}}\right)\left(\prod_{v\in V_2(\tau)}\gamma^{-2h_{v'}}e^{-\kappa C_2\gamma^{-h_v/\tau}/2^{-h_v+3}}\right)\,.\end{aligned} \tag{8.94}$$

Choosing γ so that $\gamma^{1/\tau}/2 > 1$ and using that, if n is the number of endpoints, the number of nontrivial vertices is bounded by $2n$ (see Appendix A1), we have that, for $\alpha = 4, 2', 2$, there exists a constant C such that

8.62
$$\prod_{v\in V_\alpha(\tau)} \gamma^{-h_v} e^{-\frac{\kappa C_2\gamma^{-h_v/\tau}}{2^{-h_v+3}}} \le C^n\,. \tag{8.95}$$

By a standard calculation (see Appendix A6)

8.63
$$\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\{P_v\}}\left(\prod_{v\notin V_{\rm f}(\tau)}\gamma^{-|P_v|/8}\right) \le C^n\,, \tag{8.96}$$

for some constant C.

Formula (8.93) is the analogue of *Bryuno lemma* for the problem we are considering: it ensures that the small denominator problem arising in the series, for the incommensurability of the potential φ_x, can be controlled by taking into account the Diophantine condition. The same rôle is played by the original Bryuno lemma in the proof of the convergence of the Lindstedt series (see [G] and [GM] for a discussion). Note however that while the Lindstedt series in classical mechanics have no loops, here there are loops and one has to use the Grahm-Hadamard inquality.

The formal proof can be found in Appendix A6. The idea is quite simple. We can associate with each cluster G_v with four or two external lines an exponential factor $e^{-\kappa|N_v|/2^{-h_v+3}}$, which is indeed quite small if $|h_v|$ is large. It is due to the analiticity of the incommensurate potential φ_x. But $|N_v|$ has to be very large because of the Diophantine condition, as noted in the lemma in §8.2, and the resulting factor compensates the "bad" factors γ^{-h_v} or γ^{-2h_v} due to the power counting. The relevant or marginal terms in (8.21) and (8.20) are the analogue of the *resonances* in KAM theory.

sec.9

9. Relationship between lattice and continuum models

p.9.1 **9.1.** *Continuum models.* By the Renormalization group methods we have described in the preceding sections one can treat equally systems of fermions on the continuum or on a lattice; we applied such methods to fermions on a lattice but this was done only for fixing ideas. This feature of such methods should be not undervalued; if from one hand it is "physically reasonable" that, as far as the low energy properties are concerned and for weak interactions, the qualitative behaviour for fermions on a lattice or on the continuum is the same, it is also true on the other hand that the methods commonly used are very sensitive to the fact that the fermions are on a lattice or on the continuum. For instance the continuous Renormalization group of [S] can be used only in the continuum case (with linearized dispersion relation), and so also the bosonization techniques. All the attempts by these techniques to study models defined on a lattice, like Hubbard-models or spin chains, have to approximate the lattice with a continuum, see for istance [Af])). On the other hand the exact solutions based on the *Bethe ansatz*, like the one of the Hubbard model by [LW] or the solution of the XYZ chain, are based on the presence of a lattice.

Let us consider the simplest interacting fermionic model on the continuum, with Hamiltonian

9.1
$$H = H_0 + \lambda V + \nu N_0, \tag{9.1}$$

with H_0, N_0 given by (2.2) and V given by (2.7). Following the scheme in §8 we decompose the Grassman variables in the sum of a ultraviolet part and an infrared one. Suppose that we can prove that after the ultraviolet integration one get $\mathcal{V}^{(0)}$ similar to (8.7) with $u = 0$ and the sums over x replaced by integrals. It is clear that, as far as the integration of $\psi^{(\text{i.r.})}$ is considered the presence of the lattice or of the continuum plays essentially no role. In fact the differences in the expansion for the infrared part of the effective potential due the fact that the fermions are on the lattice or on the continuum are only that in one case the δ-functions of the conservation of momenta are defined modulo 2π and in other case not; this can lead a difference in the beta function. However this difference is present only on trees containing $\mathcal{R}\mathcal{V}^{(0)}$ and such terms are $O(\gamma^{\eta h})$ for some $0 < \eta < 1$. Moreover the integrations over k are in one case for $|k| \le \pi$ and in the other case $k \in \mathbb{R}$; however the presence of the cut-off compact support functions makes the integrals in both case extending over the same interval.

p.9.2 **9.2.** *The ultraviolet problem.* What is really different is the ultraviolet integration, and this is related to the fact that the behaviour of the Schwinger functions for small distances or large momenta is very different in lattice or continuum models.

The analysis of the ultraviolet part of the model (9.1) was done in [BGPS] using a tree analysis similar to the one (with $\mathcal{R} = 1$) discussed above. One decomposes the ultraviolet part of the propagator

9.2
$$g^{(\text{u.v.})}(\mathbf{x}-\mathbf{y}) = \sum_{h=0}^{\infty} C^{(h)}(\mathbf{x}-\mathbf{y}), \tag{9.2}$$

with

9.3
$$C^{(h)}(\mathbf{x}-\mathbf{y}) = \gamma^{\frac{h}{2}} \bar{C}_h \left(\gamma^h(x_0-y_0), \gamma^{\frac{h}{2}}(x-y)\right), \tag{9.3}$$

such that

9.4
$$|\bar{C}_h(x_0, x)| \le \frac{C_N}{1 + |\mathbf{x}|^N}. \tag{9.4}$$

Then $\mathcal{V}^{(0)}(\psi^{(\text{i.r.})})$ is written as a sum over trees $\tau \in \mathcal{T}_{n,0}$ which is bounded, if $|\nu| \le C|\lambda|$, by

9.5
$$C^n \, |\lambda|^n \sum_{\tau \in \mathcal{T}_{n,0}} \prod_{v \in \tau} \gamma^{(h_v - h_{v'}) D_v / 4} \, , \tag{9.5}$$

with

9.6
$$D_v = |P_v| + 2n_v^4 + 4n_v^2 - 6 \, , \tag{9.6}$$

and $|P_v|$ is the number of external fields of the cluster v while n_v^4, n_v^2 are the number of λ or ν vertices inside the cluster G_v.

We have seen that the sum over trees can be done of $D_v > 0$; this is what happens in this case except in a finite number of cases, namely
(1) $|P_v| = 2$, $n_v^4 = 2$, $n_v^2 = 0$;
(2) $|P_v| = 2$, $n_v^4 = 1$, $n_v^2 = 0$.

However an explicit analysis of the above cases shows that the bounds can be improved and $D_v > 0$ also in that cases, see [BGPS]. Note that the ultraviolet part of the theory is a *superrenormalizable* theory, while the infrared part is a *renormalizable* one.

p.9.3 **9.3.** *The Luttinger model and the ultraviolet problem.* A similar result holds for the ultraviolet part of the Luttinger model, see [GS]. The analysis is more difficult, for the weaker decay properties of the ultraviolet propagator of the Luttinger model, and it is inspired by a proof for the ultraviolet *Yukawa*$_2$ model in [Le].

We do not present here the details of the proof (also because we did not even give the exact definition the Hamiltonian of the Luttinger model), and, as elsewhere in this paper, we defer to the original paper, [GS]: here we confine ourselves to outline the idea of the proof.

By analyzing carefully the structure of the clusters of the Feynman diagrams, as done for the infrared problem, one can easily identify the contributions which are responsible for bad dimensional bounds as the clusters with two and four external lines (the scaling properties of the propagator are exactly the same both in the ultraviolet and in the infrared region). However, by using the connectedness properties of the Feynman diagrams and the non-locality of the interaction, one can easily realize that not all clusters present the same problems: to say better the dimensional bounds can be improved for a suitable subclass of clusters with two and four external lines. As far as the remaining ones are concerned, one can then use the symmetry properties of the propagator of the Luttinger model (it is an odd function), to show that there are cancellations such that the dominant terms (which would lead to divergences when summing all scales) are in fact vanishing.

The real difficulty arises when trying to take care of the cancellations without bounding separately the single Feynman diagrams (for which the above argument simply applies), but directly working with the truncated expectations in order to use the Garm-Hadamard inequality to obtain summability. Then one has to disentagle the classes of graphs lumped together in the definition of truncated expecation in order to still keep the good ultraviolet bound valid for them, but not too much to loose the good combinatorial behaviour of the estimates: with respect to [Le] the decomposition of the graphs has to be carried out to a much deeper extent.

sec.10 ## 10. Hidden symmetries and flow equations

In the preceding sections we have defined an expansion for the effective potential which is convergent if the running coupling constants are small, see Theorem 1 and (8.76). An easy consequence of its proof is that, if $\sigma_k = 0$ for any k than the running coupling constants $\vec{v}_h$ are analytic as a function of $\vec{v}_k$, $k > h$, and so it is the effective potential.

Before considering the problem of proving that the running coupling constants verify (8.76) if the perturbative parameters λ, u are small enough, we mention that the value of $\bar\varepsilon$ obtained collecting patiently all the numerical constants obtained in the bounds is, in adimensional units, very small and "unrealistic". It is likely that the value of $\bar\varepsilon$ could be greatly improved, using perhaps computer assisted proof techniques. Again the analogy with classical mechanics is helpful: in the classical estimate of the convergence radius of KAM tori (whose perturbative expansion, as we have seen, is so similar to the perturbative series of quantum field theory, except for the presence of loops) one gets a really unappliable bounds for the convergence radius, which can however be largely improved till a realistic value using computer assisted proofs. So it is reasonable to assume that one can improve largely the estimate on $\bar\varepsilon$. However from the exact solutions we can see that one cannot enlarge λ indefinitely; for istance if $u = 0$ and $v(x-y) = \delta_{|x-y|,1}$ the model reduces to the XXZ spin chain for which it is known, see [MW], that the spectrum has a gap for $\lambda \le -1$, so there is no hope that our series converges for any values of λ. The same holds for the Luttinger model, in which the solution in [ML] is valid only for $\lambda \ge -\hat v(0)/4\pi$. Note also that $\bar\varepsilon$ depends in a critical way by p_F and it is vanishing if $p_F = 0, \pi$.

The validity of (8.76) is a non trivial property and it is essentially due to the fact that the Holstein-Hubbard model is "close" in a Renormalization group sense to an exactly solvable model, the *Luttinger model*, verifying many important symmetries (Lorentz invariance, gauge invariance, etc) which are not verified by the Holstein-Hubbard Hamiltonian (we are speaking here of this model only for fixing the ideas but the same considerations holds for a class of models, see §13, if the fermions are *spinless*). One can say that symmetries are "hidden". This kind of ideas are very old; they date back to Tomonaga, which discovered that $d = 1$ non relativistic fermions can be "approximated" by two types of fermions with linear dispersion relation, and this leads Luttinger to propose his model [L]. This idea was so successfull that from the '70 the study of $d = 1$ metals is done directly in terms of this fermions with linear dispersion relation, see [S] and [V]. The new point in the approach we are discussing is that we can really check the validity of this approximation in a quantitative way, obtaining rigorous estimates on the size of the corrections. In fact, while the validity of such approximation is usually giustified using qualitative arguments coming from Renormalization Group (for istance saying that many models are in the same "class of universality" than the Luttinger model, that thay differ for "irrelevant terms" from the Luttinger model and so on), in the approach we are discussing it is possible to give to such arguments a rigorous meaning and they can be substantiated by rigorous bouds.

To implement the above considerations we consider the integration (with an ultraviolet cut-off, as there is no $h = 1$):

6.1*a*
$$P_{(l)}(d\psi^{(\le 0)}) = \prod_{\mathbf{k}'}\prod_{\omega=\pm1} \frac{d\hat\psi^{(\le0)+}_{\mathbf{k}',\omega} d\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} \cdot \\ \cdot\ \exp\left\{ -\frac{1}{L\beta}\sum_{\omega=\pm1}\sum_{\mathbf{k}'} C_0(\mathbf{k}')\big(-ik_0 + \omega v_0 k'\big)\hat\psi^{(\le0)+}_{\mathbf{k}',\omega}\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}\right\}, \tag{10.1}$$

where $C_0^{-1} = \sum_{k=h_\beta}^0 f_k$. We can perform the integration

6.1*aa*
$$\int P_{(l)}(d\psi^{(\le0)}) e^{-\lambda V_\lambda(\psi^{(\le0)})} \tag{10.2}$$

with

6.*x*
$$V_\lambda = \int d\mathbf{x}\psi^{+,\le0}_{\mathbf{x},1}\psi^{+,\le0}_{\mathbf{x},-1}\psi^{-,\le0}_{\mathbf{x},-1}\psi^{-,\le0}_{\mathbf{x},1} \tag{10.3}$$

We can integrate (10.2) using a procedure similar to the one discussed in the previos sections but:
1)$\sigma_h = 0$ for any h, as the propagator is diagonal in the ω-indices
2)$\nu_h = 0$ as the propagator is an odd function as a function of $\mathbf{x}$.

This means, see §5, that there are only two running coupling constant in this theory, namely δ_h, λ_h. We will call the theory defined by the integration (10.1) *infrared Luttinger model*; note that, contrary to the true

Luttinger model (obtained from (10.1) replacing C_0^{-1} with 1 and Wick ordering V_λ), the infrared Luttinger model is *not* exactly solvable.

Returning to the beta function of the Hubbard-Holstein model (8.1), the flow equations for the running coupling constants $\vec{v}_h$ are given by

$$\nu_{h-1} = \gamma\nu_h + G^h_\nu(\vec{v}_h, \dots, \vec{v}_0)$$

$$\lambda_{h-1} = \lambda_h + G^h_\lambda(\vec{v}_h, \dots, \vec{v}_0)$$

$$\delta_{h-1} = \delta_h + G^h_\delta(\vec{v}_h, \dots, \vec{v}_0)$$

$$\sigma_{h-1} = \sigma_h + G^h_\sigma(\vec{v}_h, \dots, \vec{v}_0)$$

$$\frac{Z_{h-1}}{Z_h} = 1 + G^h_z(\vec{v}_h, \dots, \vec{v}_0)$$

Let us call $\mu_h = (\lambda_h, \delta_h)$ the running coupling constants appearing in the Renormalization Group analysis of the infrared Luttinger model. We want to write the $\beta^{(h)}_i$ functions in a Luttinger model part plus a "correction".

We know from Lemma 2 that we can write each propagator $g^{(k)}_{\omega,\omega'}$ as a term indipendent from σ_k, given by $\delta_{\omega,\omega'}[g^{(k)}_{0,\omega} + C^{(k)}_{1,\omega}]$ plus a term $\delta_{\omega,-\omega'}g^{(k)}_{\omega,-\omega} + \delta_{\omega,\omega'}C^{(k)}_{2,\omega}$; this second addend verifies the same bound of the first one times $\frac{|\sigma_k|}{\gamma^k}$. We can write then, if $i = \nu, \sigma, \mu, z$

xz
$$G^h_i(\mu_h, \nu_h, \sigma_h; \dots; \mu_0, \nu_0, \sigma_0) = G^{1,h}_i(\mu_h, \nu_h; \dots; \mu_0, \nu_0) + G^{2,h}_i(\mu_h, \nu_h, \sigma_h; \dots; \mu_0, \nu_0, \sigma_0) \tag{10.4}$$

where the first addend is σ independent, and by symmetry reasons

$$G^{1,h}_\sigma(\mu_h, \nu_h; \dots; \mu_0, \nu_0) = 0$$

Morever

ca14
$$|G^{2,h}_\sigma(\mu_h, \nu_h, \sigma_h; \dots; \mu_0, \nu_0, \sigma_0)| \le C|\lambda_h\sigma_h| \tag{10.5}$$

and for $i = \nu, \mu$

lll
$$|G^{2,h}_i(\mu_h, \nu_h, \sigma_h; \dots; \mu_0, \nu_0, \sigma_0)| \le C|\sigma_h|\gamma^{-h}|\lambda_h|^2 \tag{10.6}$$

as a consequence of the bounds (8.59),(8.60) of the propagator $\delta_{\omega',-\omega}g^{(k)}_{\omega,\omega'} + \delta_{\omega,\omega'}C^{(k)}_{2,\omega}$ and of the short memory property, see below. As in the infrared Luttinger model there is no a running coupling constant ν_h it is convenient to write, if $i = \mu, \nu$

ca13
$$G^{1,h}_i(\mu_h, \nu_h; \dots; \mu_0, \nu_0) = \bar{G}^{1,h}_i(\mu_h; \dots; \mu_0) + \hat{G}^{1,h}_i(\mu_h, \nu_h; \dots; \mu_0, \nu_0) \tag{10.7}$$

where the first term in the r.h.s. of (10.7) is obtained putting $\nu_k = 0$, $k \ge h$ in the l.h.s. Finally we write

ca14a
$$\bar{G}^{1,h}_i(\mu_h; \dots; \mu_0) = \bar{G}^{1,h,l}_i(\mu_h; \dots; \mu_0) + \bar{G}^{1,h,nl}_i(\mu_h; \dots; \mu_0) \tag{10.8}$$

where $\bar{G}^{1,h,l}_i(\mu_h; \dots; \mu_0)$ involves only propagators $g^{(k)}_{0,\omega}(\mathbf{x}; \mathbf{y})$ and it coincides with beta function of the infrared Luttinger model. It is easy to see, from the oddness of $g^{(h)}_{0,\omega}(\mathbf{x}; \mathbf{y})$ that

ccc
$$\bar{G}^{1,h}_\nu(\mu_h; \dots; \mu_0) = 0 \tag{10.9}$$

On the other hand

cccc
$$|\bar{G}^{1,h,nl}_\mu(\mu_h; \dots; \mu_0)| \le C\gamma^{\eta h}|\lambda_h|^2 \, . \tag{10.10}$$

In fact each contribution to $\bar G_i^{1,h,nL}$ has a propagator $C_{1,\omega}^{(k)}$ replacing a $g_{0,\omega}^{(k)}$ in the analogous contribution to $G_i^{1,h,l}$; one gains then, with respect to the bounds for $G_i^{1,h,l}$, a factor γ^k for some $k \geq h$ (see Lemma 2 in §8). One then uses an immediate consequence of (8.87), saying that any contribution to the effective potential associated to a tree with a vertex at scale k is bounded by $C^n \varepsilon_h^n \gamma^{\eta(h-k)}$, for some $\eta < 1$; this property is often called *short memory*, as the exponential factor decreases the contribution from trees contributing to the effective potential with scale h and involving integration of fields with scale k very far from $k \gg h$.

Let us assume that

beta
$$\bar G_\mu^{1,h,l}(\mu_h, \dots, \mu_h) = 0 \tag{10.11}$$

which means that the infrared Luttinger model Beta function with equal arguments is vanishing. Of course one can check this statement at the first perturbative orders but to really prove (10.11) one needs a non-perturbative argument, see the end of §11. Assuming (10.11), one can check inductively that the running coupling constants remain small, if the counterterm ν is chosen properly. We do not this, see [BGPS], [M1] and [BeM], but we simply give some idea of the proof.

The first step is to choose ν in a proper way so that ν_h is bounded. We have seen in §7 that the interaction moves the singularity, so producing divergences if the counterterm is not chosen properly. The RG flow equation is given by

it
$$\nu_{h-1} = \gamma \nu_h + \beta_\nu^{(h)} \tag{10.12}$$

with, by (10.9), $|\beta_\nu^{(h)}| \leq C\lambda_h^2[\gamma^{\eta h} + \sigma_h \gamma^{-h}]$. By solving (10.12) by iteration

xxxu
$$\nu_h = \gamma^{-h+1}[\nu_0 + \sum_{k=h+1}^{0} \gamma^{k-2} \beta_\nu^k] \tag{10.13}$$

and fixing $\nu_0 = -\sum_{k=h^*+1}^{0} \gamma^{k-2} \beta_\nu^k$ (note that also the r.h.s. of this equation depends on ν_0, and that ν_0 is a function of ν) it is possible to show that that

jj
$$|\nu_h| \leq \lambda_h^2 C[\gamma^{\eta h} + |\sigma_h| \gamma^{-h}] \tag{10.14}$$

By (10.14) we see that the bound for $\hat G_i^{(1,h)}$, which by definition is given by trees with at least an ν-end-point by (10.7), has at least an extra $\gamma^{\eta h}$ or $|\sigma_h|\gamma^{-h}$ with respect to the bound for $\bar G_i^{(1,h)}$, *i.e.*

jjjjj
$$\left| \hat G_i^{1,h}(\mu_h, \nu_h; \dots; \mu_0, \nu_0) \right| \leq C|\sigma_h| \gamma^{-h} |\lambda_h|^2 , \tag{10.15}$$

By an explicit lowest order computation,

$$\frac{\sigma_{h-1}}{\sigma_h} = 1 + \lambda_h[-\beta_1 + \hat\beta_\sigma^{(h)}]$$

$$\frac{Z_{h-1}}{Z_h} = 1 + \lambda_h^2[\beta_2 + \hat\beta_z^{(h)}]$$

with β_1, β_2 positive non vanishing constants and $|\hat\beta_\sigma^{(h)}|, |\hat\beta_z^{(h)}| \leq C|\lambda_h|$. From such equations immediately follow that

ca6
$$\gamma^{\beta_1 c_1 \lambda h} \leq \frac{|\sigma_{h-1}|}{|\sigma_0|} \leq \gamma^{c_2 \beta_1 |\lambda| h} \quad \gamma^{-c_3 \beta_2 \lambda^2 h} \leq Z_{h-1} \leq \gamma^{-c_4 \beta_2 \lambda^2 h} \tag{10.16}$$

and an immediate consequence of them is an estimate for h^*.

We now consider the flow for μ_h; we want to prove that

llll
$$|\lambda_{h-1} - \lambda| < C|\lambda|^{3/2} \quad |\delta_{h-1}| \leq C|\lambda|^{3/2} . \tag{10.17}$$

Assume inductively that, for any $h < k \leq 0$

bo
$$|\mu_{k-1} - \mu_k| \leq |\lambda|^{\frac{3}{2}} \left[\gamma^{\eta k} + \frac{|\sigma_k|}{\gamma^k} \right] . \tag{10.18}$$

We can write

10.18
$$\bar{G}^{1,h}_{\mu}(\mu_h, \mu_{h+1}, \dots, \mu_0) = \bar{G}^{1,h}_{\mu}(\mu_h, \dots, \mu_h) + \sum_{k=h+1}^{0} D^{h,k} , \tag{10.19}$$

where

10.19
$$D^{h,k} = \bar{G}^{1,h}_{\mu}(\mu_h \dots, \mu_h, \mu_k, \mu_{k+1}, ..\mu_0) - \bar{G}^{1,h}_{\mu}(\mu_h, \dots, \mu_h, \mu_h, \mu_{k+1}, \dots, \mu_0) \tag{10.20}$$

and $\bar{G}^{1,h}_{\mu}(\mu_h, \dots, \mu_h)$ is estimated by (10.10) and (10.11).

From the short memory property it follows

b03
$$|D^{h,k}| \leq C|\lambda|\gamma^{-2\eta(k-h)}|\mu_k - \mu_h| \tag{10.21}$$

From (10.6), (10.7), (10.10), (10.15), (10.18) and (10.21) it follows

$$|\mu_{h-1} - \mu_h| \leq C|\lambda|^{\frac{5}{2}} \sum_{k=h+1}^{0} \gamma^{-2\eta(k-h)} \sum_{k'=h+1}^{k} [\gamma^{\eta k'} + \frac{|\sigma_{k'}|}{\gamma^{k'}}] + 2C\lambda^2 \frac{|\sigma_h|}{\gamma^h} + 2C\lambda^2 \gamma^{\eta h}$$

which immediately implies (10.18) with $k = h$.

sec.11
11. Vanishing of the Luttinger model Beta function

We have seen in §10 that an essential argument to study the flow of the running coupling constants is (10.11), stating that the beta function of the Luttinger model is vanishing. We have in fact seen in §10 that, given (10.11), the flow of the running coupling constants remains bounded; this is a remarkable properties of *spinless* fermions. In order to prove (10.11) one has to use some a nonperturbative argument; one possibility would be to use some Ward identity. Another possibility [BG1], which is the one discussed here, is to use the exact solution of the Luttinger model [ML] and the fact that all the Schwinger functions of such model can be explicitely computed. The idea of using the exact solution is really in the spirit of the renormalization group approach; one shows that a model is in the "universality class" of some special model which is exactly solvable, and takes all the possible advantage from such exact solution (see for istance [A1], in particular for the idea of "continuation"). On the other hand it is likely that one is able to prove (10.11) also directly by Ward identities without using exact solutions; this is done (with no pretense of rigor) in the context of *moltiplicative* Renormalization group in [MD].

Again we refer to the original papers, especially [BGPS], [GS] abd [BM1], for the proofs, and we give here only some ideas. The strategy is very simple. The Hamiltonian of the Luttinger model is given by

L
$$\begin{aligned} H &= H_0 + V, \\ H_0 &= \sum_{\omega=\pm 1} \int_0^L \mathrm{d}x \, \psi^+_{\omega,x} \left(i\omega\partial_x - p_F \right) \psi^+_{\omega,x}, \\ V &= \lambda \int_0^L \mathrm{d}x \mathrm{d}y \, v(x-y) \, \psi^+_{1,x}\psi^-_{1,x}\psi^+_{-1,y}\psi^-_{-1,y} + a \int_0^L \mathrm{d}x \left(\psi^+_{1,x}\psi^-_{1,x} + \psi^+_{-1,x}\psi^-_{-1,x} \right) + b \int_0^L \mathrm{d}x, \end{aligned} \tag{11.1}$$

where $|v(x-y)| \leq e^{-p_0|x-y|}$ is a short range potential and a, b have to be computed introducing an ultraviolet cut-off in (11.1) (which otherwise does not have a well defined meaning) and by imposing the the Schwinger

functions of the model are well defined uniformly in the cut-off, see [BGM]; this correspond to a Wick order in the interaction. Finally $p_F = \frac{2\pi}{L}(n_F + \frac{1}{2})$ with n_F a positive integer.

By the exact solution of [ML] it is possible to compute all the $\beta = \infty$ finite L Schwinger functions for this model, see [BeGM].

We stress that the Schwinger functions can be really computed in a rigorous way *only* (up to now) for the model with Hamiltonian (11.1). In literature the name of "Luttinger model" is improperly used for many other models with sligthly different Hamiltonians; for istance the *Thirring model*, [Th], in which $v(x-y)$ is replaced by $\delta(x-y)$ and the theory is defined with an ultraviolet cut-off which selects momenta $|k| \le 1$; for such models, as far as we know, there exists no exact solution.

One can study the Luttinger model (11.1) by Renormalization group methods. We have already discussed the Renormalization Group analysis of the infrared Luttinger model defined by (10.1),(10.2). In order to study the model (11.1) we can write as in §5 $\psi = \psi^{(u.v.)} + \psi^{(i.r.)}$, and $\psi^{(i.r.)}$ has exactly the same propagator as the field with integration $P_{(l)}$ given by (10.1); in other words, if $\tilde{P}_{(l)}(d\psi)$ is the Grassman integration of the Luttinger model (11.1)

$$\tilde{P}_{(l)}(d\psi) = P_{(l)}(d\psi^{(i.r.)})P_{(l)}(d\psi^{(u.v.)})$$

where $P_{(l)}(d\psi^{\le 0})$ is given by (10.1). It is possible to prove, see [GS], that if V is the Luttinger model interaction in (11.1), then

$$\int P_{(l)}(d\psi^{(u.v.)})e^{V(\psi^{(u.v.)}+\psi^{(i.r.)})} = e^{\lambda_0 V_\lambda(\psi^{(i.r.)})+\tilde{V}(\psi^{(i.r.)})}$$

where, if $\psi^{(i.r.)} \equiv \psi^{(\le 0)}$, $V_\lambda(\psi^{\le 0})$ is given by (10.2), λ_0 is an analytic function of λ and $\tilde{V}(\psi^{\le 0})$ is similar to (8.6) and such that

$$\mathcal{L}\tilde{V}(\psi^{\le 0}) = 0$$

This means that the only difference between the infrared Luttinger model (10.1) or the Luttinger model is by *irrelevant terms*; then the two models have the same beta function up to terms $O(\gamma^{\eta h})$ for the short memory property.

Let us call $\mu_h^{L,\beta}$ the running coupling constants of the Luttinger model, and set $\lim_{\beta\to\infty}\mu_h^{L,\beta} = \mu_h^L$ and $\lim_{L\to\infty}\mu_h^L = \mu_h$. Note also that if $p_F = \frac{2\pi}{L}(2n_F + \frac{1}{2})$ than the analogue of (5.10) is a finite sum with starting from $h_{L,\beta}$; moreover $\lim_{\beta\to\infty} h_{L,\beta} = h_L = O(\log L^{-1})$.

One can prove the following result.

LEMMA 5. *There exists an $\tilde{\varepsilon}$ such that, for $|\lambda| \le \tilde{\varepsilon}$ and for any h, μ_h is analytic as a function of λ and*

ss
$$|\mu_h - \mu_0| \le C|\lambda|^{3/2} . \tag{11.2}$$

Proof. Let us resume the proof of the above lemma referring for detailed proofs to the original papers. Let us prove first that if $|\lambda| \le \tilde{\varepsilon}$ than $|\mu_h| \le 2\tilde{\varepsilon}$ for any h. Suppose that this is not true; then there exists $h_0 > -\infty$ such that $|\mu_k| \le 2\tilde{\varepsilon}$ for $k > h_0$ but

con
$$|\mu_{h_0}| > 2\tilde{\varepsilon} \tag{11.3}$$

We will show that this gives a contradiction. Let us fix $L_0\gamma^{h_0} = 1/n$, if n is some fixed real number. By the analogue of Theorem 1 in §8 for the Luttinger model we can say that the running coupling constants at scale h are analytic in μ_k, $k > h$ if $h \ge h_0$ and $2\tilde{\varepsilon} \le \bar{\varepsilon}$. In general $h_0 \ne h_{L_0}$.

As we know from the exact solution Schwinger functions of the Luttinger model, we want to use this knowledge for showing that (11.3) is not possible. Let us consider the Luttinger model in a volume L_0 and

bobo
$$e^{W_{L_0}(\phi)} = \int \tilde{P}_{(l)}(d\psi)e^{V(\psi+\phi)} \tag{11.4}$$

and we compute the above integral *in a single step i.e.* without performing a multiscale decomposition analysis. It holds that

$$W_{L_0}(\phi) = \sum_{\omega} \int d\mathbf{x} d\mathbf{y} W_{2,L_0}(\mathbf{x}-\mathbf{y}) \phi^+_{\omega,\mathbf{x}} \phi^-_{\omega,\mathbf{y}} +$$

$$\sum_{\omega} \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 W_{4,L_0}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \phi^+_{\omega,\mathbf{x}_1} \phi^+_{-\omega,\mathbf{x}_2} \phi^-_{-\omega,\mathbf{x}_3} \phi^-_{\omega,\mathbf{x}_4} + O(\phi^6)$$

If $\hat S^T_{2,L_0,\omega}(\mathbf{k})$, $\hat S^T_{4,L_0,\omega}(\mathbf{k}_1,\dots,\mathbf{k}_4)$ are the Fourier transforms of the truncated two and four point Luttinger model Schwinger functions (*i.e.* Schwinger functions expressed by connected Feynman diagrams) in a volume L_0 then

mmm
$$\hat S^T_{2,L_0,\omega}(\mathbf{k}) = \frac{1}{-ik_0 + \omega k + \hat W_{2,irr,L_0}(\mathbf{k})} \tag{11.5}$$

$$\hat S^T_{4,L_0,\omega}(\mathbf{k}_1,\dots,\mathbf{k}_4) = \hat S^T_{2,L_0,\omega}(\mathbf{k}_1) \hat S^T_{2,L_0,-\omega}(\mathbf{k}_2) \hat S^T_{2,L_0,-\omega}(\mathbf{k}_3) \hat S^T_{2,L_0,\omega}(\mathbf{k}_4) \hat W_{4,irr,L_0}(\mathbf{k}_1,\dots,\mathbf{k}_4)$$

where $\hat W_{2,irr,L_0}, \hat W_{4,irr,L_0}$ are the *irreducible* parts of $\hat W_{2,L_0}, \hat W_{4,L_0}$, *i.e.* they are given by Feynman graphs which cannot be disconnected by cutting a single line, see for istance [AGD]. The above relations can be proved using the analyticity in λ of the r.h.s. and the l.h.s. of (11.5) for $|\lambda| \le \varepsilon_{L_0}$, with $\varepsilon_{L_0} \to 0$ for $L_0 \to \infty$, developing (11.5) in series of λ and showing the equality of the coefficients. It is straightforward then to express $\hat W_{2,irr,L_0}, \hat W_{4,irr,L_0}$ as functions of of $\hat S^T_{2,L_0,\omega}$, $\hat S^T_{4,L_0,\omega}$. Moreover it holds that

mm
$$\hat W_{4,irr,L}((0,\frac{\pi}{\tilde n L}),...,(0,\frac{\pi}{\tilde n L})) = W^{h_L}_{4,L}((0,\frac{\pi}{\tilde n L}),...,(0,\frac{\pi}{\tilde n L})) \tag{11.6}$$

with $\tilde n > 1$, and similar expressions hold for $\hat W_{2,irr,L}$ and their derivatives. The r.h.s. of (11.6) is obtained from (11.4) integrating scale by scale from 1 to h_L the fermionic integration like in §5, and noting that $g^{(k)}(\mathbf{k})$, $k \ge h_L$, is vanishing when computed at $\mathbf{k} = (0, \pi/\tilde n L)$ so that all the not irriducible contributions are vanishing.

At the end, as from §8 the running coupling constants are expressed in terms of $W^h_{4,L}(\mathbf{k}_1,..,\mathbf{k}_4)$, $W^h_{2,L}(\mathbf{k})$ computed at the Fermi surface (or, more exactly, at the admissible momenta closest to p_F), we can express the running coupling constants $Z^{L_0}_{h_{L_0}}, \delta^{L_0}_{h_{L_0}}, \lambda^{L_0}_{h_{L_0}}$ in terms of $\hat S^T_{2,L_0,\omega}$, $\hat S^T_{4,L_0,\omega}$ or their derivatives with momenta at the Fermi surface; for istance

manm
$$\frac{\tilde n L_0}{\pi} \frac{1}{\tilde Z^{L_0}_{h_{L_0}}(1+\tilde\delta^{L_0}_{h_{L_0}})} = S^T_{2,L_0,\omega}(\pi/\tilde n L_0) \tag{11.7}$$

The running coupling constants $\tilde Z^{L_0}_{h_{L_0}}, \tilde\delta^{L_0}_{h_{L_0}}, \tilde\lambda^{L_0}_{h_{L_0}}$ appearing in (11.7) are not exactly the running coupling constants $Z^{L_0}_{h_{L_0}}, \delta^{L_0}_{h_{L_0}}, \lambda^{L_0}_{h_{L_0}}$; the last ones are related to the Fourier transform of the effective potential computed at π/L_0 while the others are related to the Fourier transform of the effective potential computed at $\pi/\tilde n L_0$; however it is easy to show that $|\mu^L_{h_L} - \tilde\mu^L_{h_L}| \le C\varepsilon^2_{h_L}$. (11.7) is valid for $|\lambda| \le \varepsilon_{L_0}$; however the r.h.s is analytic in λ by looking at the explicit expression which is obtained from the exact solution and the l.h.s. by Theorem 1 as $2\tilde\varepsilon \le \bar\varepsilon$, so one extends its validity to a domain L_0-independent by using analytic continuation.

At the end by using the explicit expression of the Luttinger model Schwinger functions one finds, see [BM1]

xcx
$$|\lambda^{L_0}_{h_{L_0}} - \lambda| \le C\lambda^2 \quad |\delta^{L_0}_{h_{L_0}}| \le C\lambda^2 \tag{11.8}$$

On the other hand one can prove that, for $\varepsilon_{h_0} \le \bar\varepsilon$, the difference between $\mu^{L_0}_{h_0}$ and μ_{h_0} is such that, see [BM1],

xcx1
$$|\mu^{L_0}_{h_0} - \mu_{h_0}| \le C\varepsilon^{3/2}_{h_0+1} \frac{\gamma^{-h_0}}{L_0} \tag{11.9}$$

On the other hand it is clear, by the convergence of the beta function, that $|\mu^{L_0}_{h_0} - \mu^{L_0}_{h_{L_0}}| \le Cn\varepsilon^{3/2}_{h_0+1}$.

Let be $|\mu_0| \le \tilde\varepsilon$; we can write from (11.8),(11.9)

2223
$$|\mu_{h_0} - \mu_0| = |\mu_{h_0} - \mu^{L_0}_{h_0}| + |\mu^{L_0}_{h_0} - \mu^{L_0}_{h_{L_0}}| + |\mu^{L_0}_{h_{L_0}} - \mu_0| \le Cn\varepsilon^{3/2}_{h_0+1} + C\lambda^2 \le \frac{\tilde\varepsilon}{8} \tag{11.10}$$

in contradiction with (11.3); we have proved then that if $|\lambda| \le \tilde\varepsilon$ than $|\mu_h| \le 2\tilde\varepsilon$ for any h. This means that the running coupling constants remain inside the analyticity radius of the beta function, so that μ_h is analytic as a function of λ and (11.2) follows from (11.10). ■

In order to prove (10.11) note that

c
$$\mu_h = \mu_0 + \sum_{n=2}^{r} c^{(h)}_n \mu^n_0 + O(\mu^{r+1}_0) \; . \tag{11.11}$$

The flow equation is given by:

c1
$$\mu_{h-1} = \mu_h + \beta^L_h(\mu_h, \dots, \mu_h) + \sum_k (\mu_h - \mu_k) D^{h,k} \tag{11.12}$$

Assume *by contradiction* that there exists a $b_r \neq 0$ such that

$$\beta^L_h(\mu_h, \dots, \mu_h) = b_r(\mu_h)^r + O(\mu^{r+1}_h)$$

Insering (11.11) into (11.12) one gets

$$c^{(h-1)}_r = c^{(h)}_r + b_r + O(\gamma^{\eta h})$$

implying that $c^{(h)}_r$ is a diverging sequence, in contradiction with (11.12).

sec.12
12. The two-point Schwinger functions

p.12.1 **12.1.** *The expansion.* In naive perturbation theory (not convergent) an expansion for the correlation functions follows immediately from the expansion for the effective potential; for istance the two Schwinger function is given by

$$S(\mathbf{k}) = g(\mathbf{k}) + g(\mathbf{k}) V_2(\mathbf{k}) g(\mathbf{k})$$

where V_2 is the effective potential with two external fields. In a perturbation theory based on the Renormalization group like the one discussed in the preceding sections the relationship between the effective potential and the correlation functions is not so immediate. In fact $V^{(h)}$ has external lines with a *smaller* scale than the scale of the lines contracted to form the kernels of $V^{(h)}$; contracting the external lines of $V^{(h)}$ with fields representing the external fields one gets a contribution to the Schwinger functions, but there are many other terms contributing to the Schwinger functions that cannot be obtained in this way (the contributions in which the propagators connecting the external fields have no the smallest scale among all the propagators). We will see that new expansions are necessary for the Schwinger functions and the response functions; new *critical indices* which were not present in the theory of the effective potential will appear.

We start by the two-point Schwinger function which is given by the following functional integral, if $\phi^\pm_{\mathbf{x}}, \phi^\pm_{\mathbf{y}}$ are Grassman variables:

$$S(\mathbf{x};\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}} \partial\phi^-_{\mathbf{y}}} \mathcal{S}(\phi)|_{\phi^+=\phi^-=0}$$

with

$$e^{\mathcal{S}(\phi)} = \int P(d\psi) e^{V(\psi) + \int d\mathbf{x} [\psi^+_{\mathbf{x}} \phi^-_{\mathbf{x}} + \phi^+_{\mathbf{x}} \psi^-_{\mathbf{x}}]}$$

We proceeed as in the expansion of the effective potential integrating the fields $\psi^{(1)}, \dots \psi^{(h+1)}$ so obtaining

$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(\geq h+1)}(\phi)} \int P_{Z_h}(d\psi^{(\leq h)})$$

12.1
$$e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \phi) + \int d\mathbf{x} [\psi^+_{\mathbf{x},\omega} Q^{(h+1)}_{\omega,\omega'}(\mathbf{x}) \phi^-_{\mathbf{x},\omega'} + \phi^+_{\mathbf{x},\omega} Q^{(h+1)}_{\omega,\omega'}(\mathbf{x}) \psi^-_{\mathbf{x},\omega'}]} , \tag{12.1}$$

where $P_{Z_h}(d\psi^{(\leq h)})$ and $\mathcal{V}^{(h)}$ are given by (8.43), while $S^{(\geq h+1)}(\phi)$ denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi)$ can be written as

$$[\phi^+_\omega * G^{(h)}_{\omega,\omega'} * \frac{\partial}{\partial \psi^+_{\omega'}} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi)] + \frac{\partial}{\partial \psi^-_\omega} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi) * G^{(h)}_{\omega,\omega'} * \phi^-_\omega)] +$$

bb
$$[\phi^+_\omega * G^{(h)}_{\omega,\omega'} * \frac{\partial^2}{\partial \psi^+_{\omega'} \partial \psi^-_{\omega''}} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi) * G^{(h)}_{\omega'',\omega'''} * \phi^-_\omega] + \bar{W}^{(h)}_R \tag{12.2}$$

where

5.a
$$G^{(h)}_{\omega,\omega'} = \sum_{k \geq h+1} \frac{1}{Z_k} g^{(k)}_{\omega,\omega''} * Q^{(k)}_{\omega'',\omega'} \tag{12.3}$$

and $\bar{W}^{(h)}_R$ contains terms of higher order in ϕ. The above formula can be proved by induction, with $Q^{(2)}_{\omega,\omega'} = 0$.

Now we write in (12.2)

$$\frac{\partial}{\partial \psi^+_{\omega'}} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi) = \frac{\partial}{\partial \psi^+_{\omega'}} \mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi) + \frac{\partial}{\partial \psi^+_{\omega'}} \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi)$$

and the same decomposition is done for $\frac{\partial}{\partial \psi^-_\omega} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi)$, but not on the terms in the second line of (12.2) (the reason of this choice will be clear at the end). Note that one have to avoid that a derivative generated by the renormalization procedure is applied on the propagator $G^{(h)}$ (if this happens one do not get a gain factor $\gamma^{h_v - h_{v'}}$), and this can be ensured by choosing as a a localization point the coordinate of the field contracted in $G^{(h)}$.

In the integration of the effective potential one has to put part of the relevant part of the effective potential in the free integration; the same has to be done in the expansion for the two point Schwinger function for $\mathcal{B}^{(h)}(\psi^{(\leq h)}, \phi)$, changing Q_h. We define

5.b
$$Q^{(h)}_{\omega,\omega'} = Q^{(h+1)}_{\omega,\omega'} + z_h Z_h [\partial_t + \varepsilon(i\partial_x))] G^{(h)}_{\omega,\omega'} + s_h Z_h G^{(h)}_{\omega,-\omega'} \tag{12.4}$$

We can write the integral in the r.h.s. of (12.1) as

$$e^{-L\beta t_h} \int P_{\tilde{Z}_{h-1}}(d\psi^{\leq h})$$

5.4a
$$e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \tilde{\mathcal{B}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \phi) + \int d\mathbf{x} [\psi^+_{\mathbf{x},\omega} Q^{(h)}_{\omega,\omega'}(\mathbf{x}) \phi^-_{\mathbf{x},\omega'} + \phi^+_{\mathbf{x},\omega} Q^{(h-1)}_{\omega,\omega'}(\mathbf{x}) \psi^-_{\mathbf{x},\omega'}]} , \tag{12.5}$$

where $\tilde{\mathcal{B}}^{(h)}$ is equal to $\mathcal{B}^{(h)}$ with $\frac{\partial}{\partial \psi^+_{\omega'}} \mathcal{V}^{(h)}(\sqrt{Z_h}\psi)$ replaced by $\frac{\partial}{\partial \psi^+_{\omega'}} \tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi)$ where $\mathcal{L}\tilde{V}^{(h)} = \mathcal{L}V^{(h)} - s_h F_\sigma - z_h(F_\alpha + F_\zeta)$. Now we rescale the fields

$$e^{-L\beta t_h} \int P_{\tilde{Z}_{h-1}}(d\psi^{\leq h})$$

5.4b
$$e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)+\int d\mathbf{x}[\psi^+_{\mathbf{x},\omega}Q^{(h)}_{\omega,\omega'}(\mathbf{x})\phi^-_{\mathbf{x},\omega'}+\phi^+_{\mathbf{x},\omega}Q^{(h-1)}_{\omega,\omega'}(\mathbf{x})\psi^-_{\mathbf{x},\omega'}]}\ , \tag{12.6}$$

where $\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)=\tilde{\mathcal{B}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)$, we integrate with respect to $\psi^{(h)}$ and the procedure can be iterated.

At the end, after taking the functional derivatives with respect to $\phi^+_{\mathbf{x}},\phi^-_{\mathbf{x}}$ we get an expansion in terms of a new class of trees $\tau\in\mathcal{T}_{n,h,k}$, which are similar to the trees of the effective potential, see Fig. 8, with the following modifications.
(1) There are $n+2$ end-points and to two of them, called v_x,v_y, are associated the following function

$$\int d\mathbf{x}\psi^+_{\mathbf{x},\omega}Q^{(h)}_{\omega,\omega'}(\mathbf{x})\phi^-_{\mathbf{x},\omega'}+\int d\mathbf{x}G^{(h)}_{\omega,\omega'}*\frac{\partial}{\partial\psi^+_{\omega'}}\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi)\phi^-_{\mathbf{x},\omega'}$$

or

$$\int d\mathbf{y}\phi^+_{\mathbf{y},\omega'}Q^{(h)}_{\omega',\omega}(\mathbf{y})\psi^-_{\mathbf{y},\omega}+\int d\mathbf{y}\phi^+_{\omega,\xi}*G^{(h)}_{\omega,\omega'}*\frac{\partial}{\partial\psi^-_{\omega'}}\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi)]$$

(2) Let be v_{xy} the first vertex whose cluster contains both v_x and v_y; its scale is said $\bar{h}$ and no $\mathcal{R}$-operation is defined on the vertices on the line from v_{xy} to the root (this follows from the fact that we have made no decomposition in relevant and irrelevant part in the terms in the second line of (12.2)).
(3)There are no external lines in the root of the tree. The scale of the root is $\bar{k}$.

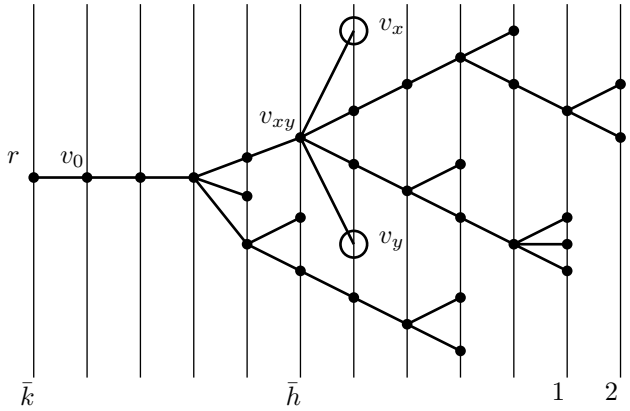


FIG. 16. A tree appearing in the graphic representation of the two-point Schwinger function (see (12.8) below). The two endpoints v_x and v_y are connected both to the vertex v_{xy} on scale $\bar{h}$.

In order to perform the bounds we need some informations on $G^{(h)}$ given by (12.3); it is easy to show that

$$|\tilde{g}^{(k)}*Q^{(k)}|\le\gamma^k\frac{C_N}{1+(\gamma^k|\mathbf{x}-\mathbf{y}|)^N}$$

This simply follows from the fact that the Fourier transform of Q^k is bounded by a constant. In fact by (12.3) and (12.4) we obtain

5.c
$$Q^{(h)}_{\omega,\omega'}=Q^{h+1}_{\omega,\omega'}+z_hZ_h[\partial_t+\varepsilon(i\partial_x))]\sum_{k\ge h+1}\frac{1}{Z_k}g^k_{\omega,\omega''}*Q^{(k)}_{\omega'',\omega'}+s_hZ_h\sum_{k\ge h+1}\frac{1}{Z_k}g^k_{\omega,\omega''}*Q^{(k)}_{\omega'',-\omega'} \tag{12.7}$$

which can be solved by iteration. In bounding the convolution $\tilde{g}^{(k)} * Q^k$ we have to evaluate $\hat{Q}^{(k)}(\mathbf{k})$ only on the support of $\tilde{f}^{(k)}(\mathbf{k})$. Considering the Fourier transform of (12.7) one obtains that only one term survive in the sum in (12.7)

$$\hat{Q}^{(h)}_{\omega,\omega'}(\mathbf{k}) = 1 + z_h Z_h \sum_{\omega''}(-ik_0 + \omega k)\frac{1}{Z_{h+1}} g^{(h+1)}_{\omega,\omega''}(\mathbf{k})\hat{Q}^{(h+1)}_{\omega'',\omega'}(\mathbf{k}) + s_h Z_h \frac{1}{Z_{h+1}}\sum_{\omega''} g^{(h+1)}_{\omega,\omega''}(\mathbf{k}) Q^{h+1}_{\omega'',-\omega'}$$

and by induction one can deduce that $|\hat{Q}^{(h)}_{\omega,\omega'}(\mathbf{k})| \le 1 + O(\varepsilon_h)$. Then one easily obtain the bound

$$|G^{(h)}(\mathbf{x},\upsilon)| \le \frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^{(h)}|\mathbf{x}-\mathbf{y}|)^N}$$

After deriving with respect to $\phi^+_{\mathbf{x}}$, $\phi^-_{\mathbf{y}}$ we obtain

onco
$$S(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'} e^{-ip_F(\omega x-\omega' y)}\sum_{\bar{h}=h^*}^{1}\tilde{g}^{(\bar{h})}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) + \sum_{\bar{h}=h^*}^{1}\sum_{\bar{k}=h^*}^{\bar{h}-1}\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}^{h,k}_n}\bar{S}_{\bar{h},\bar{k},\tau;\omega,\omega'}(\mathbf{x};\mathbf{y}) \tag{12.8}$$

where $\bar{S}_{\bar{h},\bar{k},\tau;\omega,\omega'}(\mathbf{x};\mathbf{y})$ is obtained by the expansion described above after taking the functional derivative. Calling

$$\bar{S}^{(\bar{h})}(\mathbf{x};\mathbf{y}) = \sum_{\bar{k}=h^*}^{\bar{h}-1}\sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}^{h,k}_n}\bar{S}_{\bar{h},\bar{k},\tau;\omega,\omega'}(\mathbf{x};\mathbf{y})$$

we obtain

c1km
$$|\bar{S}^{(\bar{h})}_{\omega,\omega}(\mathbf{x};\mathbf{y})| \le \frac{\gamma^{\bar{h}}}{Z_{\bar{h}}}\frac{C_N}{1+(\gamma^{\bar{h}}|\mathbf{x}-\mathbf{y}|)^N} \tag{12.9}$$

and

c2km
$$|\bar{S}^{(\bar{h})}_{\omega,-\omega}(\mathbf{x};\mathbf{y})| \le \frac{|\sigma_{\bar{h}}|}{\gamma^{\bar{h}}Z_{\bar{h}}}\frac{C_N}{1+(\gamma^{\bar{h}}|\mathbf{x}-\mathbf{y}|)^N}. \tag{12.10}$$

The proof of (12.9),(12.10) is obtained by a modification of the arguments used to bound the effective potential. In fact, as far as the bounds are corncerned, the vertices v_x and v_y are like two ν vertices with an external line (the ϕ line) and an extra $\gamma^{-\bar{h}}/\sqrt{Z_{\bar{h}}}$ factor for each one; moreover no integration over the coordinate is associated to such vertices. Another important difference is that no $\mathcal{R}$ is applied on all the vertices between v_{xy} and the root; the clusters assiciated to such vertices can be at most marginal (by definition to the cluster v_{xy} are external at least the two ϕ fields, so that if $h \neq k$ all the clusters between v_{xy} and v_0 have at most four external lines). To renormalize them we can multiply by $\gamma^{\bar{h}-\bar{k}}\gamma^{\bar{k}-\bar{h}}$; the factor $\gamma^{\bar{k}-\bar{h}}$ is enough to renormalize all the clusters between v_{xy} and v_0.

It is then natural to compare the bounds for $\bar{S}_{h,k,\tau;\omega,\omega}(\mathbf{x};\mathbf{y})$ and the bounds for a contribution to the effective potential with two external lines; with respect to the bound for the effective potential with two external lines (wich gives dimensionally a factor $\gamma^{\bar{k}}$) there is a $\gamma^{2\bar{h}}$ more for the fact that there are two integrations less (one integration kills the factor $L\beta$); moreover there is an extra factor $\frac{\gamma^{-2\bar{h}}}{Z_{\bar{h}}}\gamma^{\bar{h}-\bar{k}}$ by the preceding considerations. Collecting together such factors with a decay factor

12.11
$$\frac{C_N}{1+(\gamma^{\bar{h}}|\mathbf{x}-\mathbf{y}|)^N}, \tag{12.11}$$

which one can extract from the tree connecting v_x with v_y one gets (12.9); moreover (12.10) is obtained taking into account that there is at least a σ vertex.

A similar expansion can be obtained also for the four point Schwinger functions, simply deriving with respect to four ϕ-fields; however such expansion is not suitable for the computation of the response functions and onother one is necessary, see §14.

sec.13

13. Two-point Schwinger functions for spinless fermions

In this section we show how the Renormalization group analysis described above can be used to obtain properties for many systems of spinless fermions in one dimension. For fixing the ideas we have considered till now the model with Hamiltonian (8.1). The reason is that the properties of the other models described in this section can be easily deduced from the discussion of the model (8.1), which is in a sense the most general case, if the fermions are spinless.

p.13.1 **13.1.** *Free fermions.* Let us resume quickly the properties of the two-point Schwinger function for a system of free fermions in the continuum, with Hamiltonian $H = H_0$ given by (2.2). The eigenstates of H_0 can be easily constructed by the solutions of the Schrödinger equation

1.30a
$$-\frac{1}{2m}\frac{\partial^2}{dx^2}\psi(x) = E\psi(x) \ , \tag{13.1}$$

The n-point Schwinger function can be written using the Wick rule in terms of the two-point Schwinger function

mkm
$$g(\mathbf{x}-\mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + \frac{k^2}{2m} - \mu} \tag{13.2}$$

with $\int \frac{d\mathbf{k}}{(2\pi)^2} \equiv \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}$. It can be written as

$$g(\mathbf{x};\mathbf{y}) = -R(\mathbf{x}-\mathbf{y}) + \vartheta(x_0-y_0)e^{(x_0-y_0)p_F^2/2m}e^{-m(x-y)^2/2(x_0-y_0)}\sqrt{\frac{m}{2\pi(x_0-y_0)}},$$

where

$$R(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\int_0^{p_F} dk\cos k(x-y)e^{-\frac{x_0-y_0}{2m}(k^2-p_F^2)}$$

is a smooth function such that

$$\lim_{x_0-y_0\to 0} R(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\frac{\sin p_F(x-y)}{x-y}$$

For large $|\mathbf{x}-\mathbf{y}|$ the free Schwinger function decays as the inverse of $|\mathbf{x}-\mathbf{y}|$ times an oscillating factor. Note also that the function is singular for $\mathbf{x} = \mathbf{y}$.

Similar results can be found for $H = H_0$ in the discrete case (2.2); the large distance asymptotic behaviour of the two-point Schwinger function is the same (but only if $p_F \neq 0, \pi$), but the function is finite for $\mathbf{x} = \mathbf{y}$ (but the time derivative is singular at $\mathbf{x}_0 = \mathbf{y}_0$).

p.13.2 **13.2.** *Non interacting fermions in a periodic potential.* Let us consider a system of fermions in the continuous case subject to a periodic potential, with Hamiltonian $H = H_0 + uP$, with P given by (2.3); we fix $T = 1$ for semplicity.

Also this model is exactly solvable, the eigenstates of H being expressed in terms of the solutions of the Schrödinger equation

aa
$$[-\frac{1}{2m}\frac{\partial^2}{dx^2} + u\varphi(x)]\psi(x) = E\psi(x) \tag{13.3}$$

The theory of the solutions of such equations is rather well developed, see [T]. By making a linear combination with suitable coeffficients of two independent solutions of (13.3) one obtains solutions $\phi(k,x,u)$, called

Floquet solutions, such that $\phi(k, x+1, u) = e^{ika}\phi(k, x, u)$. If k is real they are called *Bloch waves*: they are indiced by the real number k, the *crystalline momentum*, and they verify (13.3) with $E = \varepsilon(k, u)$ which is a continuous function except for $k = n\pi$, n integer where it is generically discontinuous. The values $\Delta_n = \varepsilon((n\pi/a)^+, u) - \varepsilon((n\pi/a)^-, u)$ are called *gaps*; sometimes $\Delta_n = 0$ and in this case one speaks of *closed gaps*. The theory of Bloch waves can be without difficulty adapted to the case of the finite difference Schrödinger equation.

The two-point Schwinger function is given by

mmmm
$$S_0(\mathbf{x}, \mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\phi(k, x, u)\phi(k, -y, u)e^{ik_0(x_0 - y_0)}}{-ik_0 + (\varepsilon(k, u) - \mu)} \tag{13.4}$$

The spectral gap is equal to Δ_n when $p_F = n\pi$ and it is 0 for all the other values of p_F. For small u we have $\Delta_n = c_n u + O(u^2)$ where c_n is the n-th Fourier coefficient of $\varphi(x)$. If $p_F = n\pi$ the system is called *filled band* Fermi system. The asymptotic behaviour for large values of $|\mathbf{x} - \mathbf{y}|$ of the two-point Schwinger function depends critically on the value of the Fermi momentum; it holds, see [BM1] and [BM2].
(1) If $p_F \neq n\pi$ then, for a suitable constant C and $|\mathbf{x} - \mathbf{y}| \geq 1$

z.o
$$|S_0(\mathbf{x}, \mathbf{y})| \leq \frac{C}{|\mathbf{x} - \mathbf{y}|} \tag{13.5}$$

and for small u

$$|S_0(\mathbf{x}; \mathbf{y}) - g(\mathbf{x} - \mathbf{y})| \leq \frac{C|u|}{|\mathbf{x} - \mathbf{y}|}$$

(2) If $p_F = n\pi$ for any $N > 1$ one can find constants C_N such that if $|\mathbf{x} - \mathbf{y}| \geq 1$

z.1
$$|S_0(\mathbf{x}, \mathbf{y})| \leq \frac{1}{|\mathbf{x} - \mathbf{y}|} \frac{C_N}{1 + \Delta_n^N |\mathbf{x} - \mathbf{y}|^N} \tag{13.6}$$

Probably an optimal bound is an exponential one.

Such two cases correspond to the *metallic* or *insulating* phase of the system; in one case the ground state energy has no gap and in the other case it has a gap.

It can be of some interest to have some insights of how (13.5),(13.6), which are true for any value of u, can be derived by the Bloch waves property. A very common technique to obtain similar bounds is to shift the integration domain in the complex plane; this means that a detailed knowledge of the analytic properties of Bloch waves in the complex plane is required. A study of this problem was done in [K], and we resume quickly the results. The function $\varepsilon(k, u)$ as a function of complex k may be represented on a Riemann surface with an infinite sequence of sheets S_n, in such a way that on each S_n for k real one has the value of $\varepsilon(k, u)$ corresponding to the n-th energy band. Each sheet S_n is connected to S_{n+1} by an infinite sequence of branch points of order two given by $k_{2m} = \pm[2(j+1)\pi + ih_{2m}]$ for $j = 0, \pm 1, \ldots$ and by $k_{2m+1} = \pm[2j\pi + ih_{2m+1}]$ for $j = 0, \pm 1, \ldots$; such branch points are closer and closer to the real axis as $\lim_{n\to\infty} h_n = 0$. Then starting on a real value of k on the band n, passing around k_n and returning on the real axis, one is in the band $n+1$. Close to the branch points one has

$$\varepsilon = \varepsilon_n + \beta_n (k - k_n)^{\frac{1}{2}} + o((k - k_n)^{\frac{1}{2}})$$

Analogous properties hold for $\phi(k, x, u)$ with the only difference that the branch points are now of order 4 and close to the branch points can be written as

$$\phi(k, x, u) = \frac{A_n}{(k - k_n)^{\frac{1}{4}}} [1 + B_n (k - k_n)^{\frac{1}{2}} + o((k - k_n)^{\frac{1}{2}})$$

Finally on each S_n the functions are periodic or antiperiodic with period 2π. The functions $\varepsilon(k,u)$, $\phi(k,x,u)$ appearing in (13.4) are the restriction to the real axis of functions defined in the complex plane, with cuts from k_n to $\bar{k}_n$, once that the value corresponding to the segment $(-\pi,\pi)$ is fixed to the value of the first band. Let us return now to the problem of shifting the contour of (13.4); as the singularity are closer and closer to the real axis (unless one chooses some special periodic functions in which h_n is bounded) one can consider a path circunventing the singularities with infinitesimal circles, see [BM1]: one uses that the singularity is integrable and the periodicity properties. Then the estimates (13.5),(13.6) are obtained.

The same results can be obtained in a different way, at least if u is small, without using any property of the solutions of the Schrödinger equation. In fact one can apply the Renormalization group techniques introduced above with $\lambda = 0$, $\phi(x)$ a periodic function and $x \in \mathbb{R}$. The expansions of the preceding sections can be easily adapted (in some sense they become trivial) to the case of $H = H_0 + uP + \nu N_0$. If $\lambda = 0$ all the contributions to the effective potential are bilinear in the fields, so that the definition of localization is given by the analogue of (8.20) but the Kronecker delta is not defined mod. 2π; the running coupling constants are, if $p_F = n\pi$, $\nu_h, \sigma_h, z_h, \alpha_h$; if $p_F \neq n\pi$ they are the same but $\sigma_h = 0$. As the interaction is bilinear in the field a bound on each Feynman graph is enough to prove the convergence and there is no need of Gram-Hadamard bounds; moreover there is no small divisor problems.

In the *filled band case* $p_F = n\pi$ one can choose $\nu = 0$; this follows noting that, from (10.13), $\nu_h = \gamma^{-h+1}[\sum_{k=0}^{h+1}\gamma^{k-2}\beta_\nu^{(k)}]$ and $|\beta_\nu^{(k)}| \le C|u|^2$ and $\gamma^{-h} \le C|u|^{-1}$. It is easy to show that the running coupling constants $\vec{v}_h$ remain close $O(u^2)$ to their values at $h = 0$. From (12.8) the infrared part of the Schwinger function is given by

4.33jjjj
$$S_0(\mathbf{x},\mathbf{y}) = \sum_{h=h^*}^{1} g^{(h)}(\mathbf{x};\mathbf{y}) + u\sum_{h=h^*}^{1}\bar{S}^{(h)}(\mathbf{x};\mathbf{y}) \tag{13.7}$$

where

$$g^{(h)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm 1} e^{-i(\omega x-\omega' y)p_F} g^{(h)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})$$

and

$$|\bar{S}^{(h)}(\mathbf{x};\mathbf{y})| \le \gamma^h \frac{C_N}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^N}$$

Remember that $h^* = O(\log(|c_n u|)^{-1})$ and let be $\sigma = |c_n u|$ (we are assuming that $c_n \neq 0$). If $1 \le |\mathbf{x}-\mathbf{y}| \le (2\sigma)^{-1}$ and $h_x \ge h^*$ is such that $\gamma^{-h_x-1} < |\mathbf{x}-\mathbf{y}| \le \gamma^{-h_x}$, (13.7) gives, if $N > 1$,

4.35jjjj
$$\Big|S_0(\mathbf{x};\mathbf{y})\Big| \le C_N\sum_{h=h^*}^{h_x-1}\gamma^h + \sum_{h=h_x}^{1}\frac{C_N\gamma^h}{\gamma^{Nh}|\mathbf{x}-\mathbf{y}|^N} \le \gamma^{h_x}C_N \le \frac{C_N}{1+|\mathbf{x}-\mathbf{y}|}\ . \tag{13.8}$$

On the other hand, if $|\mathbf{x}-\mathbf{y}| \ge (2\sigma)^{-1}$, (13.7) implies that

4.34jjjj
$$\Big|S_0(\mathbf{x};\mathbf{y})\Big| \le \frac{C_N}{|\mathbf{x}-\mathbf{y}|^N}\sum_{h=h^*}^{1}\gamma^{-(N-1)h} \le \frac{C_N}{|\mathbf{x}-\mathbf{y}|^N}|\sigma|^{-N+1} \le \frac{C_N|\sigma|}{1+|\sigma|^N|\mathbf{x}-\mathbf{y}|^N}\ , \tag{13.9}$$

provided that $N > 1$. By a sligth refinement of the above bounds one obtains (13.6).

Moreover $\sum_{h=h^*}^{0} g^{(h)}(\mathbf{x};\mathbf{y})$ in (13.7) can be written, in the $L,\beta \to \infty$ as

CCM
$$\int \frac{d\mathbf{k}}{(2\pi)^2} f^{(i.r.)}(\mathbf{k})\,\tilde{\phi}(k,x,u)\,\tilde{\phi}(k,-y,u)\,\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\tilde{\varepsilon}(k,u)} \tag{13.10}$$

where $f^{(i.r.)}(\mathbf{k})$ is the numerator of the r.h.s of the third of (5.4), $\tilde{\phi}(k,x,u) = e^{-ikx}\tilde{u}(k,x,u)$ and

4.29b
$$\tilde{u}(k,x,u) = e^{i\text{sign}\,(k)p_F x}\left[\cos(p_F x)\sqrt{1-\frac{\text{sign}\,(|k|-p_F)\sigma}{\sqrt{((|k|-p_F)v_0)^2+\sigma^2}}}\right. \tag{13.11}$$

$$-i\,\mathrm{sign}\,(k)\sin(p_F x)\sqrt{1+\frac{\mathrm{sign}\,(|k|-p_F)\sigma}{\sqrt{((|k|-p_F)v_0)^2+\sigma^2}}}\;\Bigg]$$

4.29c
$$\tilde{\varepsilon}(k,u)=(|k|-p_F)^2/2m+\mathrm{sign}\,(|k|-p_F)\sqrt{v_0(|k|-p_F)^2+\sigma^2} \tag{13.12}$$

and $v_0 = p_F/m$. The first order term of (13.7) is very similar to (13.4), and $\tilde{\phi}(k,x,u)$ are just the Bloch waves computed at the first order by degenerate perturbation theory.

In the *not filled band* case $p_F \neq n\pi$ then $\sigma_h = 0$ so that the propagator is diagonal and by choosing a suitable ν_0

4.33pp
$$S_0(\mathbf{x};\mathbf{y})=\sum_{h=-\infty}^{1} g^{(h)}(\mathbf{x};\mathbf{y})+u\sum_{h=-\infty}^{1}\bar{S}^{(h)}(\mathbf{x};\mathbf{y}) \tag{13.13}$$

where

$$g^{(h)}(\mathbf{x};\mathbf{y})=\sum_{\omega,\omega'=\pm 1} e^{-i\omega(x-y)p_F} g^h_\omega(\mathbf{x};\mathbf{y})$$

where $g^{(h)}_\omega(\mathbf{x};\mathbf{y})$ is given by the analogous of (8.50) with $\sigma_h = 0$ and $|\bar{S}^{(h)}(\mathbf{x};\mathbf{y})| \le \gamma^h \frac{C_N}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^N}$. It holds if h_x is such that $\gamma^{-h_x-1} < |\mathbf{x}-\mathbf{y}| \le \gamma^{-h_x}$, (13.13) gives, if $N > 1$,

13.14
$$\left|S_0^{i.r.}(\mathbf{x};\mathbf{y})\right| \le C_N\sum_{h=-\infty}^{h_x-1}\gamma^h+\sum_{h=h_x}^{1}\frac{C_N\gamma^h}{\gamma^{Nh}|\mathbf{x}-\mathbf{y}|^N}\le \gamma^{h_x}C_N \le \frac{C_N}{1+|\mathbf{x}-\mathbf{y}|}\,. \tag{13.14}$$

A decomposition analogue to (13.7) holds. Note that if the occupation number is defined with respect to the "Bloch waves" then it is of course (tautologically) discontinuous. If we consider the occupation number with respect to the plane waves, considering the Fourier transform of the Schwinger function, there is no discontinuity, in the filled band case, while it is discontinuous in the not filled band case.

Finally one can consider fermions on a *lattice* with Hamiltonian $H = H_0 + uP + \nu N_0$ with P given by (2.4) and $\frac{p}{\pi}$ a rational number and one can prove bounds similar to (13.5),(13.6).

p.13.3 **13.3.** *Noninteracting fermions in a quasi-periodic potential.* We consider now a less trivial case in which the periodic potential is replaced by a quasi-periodic one; more exactly we prefer to study the essentially equivalent case of noninteracting fermions on a lattice with an incommensurate potential. As in the commensurate case such problem could be studied by analyzing the spectrum of the finite difference Schrödinger equation

1.29a
$$-\psi(x+1)-\psi(x-1)+u\varphi(x)\psi(x)=E\psi(x)\;, \tag{13.15}$$

where $\varphi(x)$ is defined as in §8 and p/π is irrational. In (13.15) there are two periods, the one of the potential and the intrinsic one of the lattice, and this makes the properties of (13.15) and of the continuous Schrödinger equantion with a quasi-periodic potential very similar. The eigenfunctions and the spectrum strongly depend on u, contrary to the case of periodic potential, in which the eigenfunctions are always Bloch waves whenever u is large or small. On the contrary in this case for large u there are eigenfunctions with an exponential decay for large distances; this phenomenon is called *Anderson localization* (for details, see for instance [PF], [BLT] and [S]) while for small u there are eigenfunctions which are *quasi-Bloch waves* of the form $e^{ik(E)x}\,u(x)$ with $u(x) = \bar{u}(px)$ for (13.15), $\bar{u}$ being 2π-periodic in its arguments. This is proved by using KAM techniques (see [DS], [E], [BLT], [JM] and [MP]), if p verifies a Diophantine condition, *i.e.* $||np||_{\mathbb{T}^1} \ge C_0|n|^{-\tau}$ for any $n \neq 0$ and with the additional condition that, if $k(E) \equiv k$, then
(a) k is such that $||k+np||_{\mathbb{T}^1} \ge C_0|n|^{-\tau}\ \forall n \in \mathbb{Z}\setminus\{(0)\}$,
(b) or $k = \bar{n}p$, $\bar{n} \in \mathbb{N}$.

Of course such two cases do not cover all the possible k.

Probably one can get the asymptotic behaviour of the Schwinger functions just by studying the properties of the solutions of the Schrödinger equation, as it was done for the periodic potential case; this result is

however missing in the literature. On the other hand it is possible to obtain the Schwinger functions writing them as Grassman integrals using the methods seen in the previous sections. We consider a model on a lattice with Hamiltonian

nop
$$H = H_0 + uP + \nu N_0 \tag{13.16}$$

with P given by (2.4) and p/π irrational.

Small u case. We start by the case in which the incommensurate potential is weak with respect to the kinetic energy. It is natural to distinguish the case $p_F = np$, which is analogous of the filled band case in the commensurate case, from the case $p_F \neq np$. However if we assume simply $p_F \neq np$ one cannot prove the convergence of the series, due to the small divisor problem, see §8; one needs a stronger condition, namely that $||p_F + np||_{\mathbb{T}^1} \geq C_0|n|^{-\tau}, \forall n \in \mathbb{Z} \setminus \{(0)\}$. Note that the condition $p_F = \bar{n}p \bmod 2\pi$ can be verified by a finite number of $\bar{n}$ if p/π is a rational number, but by an infinite number in the irrational case. In other words the values of p_F in $(-\pi, \pi)$ such that the system has a gap in the ground state form a *dense set*. In order to perform a rigorous analysis we have to consider L finite with periodic boundary conditions; in this way the Grassman algebra is finite dimensional and the Grassman integral are well defined. This means that we cannot choose a p/π given by an irrational number, but we have to consider a sequence of rational numbers converging uniformly to a diophantine one as the volume tends to infinity.

One can prove the following theorem, see [BGM1].

THEOREM 2. *Let us consider the Hamiltonian (13.16) with $\nu = 0$ and a sequence L_i, $i \in \mathbb{Z}^+$, such that*

$$\lim_{i\to\infty} L_i = \infty\,, \qquad \lim_{i\to\infty} p_{L_i} = p\,,$$

Suppose also that there is a positive integer $\bar{n}$ such that $p_F = \bar{n}p_{L_i} \,(\mathrm{mod}\, 2\pi)$, $\hat{\varphi}_{\bar{n}} \neq 0$, p_{L_i} satisfies the diophantine condition

1.19
$$\|2np_{L_i}\|_{\mathbb{T}^1} \geq C_0|n|^{-\tau}\,, \qquad \forall n \in \mathbb{Z} \setminus \{(0)\} \qquad |n| \leq \frac{L_i}{2}, \tag{13.17}$$

for some positive constants C_0 and τ independent of i. Set $\sigma = |u\hat{\phi}_{\bar{n}}|$. Then there exists $\varepsilon_0 > 0$, such that, if $|u| \leq \varepsilon_0$ in the limit $i \to \infty, \beta \to \infty$ for any $N > 1$ there is a constant C_N, such that

1.23uu
$$|S(\mathbf{x};\mathbf{y})| \leq \frac{1}{1+|\mathbf{x}-\mathbf{y}|}\frac{C_N}{1+(|\sigma|\,|\mathbf{x}-\mathbf{y}|)^N}\,, \tag{13.18}$$

Moreover, for $1 \leq |\mathbf{x}-\mathbf{y}| \leq |\sigma|^{-1}$

$$S(\mathbf{x},\mathbf{y}) = g(\mathbf{x}-\mathbf{y}) + C_2(\mathbf{x},\mathbf{y})$$

where $g(\mathbf{x}-\mathbf{y})$ is given by (3.4) and

$$|C_2(\mathbf{x};\mathbf{y})| \leq C\frac{\sqrt{|\sigma||\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$$

for a suitable constant C. For any i, there is a spectral gap $D \geq |\sigma|/2$ around μ_0.

The above results can be also proved specializing the analysis on the Hubbard-Holstein model to the case $\lambda = 0$. The existence of the sequence of L_i is proved in [BGM1] by choosing them as the denominators of the *best approximants*. A decomposition of the Schwinger function given by (13.7) and (13.11) holds so that the above theorem say that, for small u, the Schwinger function behavior for $p_F = \bar{n}p$ if p is rational or diophantine is essentially the same; the crucial difference is that in one case there is a finite number of p_F of the form $\bar{n}p$ while in the second case there is a dense set.

It is also possible to prove the following result.

THEOREM 3. *Let us consider the Hamiltonian (13.16) and a sequence L_i, $i \in \mathbb{Z}^+$, such that $\lim_{i\to\infty} L_i = \infty, \lim_{i\to\infty} p_{L_i} = p$, if p_{L_i} satisfies the diophantine condition (13.17) and*

1.26*xxg*
$$\|p_{F,L_i} + np_{L_i}\|_{\mathbb{T}^1} \geq C_0|n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{(0)\}, \qquad |n| \leq \frac{L_i}{2}, \tag{13.19}$$

with the same positive constants C_0 and τ. Then there exist $\varepsilon > 0$ such that, for $|u| \leq \varepsilon$, there exists a function $\nu = \nu(u)$ such that

1.28*ppp*
$$|S(\mathbf{x};\mathbf{y})| \leq \frac{C}{1+|\mathbf{x}-\mathbf{y}|} , \tag{13.20}$$

for some constant C. Moreover

$$S(\mathbf{x},\mathbf{y}) = g(\mathbf{x}-\mathbf{y}) + uC_3(\mathbf{x},\mathbf{y}),$$

where $g(\mathbf{x}-\mathbf{y})$ is given by (3.4) and $C_3(\mathbf{x};\mathbf{y})$ verifies the same bound as (13.20).

The proof follows the lines of preceding sections; in fact Lemma 1 is still valid if one assumes (13.19) instead of $p_F = \bar{n}p$. By the definition of localization, see (8.19), (8.20) and (8.21), one gets $\sigma_h = 0$ for any h. However the construction of a sequence of L_i, p_{F,L_i}, p_{L_i} verifying (13.19) seems to be much more involved and it is until no construction has been exhibited (but we think that this is only a technical problem).

In any case, contrary to the commensurate case, the results obtained are not for all the possible values of p_F; the behaviour of the system for p_F neither verifying $p_F = \bar{n}p$ or $|p_F + np\|_{\mathbb{T}^1} \geq C_0|n|^{-\tau}$, $\forall n \in \mathbb{Z} \setminus \{(0)\}$ is an open problem; likewise it is not known what happens if p is neither rational or Diophantine.

Large u case. We have seen that the asymptotic behaviour of the Schwinger functions for fermions both with an external commensurate or incommensurate potential in the small u case are similar, at least if proper diophantine conditions are imposed on the Fermi momentum. Such similarity is completely lost in the *large u* case. In this case from the study of the Schrödinger equation we expect, see for istance [PF], the phenomenon of *Anderson localization* (an exponential decay of correlation functions which is not due to the presence of a gap in the spectrum and delocalized states, but due to the fact that the states are exponentially localized). Again we write the Schwinger functions as Grassman integrals; however we develop in series of $\varepsilon = \frac{1}{u}$, considering H_0 as the perturbation. In other words we write

1.2*rr*
$$\begin{aligned} H &= \bar{H}_0 + \bar{V} , \\ \bar{H}_0 &= \sum_{x\in\Lambda} (\mu - \varphi(x))\, \psi^+_x \psi^-_x , \\ \bar{V} &= -\frac{\varepsilon}{2} \sum_{x\in\Lambda} \left[\psi^+_x \psi^-_{x+1} + \psi^+_x \psi^-_{x-1} - 2\psi^+_x \psi^-_x\right] + \nu \sum_{x\in\Lambda} \psi^+_x \psi^-_x , \end{aligned} \tag{13.21}$$

with $\bar{V} \equiv H_0$, $\bar{H}_0 = P$. The $\varepsilon = 0$ Schwinger function is given by

2.4*rr*
$$\hat{g}(x,y;k_0) = \int_{-\beta/2}^{\beta/2} d\tau\, e^{ik_0\tau} g(x,y;\tau) = \delta_{x,y}\hat{g}(x,k_0) \equiv \frac{\delta_{x,y}}{-ik_0 - \varphi(x) + \mu} . \tag{13.22}$$

So one can see the analogy with the small u case; the two propagators are the same replacing x with k and $\varphi(x) - \mu$ with $E(k)$. If $\varphi(x)$ is even one can introduce quasi-particles and one can apply RG methods similar to the one discussed for the small u case. Then in [GM2] the following theorem is proven.

THEOREM 4. *Let us consider the Hamiltonian (13.21) and let be $\varphi(x) = \bar{\varphi}(\omega x)$ an even periodic function in its argument, $\varphi(x) = \varphi(-x)$, $\bar{\varphi}(x) = \bar{\varphi}(x+1)$, and with ω verifying a Diophantine condition*

1.5*ddd*
$$\|\omega n\|_{\mathbb{T}} \geq C_0|n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} , \tag{13.23}$$

for some constants $\tau > 1$ and $C_0 > 0$. Let us define $\bar\omega \equiv \omega\bar x$ such that $\mu = \bar\varphi(\bar\omega)$ and assume that there is only one $\bar x \in (0, 1/2)$ satisfying such a condition and that $\bar\varphi'(\bar\omega) \neq 0$ (the prime denotes derivative with respect the argument).

Then there exists $\varepsilon_0 > 0$, depending on ω and $\bar\omega$, and, for $|\varepsilon| < \varepsilon_0$, a function $\nu \equiv \nu(\varepsilon) \neq 0$, such that (1) if $\bar\omega \notin \omega\mathbb{Z}$ mod 1 and the additional Diophantine condition

1.6ddd
$$\|\omega n \pm 2\bar\omega\|_{\mathbb{T}} \geq C_0 |n|^{-\tau} , \qquad \forall n \in \mathbb{Z} \setminus \{0\} , \tag{13.24}$$

is verified, then $S(\mathbf{x};\mathbf{y})$ is bounded by

1.7ddd
$$|S(\mathbf{x};\mathbf{y})| \leq \log\left(1 + \min\{|x|,|y|\}\right)^{\tau} \frac{C_N \exp\left\{-4^{-1}|x-y| \log\left|\varepsilon^{-1}\right|\right\}}{1 + \left[(1+\min\{|x|,|y|\})^{-\tau}|x_0 - y_0|\right]^N} , \tag{13.25}$$

for any $N \geq 1$ and for some constant C_N depending on N;
(2) if $2\bar\omega = (2k+1)\omega$ mod 1, $k \in \mathbb{N}$, then, for $\alpha = 2(k+1)$ and for some constant C'_N depending on N,

1.8ddd
$$|S(\mathbf{x},\mathbf{y})| \leq \log\max\{(1+\min\{|x|,|y|\})^{-\tau}, \sigma\} \frac{C'_N \exp\left\{-(4\alpha)^{-1}|x-y| \log\left|\varepsilon^{-1}\right|\right\}}{1 + \left[\max\{(1+\min\{|x|,|y|\})^{-\tau}, \sigma\}|x_0 - y_0|\right]^N} , \tag{13.26}$$

with $0 \leq \sigma \leq C\,|\varepsilon|^{\eta(k)}$, where

1.9ddd
$$\eta(k) = \begin{cases} (2k+1)/4 , & k > 1 , \\ 1 , & k = 1 , \end{cases} \tag{13.27}$$

for some constant C.

We see, under suitale conditions, that the Schwinger functions decay exponentially fast. The faster than any power decay in the small u case was due to the presence of a gap in the spectrum, while in the large u case is due the localization of the eigenstates. In the first case the decay rate is order $O(u\hat\phi_m)$, if $p_F = mp$ while in the second case is $O(\log u^{-1})$.

p.13.4 **13.4.** *Interacting spinless fermions.* The case of spinless fermions interacting only through a two-body potential was studied in [BGPS] in the continuum and in [BGL] in the lattice and can be derived by the considerations in the preceeding sections putting $u = 0$ so that $\sigma_h = 0$ for any h, the propagator becomes diagonal in the ω-indices and $h^* = -\infty$. In this case an expansion in Feynman diagrams does not lead to a convergent expansion and one has to bound directly the truncated expectation, as explained in the previous sections. By §10 it follows that, for a suitable $\nu = \nu(\lambda)$, $\lambda_h, \delta_h, \nu_h$ converge to a non trivial fixed point laying in the analiticity domain of the expansion; moreover, as $\lim_{h\to-\infty} Z_{h-1}/Z_h = \gamma^\eta$, with $\eta = a\lambda^2 + O(\lambda^3)$ (the rules for computing it at every order were explained in the preceeding sections) we can write, from §12,

4.33cc
$$S(\mathbf{x};\mathbf{y}) = \sum_{h=-\infty}^{1} \frac{g^{(h)}(\mathbf{x};\mathbf{y})}{Z_h} + \lambda \sum_{h=-\infty}^{1} \bar S^{(h)}(\mathbf{x};\mathbf{y}) \tag{13.28}$$

with

$$\left|\bar S^{(h)}(\mathbf{x};\mathbf{y})\right| \leq \frac{\gamma^h}{Z_h} \frac{C_N}{1 + (\gamma^h|\mathbf{x}-\mathbf{y}|)^N}.$$

If h_x is such that $\gamma^{-h_x-1} < |\mathbf{x}-\mathbf{y}| \leq \gamma^{-h_x}$, (13.28) gives, if $N > 1$,

4.35cc
$$\sum_{h=-\infty}^{1} |\bar S^{(h)}(\mathbf{x};\mathbf{y})| \leq C_N \sum_{h=h^*}^{h_x-1} \frac{\gamma^h}{Z_h} + \sum_{h=h_x}^{1} \frac{1}{Z_h} \frac{C_N \gamma^h}{\gamma^{Nh}|\mathbf{x}-\mathbf{y}|^N} \leq \gamma^{h_x} C_N \leq \frac{C_N}{1 + |\mathbf{x}-\mathbf{y}|^{1+\eta}} . \tag{13.29}$$

At the end the following result can be proved, see [BGPS].

THEOREM 5. *Given a Hamiltonian of the form $H = H_0 + \lambda V + \nu N_0$, with H_0, N_0, V given by (2.1) and (2.5), for spinless fermions on the continuum, one can find a $\varepsilon > 0$ such that, for $|\lambda| \le \varepsilon$, there are functions $\nu = \nu(\lambda)$, $\eta = \eta(\lambda)$ such that the two-point Schwinger function of H is given by*

mkm1
$$S(\mathbf{x};\mathbf{y}) = \frac{g(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\eta}} + \frac{\lambda A_\lambda(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{1+\eta}} \tag{13.30}$$

where $g(\mathbf{x}-\mathbf{y})$ is given by (13.2), $A_\lambda(\mathbf{x};\mathbf{y})$ is bounded by a constant, $\nu = O(\lambda)$ and $\eta = a\lambda^2 + O(\lambda^3)$, with $a > 0$.

It is natural to compare (13.30) with the large distances asymptotic behaviour of the Luttinger model (11.1); from [BeGM] and [ML] the sum of the $\omega = 1$ and $\omega = -1$ Luttinger model two point Schwinger functions is given by

$$\sum_{\omega=\pm 1} \frac{1}{2\pi} \frac{e^{i\omega p_F(x-y)}}{i\omega(x-y) + v_0^*(x_0-y_0)} \frac{1+\lambda A_\omega(\lambda)}{(x^2+y^2+(v_0^*(x_0-y_0))^2)^{\eta'}},$$

with $A_\omega(\lambda)$ is bounded by a constant and v_0^*, η' are suitable functions of λ, see [BeGM] for the explicit expression. The similarity of the above equations with (13.30) is clear, but there are also some differences: for istance p_F is changed by the interaction, while it is not changed in the Luttinger model. Moreover the behaviour for small $\mathbf{x}-\mathbf{y}$ is completely different in the two models, the dependence from p_F is much more complicate in the function $S(\mathbf{x};\mathbf{y})$ given by (13.30) than in its analogue for the Luttinger model, and so on. A more exhaustive comparison between the Luttinger model correlation functions and the one we are considering is, in the case of the density-density one, in §16 below.

An analogous theorem can be proven if the fermions are on a lattice, see [BGL]. In this case however ε_0 is proportional to $(\sin p_F)^\alpha$, if α is a positive integer, so ε_0 is vanishing for $p_F \to 0$ or $p_F \to \pi$.

p.13.5 **13.5.** *Interacting spinless fermions with a periodic potential in the not filled band case.* A result very similar to Theorem 5 holds adding to the Hamiltonian a periodic potential, in the not filled band case. The result is valid for small λ and any u and is found by realizing a Renormalization group analysis similar to the one see in §8 performing a multiscale decomposition not on $g(\mathbf{x}-\mathbf{y})$ but on $S_0(\mathbf{x};\mathbf{y})$; this means that we are considering as free hamiltonian not H_0 but $H_0 + uP$. The analysis uses many properties of the Bloch waves found in [K]. Note that $|\lambda| \le \varepsilon$ and ε is proportional to $\frac{\partial\varepsilon(k,u)}{\partial k}|_{k=p_F}$ which is vanishing for $k = n\pi$. In [BM1] it is proved the following result.

THEOREM 6. *Given a Hamiltonian of the form $H = H_0 + \lambda V + uP + \nu N_0$, with H_0, N_0, V given by (2.1) and (2.5), for spinless fermions on the continuum with $p_F \neq n\pi$, $n \in \mathbb{N}$, one can find a $\varepsilon > 0$ such that, for $|\lambda| \le \varepsilon$, there are functions $\nu = \nu(\lambda)$, $\eta = \eta(\lambda)$ such that the two point Schwinger function of H is given by*

mkm11
$$S(\mathbf{x};\mathbf{y}) = \frac{S_0(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\eta}} + \frac{\lambda A_\lambda(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{1+\eta}}\ , \tag{13.31}$$

where $S_0(\mathbf{x};\mathbf{y})$ is given by (13.4), $A_\lambda(\mathbf{x};\mathbf{y})$ is bounded by a constant, $\nu = O(\lambda)$ and $\eta = a\lambda^2 + O(\lambda^3)$, with $a > 0$.

p.13.6 **13.6.** *Interacting spinless fermions with a periodic potential in the filled band case.* Let us consider the Hamiltonian on the continuum

jkj
$$H = H_0 + \lambda V + uP + \nu N_0 \tag{13.32}$$

in filled band case $p_F = \bar{n}\pi$, $\bar{n} \in \mathbb{N}^+$. An analysis similar to the one for the Holstein-Hubbard model can be performed and the following result holds that, see [BM2] and [BM3].

THEOREM 7 *Given the Hamiltonian (13.32), assume $p_F = \bar{n}\pi$ and$\hat{\varphi}_{\bar{n}} \neq 0$. There exists $\varepsilon > 0$ and functions $\nu \equiv \nu(\lambda, u)$, $\eta_2 \equiv \eta_2(\lambda, u)$ and $\eta_3 \equiv \eta_3(\lambda, u)$, continuous for $|u|, |\lambda| \leq \varepsilon$ such that $\nu = O(\lambda)$ and $\eta_3 = \beta_1\lambda^2 + \lambda^2 O(\lambda, u, \hat{u})$, $\eta_2 = \beta_2\lambda + |\lambda| O(\lambda, u, \hat{u})$, with β_1, β_2 positive generically non vanishing constants, and such that the Schwinger function, if $|\mathbf{x} - \mathbf{y}| \geq 1$ and for any positive N, satisfies*

z.1jjj
$$|S(\mathbf{x}, \mathbf{y})| \leq \frac{1}{|\mathbf{x} - \mathbf{y}|^{1+\eta_3}} \frac{C_N}{1 + (|\Delta|\,|\mathbf{x} - \mathbf{y}|)^N} \tag{13.33}$$

if C_N is a suitable constant and

z.3oo
$$\Delta = |u\hat{\varphi}_{\bar{n}}|^{1+\eta_2}. \tag{13.34}$$

Moreover, for $1 \leq |\mathbf{x} - \mathbf{y}| \leq \Delta^{-1}$

$$S(\mathbf{x}, \mathbf{y}) = \frac{1}{|\mathbf{x} - \mathbf{y}|^{\eta_3}} [g(\mathbf{x} - \mathbf{y}) + C_2(\mathbf{x}, \mathbf{y})]$$

where $g(\mathbf{x} - \mathbf{y})$ is given by (13.2) and

$$|C_2(\mathbf{x}; \mathbf{y})| \leq C \frac{|\lambda| + \sqrt{\Delta |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}$$

for a suitable constant C.

Moreover there is a spectral gap D verifying

z.4
$$D \geq \frac{\Delta}{2} \tag{13.35}$$

Let us state some physical consequences of this theorem. There is a non vanishing spectral gap also in presence of an interaction but, at least if the interaction is attractive ($\lambda < 0$) and $u \ll e^{-\kappa_1 |\lambda|}$, it is strongly renormalized by the interaction as the ratio between the *bare gap* and the *dressed gap* is $\ll 1$ for small u and vanishing as $u \to 0$.

The two-point Schwinger function can be written as

1.21koko
$$S(\mathbf{x}; \mathbf{y}) = S_1(\mathbf{x}; \mathbf{y}) + O(\lambda, u, \hat{u}) S_2(\mathbf{x}; \mathbf{y}) \,, \tag{13.36}$$

where, looking at (12.8) replacing the h dependence with a momentum dependence

$$\int \frac{d\mathbf{k}}{(2\pi)^2} \frac{1}{Z(k)} f^{(i.r.)}(\mathbf{k})\, \tilde{\phi}(k, x, \hat{u}(k))\, \tilde{\phi}(k, -y, \hat{u}(k))\, \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + \tilde{\varepsilon}(k, \hat{u}(k))}$$

with $\tilde{\phi}(k, x, \hat{u}(k)) = e^{-ikx} \tilde{u}(k, x, \hat{u}(k))$ and

4.29bkkk
$$\tilde{u}(k, x, \hat{u}(k)) = e^{i \operatorname{sign}(k) p_F x} \left[\cos(p_F x) \sqrt{1 - \frac{\operatorname{sign}(|k| - p_F)\hat{u}(k)}{\sqrt{((|k| - p_F)v_0)^2 + \hat{u}(k)^2}}} \right. \tag{13.37}$$

$$\left. -i \operatorname{sign}(k) \sin(p_F x) \sqrt{1 + \frac{\operatorname{sign}(|k| - p_F)\hat{u}(k)}{\sqrt{((|k| - p_F)v_0)^2 + \hat{u}(k)^2}}} \right]$$

4.29ckkk
$$\tilde{\varepsilon}(k,\hat{u}(k)) = (|k|-p_F)^2/2m + \mathrm{sign}\,(|k|-p_F)\sqrt{v_0^2(|k|-p_F)^2+\hat{u}(k)^2} \tag{13.38}$$

with, if $p_F = \bar{n}p$ and $\hat{u}(k), \hat{Z}(k)$ are two bounded functions such that $|\hat{u}(k)-u| = O(u\lambda)|$, $|\hat{Z}(k)^{-1}-1| = O(\lambda)$ for $||k|-p_F| > \frac{p_F}{2}$, and

$$\hat{u}(p_F) = |u\hat{\phi}_{\bar{n}}|^{1+\eta_2} \quad \hat{Z}(p_F) = |u\hat{\phi}_{\bar{n}}|^{-\eta_3(1+\eta_2)}$$

Finally $S_1(\mathbf{x},\mathbf{y}), S_2(\mathbf{x},\mathbf{y})$ obeys to the same bound (13.33).

It is natural to compare (13.36) with (13.10), valid in the $\lambda = 0$ case; one could describe the result saying that the particles near the Fermi momentum are still Bloch waves, but dressed and of the form $\phi(\mathbf{k},\mathbf{x},\hat{u}(k))/\sqrt{\hat{Z}(k)}$, *i.e.* a sort of *interacting Bloch waves*. This extra momentum dependence is natural as we expect that the interaction changes the one-particle wavefunctions mainly for momenta near the Fermi surface. One can expect then that the spectral gap, which in the non interacting $\lambda = 0$ case is $O(u)$, is deeply renormalized by the interaction between electrons becoming $O(u^{1+\eta_2})$, so becoming much larger or much smaller, if $u \ll e^{-\kappa_1|\lambda|}$, depending on the attractive or repulsive nature of the interaction.

Similar results can be found if the fermions are on a lattice with a commensurate potential.

p.13.7 **13.7.** *Interacting spinless fermions with a incommensurate potential.* This is the Holstein-Hubbard model we discussed in §8. It is a model for interacting fermions on a lattice with a incommensurate potential. In the physical literature such systems are studied in connection with the so called *quasi-crystals*, see for instance [VMG]. It holds the following theorem, see [M1].

THEOREM 8. *Let us consider the Hamiltonian (8.1) and a sequence L_i, $i \in \mathbb{Z}^+$, such that*

$$\lim_{i\to\infty} L_i = \infty\,, \qquad \lim_{i\to\infty} p_{L_i} = p\,.$$

Suppose also that there is a positive integer $\bar{n}$ such that $p_F = \bar{n}p_{L_i} \,(\mathrm{mod}\, 2\pi)$, $\hat{\varphi}_{\bar{n}} \neq 0$, p_{L_i} satisfies the diophantine condition

13.39
$$\|2np_{L_i}\|_{\mathbb{T}^1} \geq C_0|n|^{-\tau}\,, \qquad 0 \neq n \in \mathbb{Z} \qquad |n| \leq \frac{L_i}{2}, \tag{13.39}$$

for some positive constants C_0 and τ independent of i. There exist $\varepsilon > 0$ and functions $\nu \equiv \nu(\lambda,u)$, $\eta_2 \equiv \eta_2(\lambda,u)$ and $\eta_3 = \eta_3(\lambda,u)$, continuous for $|u|, |\lambda| \leq \varepsilon$ and such that $\nu = O(\lambda)$, $\eta_3 = \beta_1\lambda^2 + \lambda^2 O(\lambda,u,\hat{u})$, $\eta_2 = \beta_2\lambda + |\lambda| O(\lambda,u,\hat{u})$, with β_1, β_2 positive generically non vanishing constants, and such that the Schwinger function, if $|\mathbf{x}-\mathbf{y}| \geq 1$ and for any positive N, satisfies

z.1jjj1
$$|S(\mathbf{x},\mathbf{y})| \leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{1+\eta_3}} \frac{C_N}{1+(\Delta\,|\mathbf{x}-\mathbf{y}|)^N} \tag{13.40}$$

if

z.3oo1
$$\Delta = |u\hat{\varphi}_{\bar{n}}|^{1+\eta_2} \tag{13.41}$$

and C_N is a constant. Moreover, for $1 \leq |\mathbf{x}-\mathbf{y}| \leq \Delta^{-1}$

$$S(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^{\eta_3}}[g(\mathbf{x};\mathbf{y}) + C_2(\mathbf{x},\mathbf{y})]$$

where $g(\mathbf{x};\mathbf{y})$ is given by (3.4) and

$$|C_2(\mathbf{x};\mathbf{y})| \leq C\frac{|\lambda| + \sqrt{\Delta|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$$

for a suitable constant C. Moreover there is a spectral gap D verifying

z.41
$$D \geq \frac{\Delta}{2} \tag{13.42}$$

The considerations done in the preceding sections hold also in this case; in same sense the quasi-Bloch waves are replaced by interacting quasi-Bloch waves.

Finally the analogous of Theorem 3 for the interacting case holds.

THEOREM 9. *Let us consider the Hamiltonian (13.16) and a sequence L_i, $i \in \mathbb{Z}^+$, such that $\lim_{i\to\infty} L_i = \infty, \lim_{i\to\infty} p_{L_i} = p$, if p_{L_i} satisfies the diophantine condition (13.39) and (13.19) then there exist an ε such that, for $|\lambda|, |u| \le \varepsilon$ there are functions $\eta(\lambda, u), \nu(\lambda, u)$ with $\eta(\lambda, u) = \beta_1\lambda^2 + \lambda^2 O(\lambda, u)$ such that*

mkm1m
$$S(\mathbf{x};\mathbf{y}) = \frac{g(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\eta}} + \frac{\lambda A_\lambda(\mathbf{x};\mathbf{y})}{1+|\mathbf{x}-\mathbf{y}|^{1+\eta}} \tag{13.43}$$

where $g(\mathbf{x};\mathbf{y})$ is given by (3.4), $A_\lambda(\mathbf{x};\mathbf{y})$ is bounded by a constant, $\nu = O(\lambda)$ and $\eta = a\lambda^2 + O(\lambda^3)$, with $a > 0$.

While a sequence of L_i verifying the conditions of the first theorem is found in [BGM1], a sequence verifying the conditions of the second theorem is not constructed at the moment.

p.13.8 **13.8.** *Open problems.* Finally let us list a list of open problems about $d = 1$ spinless fermions weakly interacting with short range.

(1) In the case of periodic potential there are two different expansion for $p_F = n\pi$ and $p_F \neq n\pi$; this second one is such that λ has to be taken smaller and smaller as $p_F \to n\pi$. One would like to know the correlation functions also for $p_F \simeq n\pi$ and λ fixed *i.e.* not vanishing as $p_F \to n\pi$.

(2) One would like to study by these methods interacting fermions with a stochastic external potential. In the $\lambda = 0$ the Schwinger function was obtained in [AG] by the properties of the solution of the Schrödinger equation. It is likely that the interaction between fermions produces a renormalization of the decay rate of the Anderson localization similar to the one for the quasi-periodic potential (see [SCP]). To prove this probably Cluster expansion techniques have to be used (see [MPR]).

(3) Another interesting case is if there is a large incommensurate potential and an interaction between fermions (Holstein-Hubbard model for large u).

sec.14

14. Density-density response function

p.14.1 **14.1.** *The expansion.* We have seen in §3 that the density-density response functions, in terms of which many important physical quantities can be expressed, are related to the four-point Schwinger functions. The expansion in §12 for the two-point Schwinger functions could be adapted to the case of the n-point Schwinger function. However, while such expansion is suitable for the analysis of the asymptotic behaviour of $S^T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ when $|\mathbf{x}_1 - \mathbf{x}_2|, |\mathbf{x}_2 - \mathbf{x}_3|, |\mathbf{x}_3 - \mathbf{x}_4|, |\mathbf{x}_1 - \mathbf{x}_4|$ are large, it cannot provide the asymptotic behaviour of the density-density correlation functions, related to four-point Schwinger functions with coordinates pairwise equal; see [M2]. The reason is that, while in the two point Schwinger function, or in the four point one with all the difference of coordinates very large, the asymptotic behaviour is described in terms of the *same* critical indices appearing in the effective potential, in the expression for the density-density correlation function there are *new* ones. We give here an idea of the expansion referring for details and proofs to [BeM].

The density-density correlation function $H_{\mathbf{x}}$ is given by

5.1yy
$$H^3_{\mathbf{x}} = \frac{\partial^2 \log \mathcal{S}(\phi)}{\partial\phi(\mathbf{x})\phi(\mathbf{0})}|_{\phi=0}\ , \tag{14.1}$$

where $\phi(\mathbf{x})$ is a bosonic external field, periodic in x and x_0, of period L and β, respectively, and

5.2yy
$$e^{\mathcal{S}(\phi)} = \int P(d\psi^{(\le 1)}) e^{-V(\psi^{(\le 1)}) + \int d\mathbf{x}\phi(\mathbf{x})\psi^{(\le 1)+}_{\mathbf{x}}\psi^{(\le 1)-}_{\mathbf{x}}} \ . \tag{14.2}$$

For fixing ideas we study (14.2) for the Hamiltonian $H_0 + \lambda V + \nu N_0$ on a lattice, and in §16 we discuss what happens for more complex Hamiltonians.

We shall evaluate the integral in the r.h.s. of (14.2) introducing a scale decomposition and performing iteratively the integration of the single scale fields, starting from the field of scale 1.

After integrating the fields $\psi^{(1)}, \dots \psi^{(h+1)}$, $0 \ge h \ge h^*$, we find

5.4yy
$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(\ge h+1)}(\phi)} \int P_{Z_h}(d\psi^{\le h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi)} \ , \tag{14.3}$$

where $P_{Z_h}(d\psi^{(\le h)})$ and $\mathcal{V}^h$ are given by (8.40) while $S^{(\ge h+1)}(\phi)$, which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\le h)}, \phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form

5.5yy
$$S^{(\ge h+1)}(\phi) = \sum_{m=1}^{\infty} \frac{1}{(L\beta)^m} \sum_{\mathbf{p}_1, \mathbf{p}_m} S_m^{(\ge h+1)}(\mathbf{p}_1, \dots, \mathbf{p}_m) \Big[\prod_{i=1}^{m} \phi(\mathbf{p}_i) \delta(\sum_{i=1}^{m} \mathbf{p}_i) \Big] \tag{14.4}$$

5.6yy
$$\begin{aligned} \mathcal{B}^{(h)}(\psi^{(\le h)}, \phi) = & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{,\boldsymbol{\omega}} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_m} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2n}} \\ & \cdot \ B^{(h)}_{m,2n,,\boldsymbol{\omega}}(\mathbf{p}_1, \dots, \mathbf{p}_m; \mathbf{k}_1, \dots, \mathbf{k}_{2n}) \Big[\prod_{i=1}^{m} \phi(\mathbf{p}_i) \Big] \Big[\prod_{i=1}^{2n} \psi^{(\le h)\sigma_i}_{\mathbf{k}_i, \omega_i} \delta(\sum_{i=1}^{m} \mathbf{p}_i + \sum_{\mathbf{k}=1}^{2n} \sigma_i \mathbf{k}_i) \Big] \ . \end{aligned} \tag{14.5}$$

It is easy to see that the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields so that the only terms in the r.h.s. of (14.5) which are not irrelevant are those with $m = 1$ and $n = 1$, which are marginal. Hence we extend the definition of the localization operator $\mathcal{L}$ (8.19), (8.20) and (8.21) so that its action on $\mathcal{B}^{(h)}(\psi^{(\le h)}, \phi)$ is trivial unless $n = m = 1$ and in that case:

locco
$$\begin{aligned} & \mathcal{L} \frac{1}{(L\beta)^3} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}} \phi(\mathbf{p}) \psi^{(\le h)+}_{\mathbf{k}_1, \omega_1} \psi^{(\le h)-}_{\mathbf{k}_2, \omega_2} B^{(h)}_{1,2,,\boldsymbol{\omega}}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2) \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{p}) \\ & \quad \mathcal{L} \frac{1}{(L\beta)^3} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}} \phi(\mathbf{p}) \psi^{(\le h)+}_{\mathbf{k}_1, \omega_1} \psi^{(\le h)-}_{\mathbf{k}_2, \omega_2} B^{(h)}_{1,2,,\boldsymbol{\omega}}((\omega_2 - \omega_1)\mathbf{p}_F; \omega_1 \mathbf{p}_F, \omega_2 \mathbf{p}_F) \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{p}). \end{aligned} \tag{14.6}$$

Then we can write

5.9yy
$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\le h)}, \phi) = \frac{Z_h^{(1)}}{Z_h} F_1^{(\le h)} + \frac{Z_h^{(2)}}{Z_h} F_2^{(\le h)} \ , \tag{14.7}$$

where $Z_h^{(1)}$ and $Z_h^{(2)}$ are real numbers, such that $Z_1^{(1)} = Z_1^{(2)} = 1$ and

5.10yy
$$F_1^{(\le h)} = \sum_{\omega = \pm 1} \int d\mathbf{x} \phi(\mathbf{x}) e^{2i\omega \mathbf{p}_F \mathbf{x}} \psi^{(\le h)+}_{\mathbf{x},\omega} \psi^{(\le h)-}_{\mathbf{x},-\omega} \ , \tag{14.8}$$

5.11yy
$$F_2^{(\le h)} = \sum_{\omega = \pm 1} \int d\mathbf{x} \phi(\mathbf{x}) \psi^{(\le h)+}_{\mathbf{x},\omega} \psi^{(\le h)-}_{\mathbf{x},\omega} \ . \tag{14.9}$$

Of course we could write the above expressions in momentum space, like in (8.27); we prefer however to write them in coordinate space to make evident that in $F_1^{(\le h)}$ there is an *oscillating factor* absent in $F_2^{(\le h)}$. Note also that $Z_h^{(1)}$ and $Z_h^{(2)}$ are *new* running coupling constants.

By using the notation of §8, we can write the integral in the r.h.s. of (14.3) as

$$\begin{aligned} & e^{-L\beta t_h} \int \tilde{P}_{Z_{h-1}}(d\psi^{(\le h)}) e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)} \; = \\ & = e^{-L\beta t_h} \int P_{Z_{h-1}}(d\psi^{(\le h-1)}) \cdot \\ & \cdot \int \tilde{P}_{Z_{h-1}}(d\psi^{(h)}) e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) - \hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)} \; , \end{aligned} \tag{14.10}$$

where $\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})$ is defined as in (8.63) and

$$\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi) = \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi) \; . \tag{14.11}$$

$\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through the analogous of (8.65), that is

$$\begin{aligned} & e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) + \mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi) - L\beta\tilde{E}_h + \tilde{S}^{(h)}(\phi)} = \\ & = \int P_{Z_{h-1}}(d\psi^{(h)}) e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)} \; . \end{aligned} \tag{14.12}$$

The definitions (14.11) and (14.7) easily imply that

$$\frac{Z^{(i)}_{h-1}}{Z^{(i)}_h} = 1 + z^{(i)}_h \; , \quad i = 1, 2 \; , \tag{14.13}$$

where $z^{(1)}_h$ and $z^{(2)}_h$ are some quantities of order ε_h, which can be written in terms of a tree expansion similar to that described in §8. It follows that

$$\mathcal{S}(\phi) = -L\beta E_{L,\beta} + \sum_{h=-\infty}^{1} \tilde{S}^{(h)}(\phi) \tag{14.14}$$

and $S^{(\ge h+1)} = \sum_{k=h+1}^{1} \tilde{S}^{(k)}(\phi)$, see (14.3); moreover deriving with respect to $\phi(\mathbf{x}), \phi(\mathbf{0})$

$$H_{\mathbf{x}} = \sum_{h=-\infty}^{1} \tilde{S}^{(h)}_2(\mathbf{x}, 0) \; . \tag{14.15}$$

The functionals $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)$ and $S^{(h)}(\phi)$ can be written in terms of a tree expansion similar to the one described in §8. We introduce, for each $n \ge 0$ and each $m \ge 1$, a family $\mathcal{T}^m_{h,n}$ of trees, which are defined as in §8.4, with some differences.
(1) If $\tau \in \mathcal{T}^m_{h,n}$, the tree has $2n+m$ (instead of $2n$) endpoints. Moreover, among the $2n+m$ endpoints, there are n endpoints, which we call *normal endpoints*, which are associated with a contribution to the effective potential on scale $h_v - 1$. The m remaining endpoints, which we call *special endpoints*, are associated with a local term of the form (14.8) or (14.9); we shall say that they are of type $Z^{(1)}$ or $Z^{(2)}$, respectively.
(2) There are clusters with ϕ external fields, and on such clusters the $\mathcal{R} = 1 - \mathcal{L}$ operation, if $\mathcal{L}$ is defined as in (14.6), acts if there is only one ϕ external field and two external ψ fields. As dimensionally a ϕ field is like a couple of ψ fields, (14.7) is enough to get a convergent renormalized expansion.

14.2. *The results.* The running coupling constant $Z^{(1)}_h$ verifies a flow equation of the form

$$\frac{Z^{(1)}_{h-1}}{Z_{h-1}} = \frac{Z^{(1)}_h}{Z_h}[\beta_3\lambda_h + \beta^{(h)}_3(\vec{v}_h, \dots, \vec{v}_0)]$$

with $\beta_3 > 0$ and $|\beta_3^{(h)}(\vec{v}_h, \dots, \vec{v}_0)| \le C|\lambda_h|^2$. By performing an analysis similar to the one in §10 for the running coupling constants one can prove, $c_2 \le c_1$ are constants

5.26yy
$$\gamma^{-c_2\beta_3\lambda_1 h} \le \frac{Z_h^{(1)}}{Z_h} \le \gamma^{-c_1\beta_3\lambda_1 h} . \tag{14.16}$$

A similar results holds of course also for $\frac{Z_h^{(2)}}{Z_h}$, namely

$$\frac{Z_{h-1}^{(2)}}{Z_{h-1}} = \frac{Z_h^{(2)}}{Z_h}[\beta_4\lambda_h + \beta_4^{(h)}(\vec{v}_h, \dots, \vec{v}_0)]$$

and it is easy to check, by explicit calculation, that one has $\beta_4 = 0$. We shall show in the next section that we can decompose $\beta_4^{(h)}$ in a Luttinger model part plus a correction, and that the Luttinger model part is vanishing. This will be proved by a *Ward identity*. In particular one can show that

5.29yy
$$\gamma^{-C|\lambda|} \le \frac{Z_h^{(2)}}{Z_h} \le \gamma^{C|\lambda|} . \tag{14.17}$$

This means that there is a remarkable *cancellation* such that the ratio between $Z_h^{(2)}$ and Z_h remains near to 1: they are two diverging quantities given by an (apparently) different perturbative series, however such two series are equal to any order up to irrelevant terms.

At the end one finds for $H = H_0 + \lambda V + \nu N$, if the conditions of Theorem 1 are verified *i.e.* λ small enough, $p_F \neq 0, \pi$ and ν chosen in a proper way

5.32yy
$$\begin{aligned} H_{\mathbf{x}} = & \sum_{h,k=-\infty}^{1} \sum_{\omega=\pm 1} \Big\{ e^{2i\omega p_F x} \cdot \\ & \cdot \frac{(Z_{h\vee k}^{(1)})^2}{Z_{h-1}Z_{k-1}} g_{\omega,\omega}^{(h)}(-\omega\mathbf{x}) g_{-\omega,\omega}^{(k)}(-\omega\mathbf{x}) - \frac{(Z_{h\vee k}^{(2)})^2}{Z_{h-1}Z_{k-1}} g_{\omega,\omega}^{(h)}(-\omega\mathbf{x}) g_{\omega,\omega}^{(k)}(\omega\mathbf{x}) + \\ & + \sum_{h=-\infty}^{1} \left\{ \left(\frac{Z_h^{(1)}}{Z_h}\right)^2 G_1^{(h)}(\mathbf{x}) + \left(\frac{Z_h^{(2)}}{Z_h}\right)^2 G_2^{(h)}(\mathbf{x}) + \frac{Z_h^{(1)} Z_h^{(2)}}{Z_h^2} G_3^{(h)}(\mathbf{x}) \right\} , \end{aligned} \tag{14.18}$$

where $h \vee k = \max\{h, k\}$ and given any integer $N \ge 0$

5.35yy
$$|G_1^{(h)}(\mathbf{x})| + |G_2^{(h)}(\mathbf{x})| + \gamma^{-\eta h}|G_3^{(h)}(\mathbf{x})| \le C_N|\lambda| \frac{\gamma^{2h}}{1 + [\gamma^{(h)}|\mathbf{x}|]^N} , \tag{14.19}$$

for a suitable constant C_N (reacll that the propagator is diagonal in the case we are considering here). Moreover, we can write

5.35ayy
$$\begin{aligned} G_1^{(h)}(\mathbf{x}) &= \sum_{\sigma=\pm 1} e^{2i\sigma p_F x} \bar{G}_{1,\sigma}^{(h)}(\mathbf{x}) + r_1^{(h)}(\mathbf{x}) , \\ G_2^{(h)}(\mathbf{x}) &= \bar{G}_2^{(h)}(\mathbf{x}) + r_2^{(h)}(\mathbf{x}) , \end{aligned} \tag{14.20}$$

such that

5.35byy
$$|r_1^{(h)}(\mathbf{x})| + |r_2^{(h)}(\mathbf{x})| \le C_N|\lambda_1| \frac{\gamma^{(2+\eta)h}}{1 + [\gamma^h|\mathbf{x}|]^N} , \tag{14.21}$$

and, if we define $D_{m_0,m_1} = \partial_{x_0}^{m_0}\partial_x^{m_1}$ (∂_x is a discrete derivative; see Appendix A2), given any integers $m_0, m_1 \ge 0$, there exists a constant C_{N,m_0,m_1}, such that

5.35cyy
$$\sum_{\sigma=\pm 1} |D_{m_0,m_1}\bar{G}_{1,\sigma}^{(h)}(\mathbf{x})| + |D_{m_0,m_1}\bar{G}_2^{(h)}(\mathbf{x})| \le C_{N,m_0,m_1}|\lambda_1|\gamma^{2h} \frac{\gamma^{h(m_0+m_1)}}{1 + [\gamma^{(h)}|\mathbf{x}|]^N} . \tag{14.22}$$

An easy corollary of the above equations is that the density-density correlation function can be written as

sssa
$$H_{\mathbf{x}} = \cos(2p_F x)\Omega^{3,a}_{L,\beta,\mathbf{x}} + \Omega^{3,b}_{L,\beta,\mathbf{x}} + \Omega^{3,c}_{L,\beta,\mathbf{x}} \tag{14.23}$$

with

1.10*xxx*
$$|\partial^i_{\mathbf{x}}\Omega^{3,a}_{L,\beta,\mathbf{x}}| \le \frac{C_1}{1+|\mathbf{x}|^{2+2\eta_2+i}} \quad |\partial^i_{\mathbf{x}}\Omega^{3,b}_{L,\beta,\mathbf{x}}| \le \frac{C_1}{1+|\mathbf{x}|^{2+i}} \quad |\Omega^{3,c}_{L,\beta,\mathbf{x}}| \le \frac{C_1}{1+|\mathbf{x}|^{2+\tilde\gamma}} \tag{14.24}$$

where $\partial_{\mathbf{x}} = (\partial_{x_0}, \partial_x)$, $\tilde\gamma, C_1 > 0$ depend only on p_F and $\eta_2 = -b_3\lambda + O(\lambda^2)$. We will see in §16 that many properties of the density-density correlation function can be obtained from (14.18).

Finally it is interesting to compare the above expression with the density-density correlation function in the Luttinger model (11.1), obtained by the exact solution. The Luttinger model is defined in terms of two fields $\psi_{\mathbf{x},\omega}$, $\omega = \pm 1$, and one expects that the large distance asymptotic behaviour of $\Omega^3(\mathbf{x})$ is qualitatively similar to that of the truncated correlation of the operator $\rho_{\mathbf{x}} = \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}$, where $\psi^\sigma_{\mathbf{x}} = \sum_\omega \exp(i\sigma\omega p_F x)\psi_{\mathbf{x},\omega}$. There is apparently a problem, since the expectation of $\rho_{\mathbf{x}}$ is infinite; however, it is possible to see that there exists the limit, as $\varepsilon_1, \varepsilon_2 \to 0^+$, of $[< \rho_{\mathbf{x},\varepsilon_1}\rho_{\mathbf{y},\varepsilon_2} > - < \rho_{\mathbf{x},\varepsilon_1} >< \rho_{\mathbf{y},\varepsilon_2} >]$, where $\rho_{\mathbf{x},\varepsilon} = \psi^+_{(x,x_0+\varepsilon)}\psi^-_{(x,x_0)}$, and it is natural to take this quantity, let us call it $G(\mathbf{x}-\mathbf{y})$, as the truncated correlation of $\rho_{\mathbf{x}}$. From (2.5) of [BeGM] (by inserting in the lst sum a $(-\varepsilon_i\varepsilon_j)$, missing for a typo), it follows that, for $|\mathbf{x}| \to \infty$

1.9*lllll*
$$G(\mathbf{x}) \simeq [1+\lambda a_1(\lambda)]\frac{\cos(2p_F x)}{2\pi^2[(v_0^* x_0)^2 + x^2]^{1+\lambda a_3(\lambda)}} + \frac{(v_0 x_0)^2 - x^2}{2\pi^2[(v_0 x_0)^2 + x^2]^2}\ , \tag{14.25}$$

where $v_0^* = v_0[1+\lambda a_2(\lambda)]$ and $a_i(\lambda)$, $i = 1,2,3$, are bounded functions. Note that, in the second term in the r.h.s. of (2.7), the bare Fermi velocity v_0 appears, instead of the renormalized one, v_0^*, as one could maybe expect.

*sec.*15
15. Approximate Ward identities

In order to prove (14.17) we can reason as in §10 and we can write the Beta function for $Z_h^{(2)}/Z_h$ as sum of several terms, like in (10.7), (10.5); one of them coincides with the beta function of the infrared Luttinger model and is the crucial one in order to control the flow, while the others have a little effect on the flow of $Z_h^{(2)}/Z_h$. Then one is lead to consider the infrared Luttinger model beta function for $Z_h^{(2)}/Z_h$ in order to prove the analogous of (10.11); once that we have proved this the flow for the original model can be controlled repeating the consideration in §10. In this section we give a sketch of the proof of (14.17), following [BeM]. Let us consider the infrared Luttinger model with integration given by (10.1), and let be $Z_h^{(2),l}$ and Z_h^l the analogous of $Z_h^{(2)}$ and Z_h for such model.

Let us introduce a new external field $J_{\mathbf{x}}$, commuting with the fields ϕ^σ and $\psi^{[h,0]\sigma}$, and let us consider the functional

6.15*yy*
$$\mathcal{W}(\phi, J) = -\log \int P^{(h)}_{(l)}(d\psi^{[h,0]}) e^{-V(\psi^{[h,0]}+\phi) + \int d\mathbf{x} J_{\mathbf{x}} \sum_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}}\ . \tag{15.1}$$

with $P^{(h)}_{(l)}(d\psi^{[h,0]})$ defined by (10.1) with C_0^{-1} replaced by $C_{h,0}^{-1} = \sum_{k=h}^0 f_k$. We also define the functions

6.16*yy*
$$\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\omega}\partial\phi^-_{\mathbf{y},\omega}}\mathcal{W}(\phi, J)\Big|_{\phi=J=0}\ , \tag{15.2}$$

6.17*yy*
$$\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x}}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}}\mathcal{W}(\phi, J)|_{\phi=J=0}\ . \tag{15.3}$$

These functions have here the role of the *self-energy* and the *vertex part* in the usual treatment of the Ward identities. However, they do not coincide with them, because the corresponding Feynman graphs expansions

are not restricted to the one particle irreducible graphs. However, their Fourier transforms at zero external momenta, which are the interesting quantities in the limit $L, \beta \to \infty$, are the same; in fact, because of the support properties of the fermion fields, the propagators vanish at zero momentum, hence the one particle reducible graphs give no contribution at that quantities.

Let us define

6.25yy
$$\tilde{Z}_h^{(2)} = 1 + \hat{\Gamma}_{h,\omega}(\mathbf{0}, \mathbf{0}) \; , \tag{15.4}$$

6.26yy
$$\tilde{Z}_h = 1 + \hat{\Sigma}_{h,\omega}(\mathbf{0}) \; , \tag{15.5}$$

If we did not perform any anomalous integration, $\tilde{Z}_h^{(2)}$ and $\tilde{Z}_h$ coincide with $Z_h^{(2),l}$ and Z_h^l; the anomalous integration makes them a sligthly different but one can prove that, see [BeM]

$$|\frac{\tilde{Z}_h^{(2)}}{Z_h^{(2),l}} - 1| \le C|\lambda| \quad |\frac{\tilde{Z}_h}{Z_h^l} - 1| \le C|\lambda|$$

Let us consider a Feynman graph expansion for $\tilde{Z}_h^{(2)}$ and $\tilde{Z}_h$ similar to the one discussed in §7 with propagator $C_{h,0}^{-1}(\mathbf{k}')/ - ik_0 + \omega v_0 k$; it is easy to see that the graphs contributing to Z_h have two external fermionic lines and a derived internal propagator (in momentum space) while the graphs contributing to $\tilde{Z}_h^{(2)}$ have two external fermionic lines and an external bosonic lines, representing the external field ϕ. If we proceed formally replacing the propagator $C_{h,0}^{-1}(\mathbf{k}')/ - ik_0 + \omega v_0 k$ with $g_{\omega,F}(\mathbf{k}) \equiv 1/ - ik_0 + \omega v_0 k$ *i.e. neglecting the infrared and ulraviolet cut-offs* one finds the formal equality of the two expansions, as a consequence of the equality $\partial_k g_{\omega,F}(\mathbf{k}) = -\omega[g_{\omega,F}(\mathbf{k})]^2$; see Fig. 17. Of course such argument is only formal, as the two expansion are both meaningless neglecting the cut-offs, but it suggests that formally $\tilde{Z}_h^{(2)} = \tilde{Z}_h$.

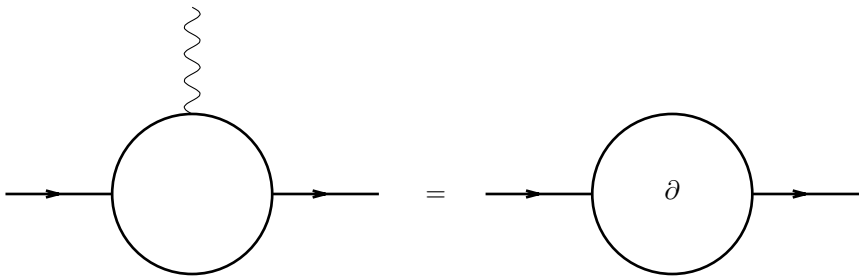


FIG. 17Graphic representation of the formal Ward identity.

What it possible to prove in a rigorous way is

ward
$$\tilde{Z}_h = \tilde{Z}_h^{(2)} + \delta\tilde{Z}_h^{(2)} \tag{15.6}$$

with

ward1
$$|\delta\tilde{Z}_h^{(2)}| \le C|\lambda| Z_h^{(2)} \tag{15.7}$$

This shows that the corrections due to the presence of cut-off functions to the "formal" Ward identity $\tilde{Z}_h^{(2)} = \tilde{Z}_h$ are bounded by a diverging quantity as $h \to -\infty$, so that the cancellations seen above seems to capture only the "leading-log" behaviour.

In order to prove (15.6) we will find convenient to write the integration in (15.1) in terms of the space-time field variables

6.3yy
$$P_{(l)}^{(h)}(d\psi^{[h,0]}) = \mathcal{D}\psi^{[h,0]} \exp\Big[- \sum_{\omega} \int d\mathbf{x} \; \psi_{\mathbf{x},\omega}^{[h,0]+} D_{\omega}^{[h,0]} \psi_{\mathbf{x},\omega}^{[h,0]-} \Big] \; , \tag{15.8}$$

where

6.4yy
$$D^{[h,0]}_{\omega}\psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta}\sum_{\mathbf{k}'} e^{i\sigma\mathbf{k}'\mathbf{x}} C_{h,0}(\mathbf{k}')(i\sigma k_0 - \omega\sigma v_0 k')\hat{\psi}^{[h,0]\sigma}_{\mathbf{k}',\omega} \ . \tag{15.9}$$

$D^{[h,0]}_{\omega}$ has to be thought as a "regularization" of the linear differential operator

6.5yy
$$D_{\omega} = \frac{\partial}{\partial x_0} + i\omega v_0 \frac{\partial}{\partial x} \ . \tag{15.10}$$

Let us now introduce the external field variables $\phi^{\sigma}_{\mathbf{x},\omega}$, $\omega = \pm 1$, anticommuting with themselves and $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$, and let us define

6.6yy
$$U(\phi) = -\log \int P^{(h)}_{(l)}(d\psi^{[h,0]}) e^{-V(\psi^{[h,0]}+\phi)} \ , \tag{15.11}$$

If we perform the *gauge transformation*

6.7yy
$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} \to e^{i\sigma\alpha_{\mathbf{x}}}\psi^{[h,0]\sigma}_{\mathbf{x},\omega} \ , \tag{15.12}$$

and we define $(e^{-i\alpha}\phi)^{\sigma}_{\mathbf{x},\omega} = e^{-i\sigma\alpha_{\mathbf{x}}}\phi^{\sigma}_{\mathbf{x},\omega}$, we get

6.8yy
$$\begin{aligned} U(\phi) = -\log \int P^{(h)}_{(l)}(d\psi^{[h,0]}) \exp\Big\{ - V(\psi^{[h,0]} + e^{-i\alpha}\phi) - \\ - \sum_{\omega}\int d\mathbf{x}\ \psi^{[h,0]+}_{\mathbf{x},\omega}\Big(e^{i\alpha_{\mathbf{x}}} D^{[h,0]}_{\omega} e^{-i\alpha_{\mathbf{x}}} - D^{[h,0]}_{\omega}\Big)\psi^{[h,0]-}_{\mathbf{x},\omega}\Big\} \ . \end{aligned} \tag{15.13}$$

Note that the integration $P^{(h)}_{(l)}(d\psi^{[h,0]})$ is not Gauge invariant due to the presence of the cut-off.

Since $U(\phi)$ is independent of α, the functional derivative of the r.h.s. of (15.13) w.r.t. $\alpha_{\mathbf{x}}$ is equal to 0 for any $\mathbf{x}$. Hence, we find the following identity:

6.9yy
$$\sum_{\omega}\left[-\phi^{+}_{\mathbf{x},\omega}\frac{\partial U}{\partial \phi^{+}_{\mathbf{x},\omega}} + \frac{\partial U}{\partial \phi^{-}_{\mathbf{x},\omega}}\phi^{-}_{\mathbf{x},\omega} + \frac{1}{Z(\phi)}\int P^{(L,h)}(d\psi^{[h,0]}) T_{\mathbf{x},\omega}\ e^{-V^{(L)}(\psi^{[h,0]}+\phi)}\right] = 0 \ , \tag{15.14}$$

where

6.10yy
$$Z(\phi) = \int P^{(h)}_{(l)}(d\psi^{[h,0]}) e^{-V(\psi^{[h,0]}+\phi)} \ , \tag{15.15}$$

6.11yy
$$\begin{aligned} T_{\mathbf{x},\omega} &= \psi^{[h,0]+}_{\mathbf{x},\omega}[D^{[h,0]}_{\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}] + [D^{[h,0]}_{\omega}\psi^{[h,0]+}_{\mathbf{x},\omega}]\psi^{[h,0]-}_{\mathbf{x},\omega} = \\ &= \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}}\hat{\psi}^{[h,0],+}_{\mathbf{k},\omega}[C_{h,0}(\mathbf{p}+\mathbf{k})D_{\omega}(\mathbf{p}+\mathbf{k}) - C_{h,0}(\mathbf{k})D_{\omega}(\mathbf{k})]\hat{\psi}^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \ , \end{aligned} \tag{15.16}$$

6.12yy
$$D_{\omega}(\mathbf{k}) = -ik_0 + \omega v_0 k \ . \tag{15.17}$$

Note that (15.16) can be rewritten as

6.13yy
$$T_{\mathbf{x},\omega} = D_{\omega}[\psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}] + \delta T_{\mathbf{x},\omega} \ , \tag{15.18}$$

where

6.14yy
$$\begin{aligned} \delta T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}}\hat{\psi}^{[h,0],+}_{\mathbf{k},\omega} \ \cdot \\ \cdot\ \big\{[C_{h,0}(\mathbf{p}+\mathbf{k})-1]D_{\omega}(\mathbf{p}+\mathbf{k}) - [C_{h,0}(\mathbf{k})-1]D_{\omega}(\mathbf{k})\big\}\hat{\psi}^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \ . \end{aligned} \tag{15.19}$$

It follows that, if $C_{h,0}$ is substituted with 1, that is if we consider the formal theory without any ultraviolet and infrared cutoff, $T_{\mathbf{x},\omega} = D_{\omega}[\psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}]$ and we would get the usual Ward identities. The presence of

the cutoffs make the analysis a bit more involved and adds some corrections to the Ward identities, which however, for λ small enough, can be controlled by the same type of multiscale analysis, that we used before.

If we derive the l.h.s. of (15.14) with respect to $\phi^+_{\mathbf{y},\omega}$ and to $\phi^-_{\mathbf{z},\omega}$ and we put $\phi = 0$, we get

6.18yy
$$0 = -\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\ < \left[\frac{\partial^2 V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}\partial\psi^{[h,0]-}_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^{[h,0]-}_{\mathbf{z},\omega}}\right] \; ; \; \sum_{\tilde\omega}\left[D_{\tilde\omega}(\psi^{[h,0]+}_{\mathbf{x},\tilde\omega}\psi^{[h,0]-}_{\mathbf{x},\tilde\omega}) + \delta T_{\mathbf{x},\tilde\omega}\right] >^T , \tag{15.20}$$

where $< \cdot;\cdot >^T$ denotes the truncated expectation w.r.t. the measure $Z(0)^{-1}P^{(L,h)}(d\psi^{[h,0]})\; e^{-V^{(L)}(\psi^{[h,0]})}$.

By using the definitions (15.2) and (15.3), equation (15.20) can be rewritten as

6.19yy
$$0 = -\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\ - \sum_{\tilde\omega} D_{\mathbf{x},\tilde\omega}\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) - \Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) , \tag{15.21}$$

where

6.20yy
$$\Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) =< \left[\frac{\partial^2 V}{\partial\psi^{+}_{\mathbf{y},\omega}\partial\psi^{-}_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^{+}_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^{-}_{\mathbf{z},\omega}}\right] \; ; \; \sum_{\tilde\omega}\delta T_{\mathbf{x},\tilde\omega} >^T \quad . \tag{15.22}$$

In terms of the Fourier transforms (15.21) can be written as

6.24yy
$$0 = \hat\Sigma_{h,\omega}(\mathbf{k}-\mathbf{p}) - \hat\Sigma_{h,\omega}(\mathbf{k}) + \sum_{\tilde\omega}(-ip_0 + \tilde\omega p)\hat\Gamma_{h,\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) + \hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k}) . \tag{15.23}$$

and by (15.4), (15.5) we get (15.6). In order to bound the correction term $\hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k})$ one can define for it a renormalized multscale expansion similar to the one of the density-density correlation function, see [BeM] and the bound (15.7) can be proved.

Note finally that we have treated in a different way the vanishing of the Luttinger model part of the beta function for λ_h and for $\frac{Z_h^2}{Z_h}$: in the first case we have used the exact solution of the Luttinger model, and in the second one a Ward identity. It is very likely that a proof of the vanishing of the Luttinger model part of the beta function for λ_h by the use of Ward identities is also possible.

sec.16
16. Spin chains

We apply the results of the preceding two sections for obtaining the spin-spin correlation function in the direction of the magnetic field of the Heisenberg XYZ model (2.10), for small anisotropy u and J_3, see [M2], [BeM]. This means that we have to generaliza the analysis in §14, see (2.15), to the lattice hamiltonian $H_0 + \lambda V + \nu N_0 + uB$ with

$$B = -\frac{1}{2}\sum_x [\psi^+_x\psi^+_{x+1} + \psi^-_{x+1}\psi^-_x]$$

Of course similar results hold for density-density correlation functions of all the models discussed in §13.

The Heisenberg XYZ chain has been the subject of a very active research over many years with a variety of methods. A first class of results is based on the *exact solutions*. If one of the three parameters is vanishing (e.g. $J_3 = 0$), the model is called XY *chain*. Its solution is based on the fact that the Hamiltonian in the fermionic form is quadratic in the fermionic fields, so that it can be diagonalized (see [LSM], [LSM1]) by a Bogoliubov transformation. The equal time correlation functions $\Omega^\alpha_{(x,0)}$ were explicitly calculated in [Mc] (even at finite L and β), in the case $h = 0$, that is $p_F = \pi/2$. Note that, while $\Omega^3_{\mathbf{x}}$ coincides with the

correlation function of the density in the fermionic representation of the model, $\Omega^1_{\mathbf{x}}$ and $\Omega^2_{\mathbf{x}}$ are given by quite complicated expressions. It turns out that $\Omega^3_{(x,0)}$ is of the following form:

1.7*llllll*
$$\Omega^3_{(x,0)} = -\frac{\alpha^{|x|}}{\pi^2 x^2}\sin^2\left(\frac{\pi x}{2}\right) F(-|x|\log\alpha, |x|)\ , \quad \alpha = (1-|u|)/(1+|u|)\ , \tag{16.1}$$

where $F(\gamma, n)$ is a bounded function, such that, if $\gamma \le 1$, $F(\gamma, n) = 1 + O(\gamma\log\gamma) + O(1/n)$, while, if $\gamma \ge 1$ and $n \ge 2\gamma$, $F(\gamma, n) = \pi/2 + O(1/\gamma)$.

For $|h| > 0$, it is not possible to get a so explicit expression for $\Omega^3_{(x,0)}$. However, it is not difficult to prove that, if $|u| < \sin p_F$, $|\Omega^3_{(x,0)}| \le \alpha^{|x|}$ and, if $x \ne 0$ and $|ux| \le 1$

1.7*allllll*
$$\Omega^3_{(x,0)} = -\frac{1}{\pi^2 x^2}\sin^2(p_F x)[1 + O(|ux|\log|ux|) + O(1/|x|)]\ . \tag{16.2}$$

Note that, if $u = 0$, a very easy calculation shows that $\Omega^3_{(x,0)} = -(\pi^2 x^2)^{-2}\sin^2(p_F x)$.

We want to stress that the only case in which the correlation functions and their asymptotic behaviour can be computed explicitly in a rigorous way is just the $J_3 = 0$ case.

If two parameters are equal (e.g. $J_1 = J_2$) and there is no external magnetic field ($h = 0$), but $J_3 \ne 0$, the model is called XXZ model. It was solved in [YY] via the *Bethe-ansatz*, in the sense that the Hamiltonian was diagonalized and the ground state energy was computed. However, it was not possible till now to obtain the correlation functions from the exact solution. Such solution is contained in the general solution of the XYZ model by Baxter [B], but again *only in the case of zero magnetic field*. The ground state energy was computed, showing for istance that the ground state may have a gap in the spectrum which, if $J_1 - J_2$ and J_3 are not too large, is given approximately by (see [LP])

1.8*llllll*
$$\Delta = 8\pi\frac{\sin\mu}{\mu}|J_1|\left(\frac{|J_1^2 - J_2^2|}{16(J_1^2 - J_3^2)}\right)^{\frac{\pi}{2\mu}} \tag{16.3}$$

with $\cos\mu = -J_3/J_1$.

The solution is based on the fact that the XYZ chain with periodic boundary conditions is equivalent to the *eight vertex* model, in the sense that H is proportional to the logarithmic derivative with respect to a parameter of the eight vertex tranfer matrix, if a suitable identification of the parameters is done, see [S], [B]. The eight vertex model is obtained by putting arrows in a suitable way on a bidimensional lattice with M rows, L columns and periodic boundary conditions. There are eight allowed vertices, and with each of them an energy is associated in a suitable way (there are four different values of the energy). With the above choice of the parameters and $T - T_c < 0$ and small, $u = O(|T - T_c|)$, so that the critical temperature of the eight vertex model corresponds to no anysotropy in the XYZ chain. Moreover, see [JKM], the correlation function C_x between two vertical arrows in a row, separated by x vertices, in the limit $M \to \infty$, is given by $C_x =< S_0^2 S_x^2 >$. Again an explicit expression for the correlation functions cannot be derived for the XYZ or the eight vertex model. In [JKM] the correlation length of C_x was computed heuristically under some physical assumptions (an exact computation is difficult because it does not depend only on the largest and the next to the largest eigenvalues). The result is $\xi^{-1} = (T - T_c)^{\frac{\pi}{2\mu}}$, if ξ is the correlation length. One sees that the critical index of the correlation length is *non universal*.

Another interesting observation is that the XYZ model is equivalent to two bidimensional interpenetrating Ising lattices with nearest-neighbor coupling, interacting via a four spins coupling (which is proportional to J_3). The *four spin correlation function* is identical to C_x. In the decoupling limit $J_3 = 0$ the two Ising lattices are independent and one can see that the Ising model solution can be reduced to the diagonalization, via a Bogoliubov transformation, of a quadratic Fermi Hamiltonian, see [LSM1].

One can presume that the large distance asymptotic behaviour of $\Omega^3_{\mathbf{x}}$ is similar to the density-density Luttinger model correlation function (14.25), if $J_1 = J_2$, to the large distance asymptotic behaviour of $\Omega^3_{\mathbf{x}}$,

if some "reasonable" relationship between the parameters of the two models is assumed (one can make for istance the substitutions $\lambda \to -J_3$, $p_F \to \arccos(J_3 - h)$, $p_0^{-1} \to a = 1$, if a the chain step and p_0^{-1} is the potential range). Of course such identification is completely arbitrary, but one can hope that for large distances the function $\Omega^3_{\mathbf{x}}$ has something to do with the density-density Luttinger model correlation function. If $J_1 \neq J_2$, there is no solvable model suitable for a similar analysis. As we said before, $\Omega^3_{\mathbf{x}}$ can be obtained from the exact solution only in the case $J_3 = 0$, when the fermionic theory is a non interacting one. In particular, if $\mathbf{x} = (x, 0)$ and $|ux| << 1$, (16.2) and a more detailed analysis of the "small" terms in the r.h.s. (in order to prove that their derivatives of order n dacay as $|x|^{-n}$), show that $\Omega^3_{L,\beta,\mathbf{x}}$ is a sum of oscillating functions with frequency $(np_F)/\pi \bmod 1$, $n = 0, \pm 1$. The frequencies are then measured by p_F, so they depend only on the external magnetic field h.

If $J_3 \neq 0$, a similar property is satisfied for the leading terms in the asymptotic behaviour but the value of p_F depends in general also on u and J_3. For example, if $u = 0$, the Hamiltonian is equal, up to a constant, to the Hamiltonian of a free fermion gas with Fermi momentum $p_F = \arccos(J_3 - h)$ plus an interaction term proportional to J_3. As it is well known, the interaction modifies the Fermi momentum of the system by terms of order J_3 and it is convenient, in order to study the interacting model, to fix the Fermi momentum to an interaction independent value, by adding a counterterm to the hamiltonian. We proceed here in a similar way, that is we fix p_F and h_0 so that

1.10*aaa*
$$h = h_0 - \nu \ , \qquad \cos p_F = J_3 - h_0 \tag{16.4}$$

and we look for a value of ν, depending on u, J_3, h_0, such that, as in the $J_3 = 0$ case, the leading terms in the asymptotic behaviour of $\Omega^3_{L,\beta}(\mathbf{x})$ can be represented as a sum of oscillating functions with frequencies $(np_F)/\pi \bmod 1$, $n = 0, \pm 1$.

As we shall see, we can realize this program only if J_3 is small enough and it turns out that ν is of order J_3. It follows that we can only consider magnetic fields such that $|h| < 1$. Moreover, it is clear that the equation $h = h_0 - \nu(u, J_3, h_0)$ can be inverted, once the function $\nu(u, J_3, h_0)$ has been determined, so that p_F is indeed a function of the parameters appearing in the original model.

If $J_1 = J_2$, it is conjectured, on the base of heuristic calculations, that to fix p_F is equivalent to the impose the condition that, in the limit $L, \beta \to \infty$, the density is fixed ("Luttinger Theorem") to the free model value $\rho = p_F/\pi$. Remembering that $\rho - \frac{1}{2}$ is the magnetization in the 3-direction for the original spin variables, this would mean that to fix p_F is equivalent to fix the magnetization in the 3 direction, by suitably choosing the magnetic field.

If $J_1 \neq J_2$, there is in any case no simple relation between p_F and the mean magnetization, as one can see directly in the case $J_3 = 0$, where one can do explicit calculations. The only exception is the case $p_F = \pi/2$, where one can see that, in the limit $L \to \infty$, $\nu = J_3$ (so that $h = 0$ by (16.4)) and that $< S^3_x >= 0$.

By using the results in §14, §15 and adapting the scheme followed in §8 one can prove, see [M2],[BeM], the following result.

THEOREM 9. *Suppose that $v_0 = \sin p_F \geq \bar{v}_0 > 0$, for some value of $\bar{v}_0$ fixed once for all, and let us define $a_0 = \min\{p_F/2, (\pi - p_F)/2\}$; then the following is true.*

a) There exists a constant ε, such that, if $(u, J_3) \in \mathcal{A}$, with

1.10*blllll*
$$\mathcal{A} = \{(u, J_3) : |u| \leq \frac{a_0}{8(1+\sqrt{2})} \ , |J_3| \leq \varepsilon\} \ , \tag{16.5}$$

it is possible to choose ν, so that $|\nu| \leq c|J_3|$, for some constant c independent of L, β, u, J_3, p_F, and the spin correlation function $\Omega^3_{L,\beta}(\mathbf{x})$ is a bounded (uniformly in L, β, p_F and $(u, J_3) \in \mathcal{A}$) function of $\mathbf{x} = (x, x_0)$, $x = 1, \dots, L$, $x_0 \in [0, \beta]$, periodic in x and x_0 of period L and β respectively, continuous as a function of x_0.

b) We can write

1.10*dllllll*
$$\Omega^3(\mathbf{x}) = \cos(2p_F x)\Omega^{3,a}(\mathbf{x}) + \Omega^{3,b}(\mathbf{x}) + \Omega^{3,c}(\mathbf{x}) \ , \tag{16.6}$$

with $\Omega^{3,i}(\mathbf{x})$, $i = a, b, c$, continuous bounded functions, which are infinitely times differentiable as functions of x_0, if $i = a, b$. Moreover, there exist two constants η_1 and η_2 of the form

1.13*lllll*
$$\eta_1 = -a_1 J_3 + O(J_3^2), \qquad \eta_2 = a_2 J_3 + O(J_3^2)\ , \tag{16.7}$$

a_1 and a_2 being positive constants, uniformly bounded in L, β, p_F and $(u, J_3) \in \mathcal{A}$, such that the following is true.

Then, given any positive integers n and N, there exist positive constants $\tilde{\gamma} < 1$ and $C_{n,N}$, independent of L, β, p_F and $(u, J_3) \in \mathcal{A}$, so that, for any integers $n_0, n_1 \geq 0$ and putting $n = n_0 + n_1$,

1.10*lllll*
$$|\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} \Omega^{3,a}(\mathbf{x})| \leq \frac{1}{|\mathbf{x}|^{2+2\eta_2+n}} \frac{C_{n,N}}{1 + [\Delta|\mathbf{x}|]^N}\ , \tag{16.8}$$

1.101
$$|\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} \Omega^{3,b}(\mathbf{x})| \leq \frac{1}{|\mathbf{x}|^{2+n}} \frac{C_{n,N}}{1 + [\Delta|\mathbf{x}|]^N}\ , \tag{16.9}$$

1.102*lllll*
$$|\Omega^{3,c}(\mathbf{x})| \leq \frac{1}{|\mathbf{x}|^2} \left[\frac{1}{|\mathbf{x}|^{\tilde{\gamma}}} + \frac{(\Delta|\mathbf{x}|)^{1/2}}{|\mathbf{x}|^{\min(0,2\eta_2)}} \right] \frac{C_{0,N}}{1 + [\Delta|\mathbf{x}|]^N}\ , \tag{16.10}$$

where $\bar{\partial}_x$ denotes the discrete derivative and

1.10*clllll*
$$\Delta = \max\{|u|^{1+\eta_1}, \sqrt{(v_0\beta)^{-2} + L^{-2}}\}\ . \tag{16.11}$$

c) $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are even functions of $\mathbf{x}$ and there exists a constant δ^*, of order J_3, such that, if $1 \leq |\mathbf{x}| \leq \Delta^{-1}$ and $v_0^* = v_0(1 + \delta^*)$

1.10*glllll*
$$\begin{aligned} \Omega^{3,a}(\mathbf{x}) &= \frac{1 + A_1(\mathbf{x})}{2\pi^2[x^2 + (v_0^* x_0)^2]^{1+\eta_2}}\ , \\ \Omega^{3,b}(\mathbf{x}) &= \frac{1}{2\pi^2[x^2 + (v_0 x_0)^2]^2} \Big\{ \frac{x_0^2 - (x/v_0)^2}{[x^2 + (v_0 x_0)^2]^2} + A_2(\mathbf{x}) \Big\};, \end{aligned} \tag{16.12}$$

1.10*hlllll*
$$|A_i(\mathbf{x})| \leq c_1\{|J_3| + (\Delta|\mathbf{x}|)^{1/2}\}\ , \tag{16.13}$$

for some constant c_1.

The function $\Omega^{3,a}(\mathbf{x})$ is the restriction to $\mathbb{Z} \times \mathbb{R}$ of a function on $\mathbb{R}^2$, satisfying the symmetry relation

1.10*klllll*
$$\Omega^{3,a}(x, x_0) = \Omega^{3,a}\Big(x_0 v_0^*, \frac{x}{v_0^*}\Big)\ . \tag{16.14}$$

d) Let $\hat{\Omega}^3(\mathbf{k})$, $\mathbf{k} = (k, k_0) \in [-\pi, \pi] \times \mathbb{R}^1$, the Fourier transform of $\Omega^3(\mathbf{x})$. For any fixed $\mathbf{k}$ with $\mathbf{k} \neq (0,0), (\pm 2p_F, 0)$, $\hat{\Omega}^3(\mathbf{k})$ is uniformly bounded as $u \to 0$; moreover, for some constant c_2, c_2',

1.11*alllll*
$$\begin{aligned} |\hat{\Omega}^3(0,0)| &\leq c_2' + c_2|J_3| \log \frac{1}{\Delta}\ , \\ |\hat{\Omega}^3(\pm 2p_F, 0)| &\leq c_2 \frac{1 - \Delta^{\eta_2}}{\eta_2}\ . \end{aligned} \tag{16.15}$$

Finally, if $u = 0$, $\hat{\Omega}^3(\mathbf{k})$ is at most logarithmically divergent at $\mathbf{k} = (0,0)$ for any J_3, and, at $\mathbf{k} = (\pm 2p_F, 0)$, it is singular only if $J_3 < 0$; in this case it diverges as $|\mathbf{k} - (\pm 2p_F, 0)|^{\eta_2}/|\eta_2|$.

e) Let $G(x) = \Omega^3(x, 0)$ and $\hat{G}(k)$ its Fourier transform. For any fixed $k \neq 0, \pm 2p_F$, $\hat{G}(k)$ is uniformly bounded as $u \to 0$, together with its first derivative; moreover

1.11*blllll*
$$\begin{aligned} |\partial_k \hat{G}(0)| &\leq c_2\ , \\ |\partial_k \hat{G}(\pm 2p_F)| &\leq c_2(1 + \Delta^{\eta_2})\ . \end{aligned} \tag{16.16}$$

Finally, if $u = 0$, $\partial_k \hat{G}(k)$ *has a first order discontinuity at* $k = 0$, *with a jump equal to* $1 + O(J_3)$, *and, at* $k = \pm 2p_F$, *it is singular only if* $J_3 < 0$; *in this case it diverges as* $|k - (\pm 2p_F)|^{\eta_2}$.

We comment the above very elaborated theorem.

a)The above theorem holds for any magnetic field h such that $\sin p_F > 0$, if $p_F = h - J_3$. Remember that the exact solution [B] is valid only for $h = 0$. Moreover u has not to be small, see (16.5), and the only small parameter is J_3; however the interesting (and more difficult) case is when u is small.

b)A naive estimate of ε is $\varepsilon = c(\sin p_F)^\alpha$, with c, α positive numbers; in other words we must take smaller and smaller J_3 for p_F closer and closer to 0 or π, *i.e.* for magnetic fields of size close to 1. It is unclear at the moment if this is only a technical problem or a property of the model.

c)If $J_1 \neq J_2$ and $J_3 \neq 0$ one can distinguish, like in the $J_3 = 0$ case (16.1), two regions in the behaviour of the correlation function $\Omega^3(\mathbf{x})$, discriminated by an intrinsic length which is given approximately by the inverse of spectral gap. In the first region the bounds for the correlation function are the same as in the gapless $J_1 = J_2$ case, while in the second region there is a faster than any power decay with rate given essentially by the gap size, which is $O(|u|^{1+\eta_1})$, see (16.11), in agreement with (16.3), found by the exact solution. The interaction J_3 has the effect that the gap becomes anomalous and it aquires a *critical index* η_1; the ratio between the renormalized and bare gap is very small or very large, if u is small, depending on the sign of J_3. In the first region one can obtain the large distance asymptotyc behaviour of $\Omega^3(\mathbf{x})$, see (16.12),(16.13); in the second region only an upper bound is obtained, but even in the $J_3 = 0$ case we are not able to obtain more from the exact solution if $h \neq 0$. If $u = 0$ only the first region is present as the spectral gap is vanishing.

d)It is useful to compare the expression for the large distance behaviour of $\Omega^3(\mathbf{x})$ in the case $u = 0$ with its analogous for the Luttinger model (2.7). A first difference is that, while in the Luttinger model the Fermi momentum is independent of the interaction, in the XYZ model in general *it is changed non trivially* by the interaction, unless the magnetic external field is zero, *i.e.* $p_F = \frac{\pi}{2}$. The reason is that the Luttinger model has special parity properties which are not satisfied by the XYZ chain (except if the magnetic field is vanishing).

e)Another peculiar property of the Luttinger model correlation function is that the dependence on p_F of the correlation function is only by the factor $\cos(2p_F x)$; this is true not only asymptotically (*i.e.* it is true not only in (14.25) but in the complete expression in [BGM]) and is due to a special symmetry of the Luttinger model (the Fermi momentum disappears from the Hamiltonian if a suitable redefinition of the fermionic fields is done, see [BGM]). This is of course not true in the XYZ model and in fact the dependence from p_F of $\Omega^3(\mathbf{x})$ is very complicated. However we will see that $\Omega^3(\mathbf{x})$ can be written as sum of three terms, see (16.6), and from (16.17),(16.9) we have that the derivatives of the first two terms verify the same bounds as their analogue of the Luttinger model (which were p_F independent). This is not true for the third term $\Omega^{3,c}(\mathbf{x})$, in which there are possibly oscillating terms making false a bound on the derivatives like (16.17),(16.9). However we can prove that such term is smaller for large distances, see (16.10) (note that $\tilde{\gamma}$ is J_3 and u independent, contrary to η_2). Of course this is true only for small J_3 and it could be that such third term plays an important role for larger J_3. If we compare (16.12) with $u = 0$ with (14.25) we see that the expressions differ essentially for the factors $A_i(\mathbf{x})$, containing terms of higher order in our expansion. We can prove that $A_i(\mathbf{x})$ verify (16.13) and that the derivatives verify a bound like (16.8),(16.9) which means that the higher order terms verify the same bound as the first order terms, or the same bound as their analogue of the Luttinger model. However the first order terms, or (14.25), have subtle symmetry properties which are very important in analizing the Fourier transform. We are able to prove that $A_1(\mathbf{x})$ verifies (16.14), which says essentially that v_0^* is the *renormalized Fermi velocity*; in fact the decomposition of Ω^{3a} in the form (16.12) decomposition of Ω^{3a} in the form (16.12) with $A_1(\mathbf{x})$ verifying (16.13) is not unique, as one can replace v_0^* with any velocity $\bar{v}_0^*$ of the form $\bar{v}_0^* = v_0^*(1 + O(\lambda))$ and an expression similar to (16.12) with $A_1(\mathbf{x})$ verifying (16.13) is still found; however with $\bar{v}_0^*$ the property (16.14) it is not true, unless $\bar{v}_0^* = v_0^*$, and this allows us to say that v_0^* is the renormalized Fermi velocity. We are not able however to prove a similar properties for $A_2(\mathbf{x})$, see below.

f)Another important property of the Luttinger model correlation function is *the fact that the not oscillating term does not acquire a critical index*, contrary to what happens for the term oscillating with frequency $\frac{p_F}{\pi}$. In the Luttinger model the not oscillating term of the correlation function is exactly (*i.e.* not asymptotically) equal to the non interacting one. Again in the XYZ model this is not true, but one is naturally lead to the conjecture that still the critical index of $\Omega^{3,b}_{L,\beta}(\mathbf{x})$ is vanishing, see for istance [Sp]. In our expansion, we have a series also for the critical index of $\Omega^{3,b}_{L,\beta}(\mathbf{x})$, and while an explicit computation of the first orders gives a vanishing result, it is not obvious that this is true at any order. However due to some hidden symmetries of the model (*i.e.* symmetries enjoyed approximately by the relevant part of the effective action) we can prove that all the coefficients are vanishing proving a Ward identity. We want to stress that this is, at our knowledge, the first example in which an approximate Ward identities is proved in a rigorous way. The Ward identity we find is *not* the same obtained neglecting the regularizations and proceeding formally.

g) The above properties can be used to study the equal time density correlation Fourier transform; if $J_3 = 0$ its first derivative at $k = \pm 2p_F$ is logarithmically divergent at $u = 0$ and it is finite at $k = 0$; if $J_3 \neq 0$ the behaviour of the first derivative at $k = \pm 2p_F$ is completely different, as it is finite if $J_3 > 0$ while it has a power like singularity, if $u = 0$, if $J_3 < 0$ see item e in the Theorem. This is due to the fact that the critical index η_2 appearing in the oscillating term in $\Omega^{3}_{L,\beta}(\mathbf{x})$ has the same sign of J_3 (note that η_2 has nothing to do with the critical index η appearing in the two point fermionic Schwinger function, which is $O((J_3)^2)$). On the other hand the equal time density correlation Fourier transform near $k = 0$ of the Luttinger, XYZ or of the free fermionic gas ($J_1 = J_2$, $J_3 = 0$) behaves in the same way (see also [Sp] for a heuristic explanation). This is due to a parity cancellation in the expansion eliminating the apparent dimensional logarithmic divergence.

h)From (14.25) in the $u = 0$ case we can see that the (bidimensional) Fourier transform can can be singular only at $\mathbf{k} = (0, 0)$ and $\mathbf{k} = (\pm 2p_F, 0)$. If $J_3 = 0$ the singularity is logharithmic at $\mathbf{k} = (\pm 2p_F, 0)$, but there is no singularity if $J_3 > 0$ and there is a power like singularity if $J_3 < 0$, see item d in the Theorem. Then the singularity at $\mathbf{k} = (\pm 2p_F, 0)$ is of the same type as in the Luttinger model, see (14.25). However, we can not conclude that the same is true for the Fourier transform at $\mathbf{k} = 0$, which is bounded in the Luttinger model, while we can not exclude a logarithmic divergence. In order to get such a stronger result, it would be sufficient to prove that the function $\Omega^{3,b}(\mathbf{x})$ is odd in the exchange of (x, x_0) with $(x_0 v, x/v)$, for some v; this property is true for the leading term corresponding to $\Omega^{3,b}(\mathbf{x})$ in (14.25), with $v = v_0$, but seems impossible to prove on the base of our expansion. We can only see this symmetry for the leading term, with $v = v_0^*$

i) Note that our theorem cannot be proved by building a multiscale renormalized expansion, neither by taking as the "free model" the XY one and J_3 as the perturbative parameter, nor by taking as the free model the XXY one and u as the perturbative parameter. In fact, in order to solve the model, one cannot perform a single Bogoliubov transformation as in the $J_3 = 0$ case; the gap has a non trivial flow and one has to perform a different Bogoliubov transformation for each renormalization group integration.

l) If $u = 0$ the critical indices and ν can be computed with any prefixed precision; we write explicitly in the theorem only the first order for simplicity. However, if $u \neq 0$, they are not fixed uniquely; for what concerns ν, this means that, in the gapped case, the system is insensitive to variations of the magnetic field much smaller than the gap size.

m) Finally there is no reason for considering a nearest-neighbor Hamiltonian like (2.10); it will be clear by the following analysis that our resuls still holds for non nearest-neighbor spin hamiltonian, as such hamiltonians differ from (2.10) for *irrelevant* (in the RG sense) terms; see also [Spe] where the case $J_3 = 0$ is studied.

*sec.*17

17. Spinning fermions

*p.*17.1 **17.1.** *The repulsive case.* If the fermions are spinning, the general scheme is the same as the one discussed for spinless fermions, but new complications arise from the fact that the number of running coupling constants

is much higher. Let us consider a system of spinning fermions on a lattice in the not filled band case with Hamiltonian

cll1
$$H = H_0 + \lambda V + \nu N_0 \tag{17.1}$$

with H_0, N_0 given by (2.1), and V given by (2.5). This case was studied in [BM1] to which we refer for details.

One can define an *anomalous integration* similar to the one in §8 for spinless fermions; the localization operator is defined by (8.19),(8.20),(8.21). The spin has the effect that there are more running coupling constants; in fact the relevant part of the effective potential, which in the spinless case is given by (8.25), is ,if $p_F \neq 0, \pi$ for any integer n:

cll
$$\mathcal{L}V^{(h)} = \gamma^h \nu_h F_\nu^{(h)} + \delta_h F_z^{(h)} + g_h^1 F_1^{(h)} + g_h^2 F_2^{(h)} + g_h^4 F_4^{(h)} + \delta_{p_F,\pi/2} g_h^3 F_3^{(h)} \tag{17.2}$$

where

$$F_1^{(h)} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \sum_{\sigma,\sigma'} \sum_{\omega} \psi^{(\le h)+}_{\mathbf{k}'_1+\omega\mathbf{p}_F,\omega,\sigma} \psi^{(\le h)+}_{\mathbf{k}'_2-\omega\mathbf{p}_F,-\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_3+\omega\mathbf{p}_F,\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_4-\omega\mathbf{p}_F,-\omega,\sigma} \delta(\sum_{i=1}^4 \sigma_i \mathbf{k}'_i)$$

F2
$$F_2^{(h)} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \sum_{\sigma,\sigma'} \sum_{\omega} \psi^{(\le h)+}_{\mathbf{k}'_1+\omega\mathbf{p}_F,\omega,\sigma} \psi^{(\le h)+}_{\mathbf{k}'_2-\omega\mathbf{p}_F,-\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_3-\omega\mathbf{p}_F,-\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_4+\omega\mathbf{p}_F,\omega,\sigma} \delta(\sum_{i=1}^4 \sigma_i \mathbf{k}'_i) \tag{17.3}$$

F4
$$F_4^{(h)} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \sum_{\sigma,\sigma'} \sum_{\omega} \psi^{(\le h)+}_{\mathbf{k}'_1+\omega\mathbf{p}_F,\omega,\sigma} \psi^{(\le h)+}_{\mathbf{k}'_2+\omega\mathbf{p}_F,\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_3+\omega\mathbf{p}_F,\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_4+\omega\mathbf{p}_F,\omega,\sigma} \delta(\sum_{i=1}^4 \sigma_i \mathbf{k}'_i) \tag{17.4}$$

$$F_3^{(h)} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} \sum_{\sigma,\sigma'} \sum_{\omega} \psi^{(\le h)+}_{\mathbf{k}'_1+\omega\mathbf{p}_F,\omega,\sigma} \psi^{(\le h)+}_{\mathbf{k}'_2+\omega\mathbf{p}_F,\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_3-\omega\mathbf{p}_F,-\omega,\sigma'} \psi^{(\le h)-}_{\mathbf{k}'_4-\omega\mathbf{p}_F,-\omega,\sigma} \delta(\sum_{i=1}^4 \sigma_i \mathbf{k}'_i)$$

and

$$g_0^2 = \lambda \hat{v}(0) + O(\lambda^2) \qquad g_0^4 = \lambda \hat{v}(0) + O(\lambda^2)$$
$$g_0^1 = \lambda \hat{v}(2p_F) + O(\lambda^2) \qquad g_0^3 = \lambda \hat{v}(2p_F) + O(\lambda^2)$$

Note that $g_2^{(h)}, g_4^{(h)}$ correspond with an interaction with a small exchange of momentum and are called *forward scattering* processes; g_h^1 correspond to an interaction with a big exchange of momenta and it is called *backward scattering*. Finally g_h^3 is possible only at $p_F = \pi/2$ and it is an *Umklapp scattering*.

Of course one can obtain the analiticity of the beta function if the running coupling constant are small enough, proving a result similar to Theorem 1 in §8. However the flow of the running coupling constants is now much more complex. We consider the case $p_F \neq 0, \pi/2, \pi$; the Renormalization Group flow equations for the running coupling constants g_h^1, g_h^2, g_h^4 are given by, if $\mu_h = g_h^2, g_h^4, \delta_h$

$$g^1_{h-1} = g^1_h + g^1_h[-\beta g^1_h + \beta^1_h(\vec{v}_h, \dots, v_0)]$$

$$g^2_{h-1} = g^2_h + g^1_h[-\frac{\beta}{2} g^1_h + \hat{\beta}^h_2(\vec{v}_h, \dots, v_0)] + \beta_2^{(h)}(\mu_h, \nu_h; \dots; \mu_0, \nu_0)$$

$$g^4_{h-1} = g^4_h + g^1_h \hat{\beta}^h_4(\vec{v}_h, \dots, v_0) + \beta_4^{(h)}(\mu_h, \nu_h; \dots; \mu_0, \nu_0)$$

with $\beta > 0$ and we have written explicitely the second order terms. Note that, by trivial symmetry considerations, any contributions to g^1_{h-1} has at least a g_1 end-point. Truncating the above equations at the second order we see that $g^1_h \to 0$ if $g^1_0 > 0$ while grows exiting out of the radius of convergence of the beta function if $g^1_0 < 0$. We consider for the moment the repulsive case $\lambda \hat{v}(2p_F) > 0$. One can proceed as in §10 dividing

the Beta function in a part depending only on the Luttinger model part of the propagator $g_\omega^{(h)}$ (see Lemma 2 in §8) plus a "correction" which is smaller by a factor $\gamma^{\eta h}$. Moreover one can fix the counterterm ν so that $\nu_h = O(\gamma^{\eta h})$ so dividing, like in §10, the Beta function in a part indipendent from ν_h plus a correction smaller by a factor $\gamma^{\eta h}$. Let be $\beta_h^i(\mu_h, \nu_h; \dots, \mu_0, \nu_0)$ the function obtained by $\beta_h^i(\vec{v}_h, \dots, v_0)$ putting $g_h^1 = \nu_h = 0$; one can show, see [BM1], that if

van12
$$\beta_h^2(\mu_h, 0; \dots; \mu_h, 0) = 0 \tag{17.5}$$

van13
$$\beta_h^4(\mu_h, 0; \dots; \mu_h, 0) = 0 \tag{17.6}$$

van11
$$\beta_h^1(\mu_h, 0; \dots; \mu_h, 0) = 0 \tag{17.7}$$

van14
$$\hat\beta_h^2(\mu_h, 0; \dots; \mu_h, 0) = 0 \tag{17.8}$$

van14a
$$\hat\beta_h^4(\mu_h, 0; \dots; \mu_h, 0) = 0 \tag{17.9}$$

then it is possible to choose a counterterm ν such that, if $\lambda v(2p_F) > 0$ then

$$\nu_h \to_{h\to-\infty} 0 \quad g_h^1 \to_{h\to-\infty} 0 \quad \frac{Z_{h-1}}{Z_h} \to_{h\to-\infty} \gamma^\eta$$

and $g_h^2, g_h^4, \delta_h \to_{h\to-\infty} g_\infty^2, g_\infty^4, \delta_\infty$ with $\eta = a\lambda^2 + O(\lambda^3)$ with $a > 0$, and $g_\infty^2 = g_0^2 + O(\lambda^2)$, $g_\infty^4 = g_0^2 + O(\lambda^2)$, $\delta_\infty = O(\lambda^2)$.

In order to prove (17.5),(17.6),(17.7),(17.8) we follow essentially the same strategy for the spinless case, see §11, but in the spinning case the role of the Luttinger model is played by the *Mattis model* with Hamiltonian

$$H = \sum_{\omega=\pm1} \sum_{\sigma=\pm1/2} \int_0^L dx(1+\delta)\psi^+_{\omega,\sigma,x}(i\omega\partial_x - p_F)\psi^+_{\omega,\sigma,x} +$$

$$\sum_{\omega,\sigma} g^{2,p} \int_0^L dxdyv(x-y)\psi^+_{\omega,x,\sigma}\psi^-_{\omega,x,\sigma}\psi^+_{-\omega,y,\sigma}\psi^-_{-\omega,y,\sigma} + \sum_{\omega,\sigma} g^{2,o} \int_0^L dxdyv(x-y)[\psi^+_{\omega,x,\sigma}\psi^-_{\omega,x,\sigma}\psi^+_{-\omega,y,-\sigma}\psi^-_{-\omega,y,-\sigma}$$

$$+ \sum_{\omega,\sigma} g^{4,p} \int_0^L dxdyv(x-y)\psi^+_{\omega,x,\sigma}\psi^-_{\omega,x,\sigma}\psi^+_{\omega,y,\sigma}\psi^-_{\omega,y,\sigma} + \sum_{\omega,\sigma} g^{4,o} \int_0^L dxdyv(x-y)\psi^+_{\omega,x,\sigma}\psi^-_{\omega,x,\sigma}\psi^+_{\omega,y,-\sigma}\psi^-_{\omega,y,-\sigma}$$

Also such model is solvable, see [M], and the Schwinger functions can be explicitely computed [M0].

Reasoning as in §11 one can study the above model by Renormalization group. Let us start from the spin symmetric Mattis model $g^{2,p} = g^{2,o}$ and $g^{4,p} = g^{4,o}$ in which one obtains an expression for the relevant part of the effective potential similar to (17.2) but with $g_h^1 = g_h^3 = \nu_h = \delta_h \equiv 0$. As the finite volume Schwinger functions of the Mattis model are known we can reason exactly as in §11 and we obtain (17.5),(17.6).

In order to prove (17.7),(17.8) we study by renormalizaton group the non-spin symmetric Mattis model in which $g^{2,p} \neq g^{2,o}$ and $g^{4,p} \neq g^{4,o}$. One obtains an expression for the relevant part of the effective potential similar to (17.2) but with $g_h^1 = g_h^3 = \nu_h = 0$ but the relevant part of the effective potential is given by

$$\mathcal{L}V^{(h)} = g_h^{2,p} F_{2,p}^{(h)} + g_h^{2,o} F_{2,o}^{(h)} + g_h^{4,o} F_{4,o}^{(h)}$$

where $F_{2,p}^{(h)}$ and $F_{2,o}^{(h)}$ are given by (17.3) with $\sigma = \sigma'$ and $\sigma = -\sigma'$ respectively, and in the same way are defined $F_{p,4}^{(h)} = 0$ and $F_{o,4}^{(h)}$, see (17.4).

The beta function with all the running coupling constants computed with the same scale driving the flow of of $g_i^{\alpha,h}$ with $i = 1, 4$ and $\alpha = o, p$ of the non spin symmetric Mattis model can be written as

vvvh2
$$\sum_{n_1,..,n_4} [g_h^{4,o}]^{n_1} [g_h^{2,p}]^{n_2} [g_h^{2,o}]^{n_3} [\delta_h]^{n_4} \beta^{(h)}_{i,\alpha;n_1,\dots,n_4} \tag{17.10}$$

Again reasoning as in §(11) by the comparison of the non spin symmetric Schwinger functions of the Mattis model it follows the vanishing of (17.10) and from the independence of $g^{4,o}$, $g_h^{2,p}$, $g^{2,o}$, δ it follows that

jjj
$$\beta^{(h)}_{i,\alpha;n_1,\dots,n_4} = 0 \tag{17.11}$$

Let us return to the spin symmetric model with effective potential given by (17.2) and $g_h^{2,p} = g_h^{2,o}$, $g_h^{1,p} = g_h^{1,o}$. For the conservation of the quasi-particle and spin indices, it is not possible to have a contribution to g^2_{h-1} involving only one $g_h^{1,o}$ and any number of μ_h; then the only possibility is to have a contribution to g^2_{h-1} involving only one $g_h^{1,p}$ and any number of μ_h. But such contribution is equal to

vvvh1
$$[g_h^{o,4}]^{n_1}[g_h^{p,2}]^{n_2-1}[g_h^{o,4}]^{n_3}[g_h^{p,1}][\delta_h]^{n_4}\beta^{(h)}_{2,\alpha;n_1,\dots,n_4} \tag{17.12}$$

so it is vanishing. In fact the function $\beta^{(h)}_{2,\alpha;n_1,\dots,n_4}$ in (17.12) and (17.10) are the same as $F^{(h)}_{p,2} = F^{(h)}_{p,1}$. This proves (17.8). The same argument can be repeated for $i = 4$ so proving (17.9).

Finally let us consider the contribution to g_1^{h-1} involving only one g_h^1 and any number of μ_h. We consider a contribution to $g^{p,1}_{h-1}$; by symmetry considerations it follows that there is no contribution to $g^{p,1}_{h-1}$ involving one $g_h^{o,1}$ and any number of μ_h, and the only possibility is a contribution involving one $g_h^{p,1}$ and any number of μ_h. But replacing $g_h^{p,1}$ with $g_h^{p,2}$ and remembering that $F^{(h)}_{p,2} = F^{(h)}_{p,1}$ this contribution coincides with a contribution to $g^{p,2}_{h-1}$ so it is vanishing by (17.11). On the other hand we are considering the spin symmetric case so $g^{p,1}_{h-1} = g^{o,1}_{h-1}$ and (17.7) is proved.

At the end the following theorem can be proved (the proof in [BM1] refers to the continuum case):

THEOREM 10. *Given the Hamiltonian (17.1) for spinning fermions with $p_F \neq 0, \pi/2, \pi$, if $\lambda\hat{v}(2p_F) > 0$ there exists an $\varepsilon > 0$ such that, for $|\lambda| \leq \varepsilon$, there are functions $\nu(\lambda), \eta(\lambda)$ such that the two point Schwinger function is given by*

$$S(\mathbf{x};\mathbf{y}) = \frac{g(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\eta}} + \frac{A(\mathbf{x};\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{1+\eta}}$$

with $A(\mathbf{x};\mathbf{y})$ bounded by a constant, $\nu(\lambda) = O(\lambda)$ and $\eta = a\lambda^2 + O(\lambda^3)$, with $a > 0$.

In the half filled band case $p_F = \frac{\pi}{2}$ there is a running coupling constant more g_h^3 whose second order flow is not trivial and given by

$$g^3_{h-1} = g^3_h + \beta g^3_h(g^1_h - 2g^2_h)$$

so that the flow of the running couplng constants becomes much more complex to study.

It is quite clear that one can add to the Hamiltonian (17.1) a term uP representing the interaction with a commensurate or an incommensurate potential; in the $\lambda\hat{v}(2p_F) > 0$ and under proper conditions on p_F forbidding the comparison of extra running coupling constants (for istance if p/π a rational number we require $p_F \neq np/2$ for any integer n) one can prove results similar to their analogue in the spinless case, see §13.

The attractive case. The analysis above shows that the presence of the spin , if $p_F \neq 0, \pi/2, \pi$ and the interation is repulsive, is in some sense irrelevant, as the two point Schwinger funtion asymptotic behaviour is similar to the one in the spinless case. The situation is completely different in the attractive case $\lambda\hat{v}(2p_F) < 0$, in which the running coupling costants do not remain in the convergence radius of the series for the Beta function unless, in the infinite volume limit, the temperature is larger than $e^{\frac{-\kappa}{|\lambda\hat{v}(2p_F)|}}$ for some suitable constant κ. It is easy in fact to check that for $h \geq h_{\bar\beta} \simeq O(\log(\bar\beta^{-1}))$, with $\bar\beta \leq e^{\frac{\kappa}{|\lambda\hat{v}(2p_F)|}}$, the running coupling constants remain $O(\lambda)$. It is generally believed that the growing of the coupling $g_1^{(h)}$ in the attractive case , or of $g_3^{(h)}$ if $p_F = \pi/2$ and always in the attractive case, are related to the opening of a gap and to

exponential decay of correlations. Our result gives an upper bound on a possible gap in the ground state enrgy, saying that $|\Delta| \leq e^{\frac{-\kappa}{|\lambda \hat{v}(2p_F)|}}$.

A proof that really there is a gap in the spectrum is up to now lacking except in the remarkable case of the Hubbard model; it is a particular case of the model we are considering in which $v(x-y) = \delta_{x,y}$ and $p_F = \frac{\pi}{2}$. In this case it was proved in [LW] that the ground state has a gap for any $\lambda < 0$; moreover the ground state is such that each site is occupied by an electron and the spins are alternating (so a spin density wave with period $\frac{1}{\rho}$).

In the general situation only mean-field approximations are at our disposal; a very simple heuristic mean field argument from which one can deduce from the growing of $g_1^{(h)}$ the appearence of a gap is the following one. As g_h^1 is the instable process, this suggests that the relevant interactions involve the exchange of a momentum of order $2p_F$ so that the important term in the interaction are of the form, for $|k|, |k'| \leq p_F/4$ (say)

sds
$$\sum_{\omega,\sigma} [\frac{1}{L} \sum_k \psi^+_{k+\omega p_F,\sigma} \psi^-_{k-\omega p_F,\sigma}][\frac{1}{L} \sum_{k'} \psi^+_{k'-\omega p_F,-\sigma} \psi^-_{k'+\omega p_F,-\sigma}] \tag{17.13}$$

Making a BCS-type mean field theory we write

$$|S|e^{i\alpha} = \frac{1}{L} \sum_k < \psi^+_{k+p_F,\sigma} \psi^-_{k-p_F,\sigma} >$$

and neglecting quantum fluctuations one obtains an effective interaction $\sum_{x,\sigma} |S| \cos(2p_F x + \alpha) \psi^+_{x,\sigma} \psi^-_{x,\sigma}$, from which the existence of a gap at the Fermi surface can be deduced. In this argument there is however a flaw; it does not take into account that, if p_F/π is irrational, then it can be that $2np_F \simeq 2p_F$ mod. 2π for very large n, so that it is not *a priori* true that one the interactions exchanging momenta $O(2np_F)$ are negligible.

A more correct way to perform a mean field analysis is the following one. One can replace in the interaction (assumed local for simplicity) $\psi^+_{x,\sigma} \psi^-_{x,\sigma} \psi^+_{x,-\sigma} \psi^-_{x,-\sigma}$ two fermionic fields with a classical field

17.14
$$\psi^+_{x,\sigma} \psi^-_{x,\sigma} \to \varphi(x) + \left[\psi^+_{x,\sigma} \psi^-_{x,\sigma} - \varphi(x)\right], \tag{17.14}$$

neglecting (this is the approximation) terms quadratic in the "fluctuations" $[\psi^+_{x,\sigma} \psi^-_{x,\sigma} - \varphi(x)]$ so obtaining a model

17.15
$$H_0 + \lambda \sum_{x \in \Lambda} \varphi(x) \psi^+_{x,-\sigma} \psi^-_{x,-\sigma} - \sum_{x \in \Lambda} \varphi^2_x. \tag{17.15}$$

This model is called *variational Holstein model* and the non trivial problem is to minimize the ground-state energy with respect to φ. One arrives to the same model also considering the interaction of fermions with a phonon field, neglecting quantum fluctuations, and il will be discussed it sec.(19). We anticipate that even in this approximation the existence of periodic ground states (which can be commensurate or incommensurate depending if p_F/π is a rational or an irrational number) is not trivial (for istance is not proved for small λ and p_F/π irrational, see below). In other words even in a mean field model the existence of a gap is not proven in general in the attractive case.

*sec.*18
18. Fermions interacting with Phonon fields

*p.*18.1 **18.1.** *Interaction with a quantized phonon field.* The Hamiltonian of a system of one dimensional fermions on a lattice interacting locally with the optical modes of a quantized phonon field is given by (2.8) and (2.9). We refer to [BGL] for more details. The two-point Schwinger function can be written as

18.1
$$S(\mathbf{x};\mathbf{y}) = \frac{\int P(\mathrm{d}\Phi) \int P(\mathrm{d}\psi)\, e^{-g\tilde{V}} \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{y},\sigma}}{\int P(\mathrm{d}\Phi) \int P(\mathrm{d}\psi) e^{-g\tilde{V}}}, \tag{18.1}$$

where $P(\mathrm{d}\Phi)$ is a bosonic integration with propagator

18.2
$$v(\mathbf{x};\mathbf{y}) = \frac{1}{L\beta} \sum_{e^{ik_0\beta}=1} \sum_{e^{ikL}=1, |k|\le\pi} \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\sigma_0^2 k_0^2 + 1 + b^2 2(1-\cos k)}, \tag{18.2}$$

with

18.3
$$|v(\mathbf{x},0)| \le \frac{C(b)}{\sigma_0}\, e^{-\kappa\sigma_0^{-1}|x_0|} e^{-\kappa_2(b)|x|}, \tag{18.3}$$

and

18.4
$$\begin{aligned} \kappa_2(b) &= \begin{cases} O(b^{-1}), & \text{for } b\to\infty\ , \\ O(\log b^{-1})), & \text{for } b\to 0\ , \end{cases} \\ C(b) &= \begin{cases} O(1), & \text{for } b\to 0\ , \\ O(b^{-1}\log b), & \text{for } b\to\infty\ . \end{cases} \end{aligned} \tag{18.4}$$

Integrating out the boson fields in (18.1) we obtain

18.5
$$S(\mathbf{x};\mathbf{y}) = \frac{\int P(d\psi) e^{g^2 V} \psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{y},\sigma}}{\int P(d\psi) e^{g^2 V}}, \tag{18.5}$$

with

18.6
$$V = \frac{1}{8}\sum_{x,y\in\Lambda} \int_{-\beta/2}^{\beta/2} \mathrm{d}x_0 \int_{-\beta/2}^{\beta/2} \mathrm{d}y_0\, v(\mathbf{x}-\mathbf{y}) \psi^+_{x,\sigma}\psi^-_{x,\sigma}\psi^+_{y,\sigma'}\psi^-_{y,\sigma'}. \tag{18.6}$$

The only difference with the previously considered interacting spinless Hamiltonian is that it is not local in time; it is easy to check that this changes nothing in the previous discussion.

Then in the spinless case one can prove that the Schwinger function has an anomalous behavior; of course the convergence radius is vanishing as $b\to\infty$ (corresponding to a long range interactions, *i.e.* $p_0\to 0$); it is also vanishing if $\sigma_0\to\infty$.

In the spinning case one is in the situation of the preceding section, so results are found only for temperatures greater than $e^{-\kappa/g^2}$.

p.18.2 **18.2.** *Classical limit: the static Holstein model.* We can study the above model also in the "static" limit in which the quantum fluctuation are neglected; to corresponds to put formally $\sigma_0=\infty$, $b=0$; one gets in this way the again *variational Holstein model* found at the end of the previous section.

The ground state problem is now equivalent to find the field ϕ minimizing the ground state fermionic energy. Before discussing this model, we stress again that the relationship between the *variational Holstein model* and the models considered in this and in the preceding sections are not very understood. Surely if there is no spin the quantum fluctuation changes completely the behaviour (the static Holstein model makes no difference among spinning or spinless fermions), at least for small interactions.

sec.19

19. The variational Holstein model

p.19.1 **19.1.** *Old results.* In the two preceding sections we arrived to the *variational Holstein model* either by considering a mean field model for spinning fermions with an attractive potential or by considering a semiclassical model for phonon-fermion interaction. The problem is to find the function $\varphi(x)$ minimizing the ground state energy of a system of fermions with Hamiltonian

19.1
$$\begin{aligned} H &\equiv H_L^{\mathrm{el}} + \frac{1}{2}\sum_{x\in\Lambda}\varphi^2(x) \\ &= \sum_{x,y\in\Lambda} t_{xy}\,\psi^+_x\psi^-_y - \mu\sum_{x\in\Lambda}\psi^+_x\psi^-_x - \lambda\sum_{x\in\Lambda}\varphi(x)\psi^+_x\psi^-_x + \frac{1}{2}\sum_{x\in\Lambda}\varphi^2(x)\ . \end{aligned} \tag{19.1}$$

At finite L, the fermionic Fock space is finite dimensional, hence there is a minimum eigenvalue $E_L^{\rm el}(\varphi,\mu)$ of the operator $H_L^{\rm el}$, for each given phonon field φ and each value of μ; let $\rho_L(\varphi,\mu)$ be the corresponding fermionic density. The aim is to minimize the functional

19.2
$$F_L(\varphi,\mu) = E_L^{\rm el}(\varphi,\mu) + \frac{1}{2}\sum_{x\in\Lambda}\varphi_x^2\ , \tag{19.2}$$

subject to the condition

19.3
$$\rho_L(\varphi,\mu) = \rho_L\ , \tag{19.3}$$

where ρ_L is a fixed value of the density, converging for $L\to\infty$, say to ρ.

It is generally believed that, as a consequence of Peierls instability argument, [P] and [F], in the limit $L\to\infty$, there is a field $\varphi^{(0)}$, uniquely defined up to a spatial translation, which minimizes (19.2) with the constraint (19.3), and it is a function of the form $\bar\varphi(2\pi\rho x)$, where $\bar\varphi(u)$ is a 2π-periodic function in u. This is physically interpreted by saying that one-dimensional metals are unstable at low temperature, in the sense that they can lower their energy through a periodic distorsion of the "physical lattice" with period $1/\rho$ (in the continuous version of the model, since $1/\rho$ is not an integer in general). There are a few results about this model in literature.

(1) An exact result, [KL], makes rigorous the theory of Peierls instability for the model (19.1) in the case $\rho=\rho_L=1/2$ (*half filled band case*), for any value of λ. In fact, in this case it has been proved that there is a global minimum of $F_L(\varphi)$ of the form $\varepsilon(\lambda)(-1)^x$, where $\varepsilon(\lambda)$ is a suitable function of λ. This means that the periodicity of the ground state phonon field is 2 (recall that in our units 1 is just the lattice spacing): this phenomenon is called *dimerization*. The proof heavily relies on symmetry properties which hold only in the half filled case. As in the case of the Hubbard model, the special symmetries at $p_F=\pi/2$ play a crucial role.

(2) In [AAR,BM] Peierls instability for the Holstein model is proven assuming λ *large enough*: in that case the fermions are almost classical particles and the quantum effects are treated as perturbations. The results hold for the commensurate or incommensurate case; in particular in the incommensurate case the function $\bar\varphi(u)$, related to the minimizing field through the relation $\varphi(x)=\bar\varphi(2\pi\rho x)$, has infinite many discontinuities. On the contrary, in the small λ case, according to numerical results, $\bar\varphi(u)$ has been conjectured to be an analytic function of its argument, both for the commensurate and incommensurate cases, [AAR]. The results are closely releted to the existence of the so called "Aubry-Mather" sets in Classical Mechanics.

p.19.2 **19.2.** *New results.* We discuss here a result in [BGM2] found using the RG methods rewieved above, in the case of small λ and any p_F. A local minimum of (19.2) satisfying (19.3) must fulfill the conditions

19.4
$$\begin{aligned}\varphi(x) &= \lambda\rho_x(\varphi,\mu)\ ,\\ \rho_L\ &= \frac{1}{L}\sum_x \rho_x(\varphi,\mu)\ ,\end{aligned} \tag{19.4}$$

and

19.5
$$M_{xy}\ \equiv \delta_{xy} - \lambda\frac{\partial}{\partial\varphi_x}\rho_y(\varphi,\mu) \quad \text{is positive definite}\ . \tag{19.5}$$

If φ is a solution of (19.4), it must satisfy the condition $\hat\varphi_0 = L^{-1}\sum_x\varphi(x) = \lambda\rho_L$. On the other hand, if we define $\chi_x=\varphi(x)-\hat\varphi_0$, we can see immediately that $\rho_L(\varphi,\mu)=\rho_L(\chi,\mu+\lambda\hat\varphi_0)$. It follows that we can restrict our search of local minima of (19.2) to fields φ with zero mean, satisfying the conditions

19.6
$$\begin{aligned}\varphi(x) &= \lambda(\rho_x(\varphi,\mu)-\rho_L)\ ,\\ \rho_L\ &= \frac{1}{L}\sum_x \rho_x(\varphi,\mu)\ ,\end{aligned} \tag{19.6}$$

and the condition (19.5).

Of course, if the field $\varphi(x)$ satisfies (19.6), the same is true for the translated field $\varphi(x+n)$, for any integer n. On the other hand, one expects that the solutions of (19.6) are even with respect to some point of Λ; hence we can eliminate the trivial source of non-uniqueness described above by imposing the further condition $\varphi(x) = \varphi(-x)$. We shall then consider only fields of the form

19.7
$$\varphi(x) = \sum_{n=-[L/2]}^{[(L-1)/2]} \hat{\varphi}'_n e^{\frac{i2n\pi x}{L}} , \qquad \hat{\varphi}'_{-n} = \hat{\varphi}'_n \in \mathbb{R} , \qquad \hat{\varphi}_0 = 0 . \tag{19.7}$$

We want to consider the case of rational density, $\rho = P/Q$, P and Q relatively prime, and we want to look for solutions such that $\varphi(x) = \varphi(x+Q)$. Hence, we shall look for solutions of (19.6) with $L = L_i = iQ$, $\rho_L = \rho$, and

19.8
$$\varphi(x) = \sum_{n=-[Q/2]}^{[(Q-1)/2]} \hat{\varphi}_n e^{i2\pi\rho n x} , \qquad \hat{\varphi}_n = \hat{\varphi}_{-n} \in \mathbb{R} , \qquad \hat{\varphi}_0 = 0 . \tag{19.8}$$

Note that the condition on L allows to rewrite in a trivial way the field $\varphi(x)$ of (19.8) in the general form (19.7), by putting $\hat{\varphi}'_n = 0$ for all n such that $(2n\pi)/L \neq 2\pi\rho m, \forall m$, and by relabeling the other Fourier coefficients.

The conditions (19.6) can be easily expressed in terms of the variables $\hat{\varphi}_n$; if we define $\hat{\rho}_n$ so that

19.9
$$\rho_x(\varphi,\mu) = \sum_{n=-[Q/2]}^{[(Q-1)/2]} \hat{\rho}_n(\varphi,\mu) e^{i2n\pi\rho x} , \tag{19.9}$$

we get

19.10
$$\hat{\varphi}_n = \lambda \hat{\rho}_n(\varphi,\mu) , \qquad n \neq 0 , \quad n = -[Q/2], \dots, [(Q-1)/2] , \tag{19.10}$$

19.11
$$\hat{\rho}_0(\varphi,\mu) = \rho_L . \tag{19.11}$$

Also the minimum condition (19.5) can be expressed in terms of the Fourier coefficients; we get that the $L \times L$ matrix

19.12
$$\bar{M}_{nm} \equiv \delta_{nm} - \lambda \frac{\partial}{\partial \hat{\varphi}'_n} \hat{\rho}'_m(\varphi,\mu) \tag{19.12}$$

has to be positive definite, if the field φ satisfies (19.10) and (19.11) and $\hat{\rho}'_m(\varphi,\mu)$ is defined analogously to $\hat{\varphi}'_m$ in (19.8). Hence, if we restrict the space of phonon fields to those of the form (19.8), we have to show that the $Q \times Q$ matrix

19.13
$$\tilde{M}_{nm} \equiv \delta_{nm} - \lambda \frac{\partial}{\partial \hat{\varphi}_n} \hat{\rho}_m(\varphi,\mu) \tag{19.13}$$

has to be positive definite, if the field φ satisfies (2.10) and (19.11).

Then the following result holds.

THEOREM 11. *Let $\rho = P/Q$, with P, Q relative prime integers, $L = L_i \equiv iQ$. Then, for any positive integer N, there exist positive constants ε, $\tilde{\varepsilon}$, c and K, independent of i, ρ and N, such that, if*

19.14
$$0 \le \frac{4\pi v_0}{\log(\tilde{\varepsilon} v_0 L)} \le \lambda^2 \le \varepsilon \frac{v_0^2 (1 + \log v_0^{-1})^{-1}}{K^N N! \log(cQ/v_0^4)} , \tag{19.14}$$

where

19.15
$$v_0 = \sin(\pi\rho) , \tag{19.15}$$

there exist two solutions $\varphi^{(\pm)}$ of (19.6), with $L = L_i$, $1 - \mu = \cos(\pi\rho)$ and $\rho_L = \rho$, of the form (19.8). The matrices $\tilde{M}$ corresponding to these solutions, defined as in (19.13), are positive definite.

Moreover, the Fourier coefficients $\hat\varphi_n^{(\pm)}$ *verify, for* $|n|>1$*, the bound*

19.16
$$|\hat\varphi_n^{(\pm)}| \le \left(\frac{\lambda^2}{v_0|n|}\right)^N |\hat\varphi_1^{(\pm)}| \,. \tag{19.16}$$

Finally, $\lambda\hat\varphi_1^{(\pm)}$ *is of the form*

19.17
$$\lambda\hat\varphi_1^{(\pm)} = \pm v_0^2 \exp\Big\{ -\frac{2\pi v_0 + \beta^{(\pm)}(\lambda, L)}{\lambda^2}\Big\}\,, \tag{19.17}$$

with

19.18
$$|\beta^{(\pm)}(\lambda, L)| \le C\lambda^2\left(1+\log\frac{1}{v_0}\right)\,, \tag{19.18}$$

where C *is a suitable constant.*

The one-particle Hamiltonian corresponding to this solution has a gap of order $|\lambda\hat\varphi_1|$ *around* μ*, uniformly on* i*.*

The above theorem proves that there are two stationary points of the ground state energy in correspondence of a periodic function with period equal to the inverse of the density, if the coupling is small enough and the density is rational, and that these stationary points are local minima at least in the space of periodic functions with that period. The energies associated to such minima are different so that the ground state energy is not degenerate.

The theorem is proved by writing $\rho_x(\varphi,\mu)$ as an expansion convergent for small λ and solving the set of equations (19.10) by a contraction method. As a byproduct it is found that the $\hat\varphi_n$ are fast decaying, (see (19.16)), so that $\varphi(x)$ is really well approximated by its first harmonics (this remark is important as the number of harmonics could be very large).

The results are uniform in the volume, so they are interesting from a physical point of view (a solution defined only for $|\lambda| \le O(1/L)$ should be outside any reasonable physical value for λ). The case in [KL] for the half filled case is contained in Theorem 11, but in [KL] it is also proved that the solution is a global minimum.

Finally the lower bound in (19.14) is a large volume condition: this is not a technical condition as, if the number of Fermions is odd, there is Peierls instability only for L large enough. The upper bound for λ in (19.14) requires λ to decrease as Q increases: in particular irrational density are forbidden. This requirement is due to the discreteness of the lattice and to *Umklapp phenomena*. Note that the dependence of the maximum λ allowed on Q is not very strong as it is a logarithmic one.

We know that $\rho_x(\varphi,\mu)$ is well defined for small λ not only in the rational density case, (in which the proof is almost trivial), but also in the irrational case: in fact the small divisor problem due to the irrationality of the density can be controlled thanks to a Diophantine condition (see Theorem 2). However to solve the set of equations (19.10), it is used a contraction method which is not trivially adaptable in the latter case. The same kind of problem arises in proving the positive defineteness of $\bar M_{nm}$ in the rational case (and this is the reason why we are able to prove that the stationary points are local minima only in the space of periodic functions with prefixed period). It is not known if such problems are only technical or there is some physical reason for this to happen.

sec.20
20. Coupled Luttinger liquids

A natural question is what happens if we consider two or more fermionic chains coupled with an hopping term from one chain to another. This problem is surprisingly very difficult, as the number of running coupling constants is very high (fifteen or more, see [F]) and many of them are growing so that a rigorous

analysis in the limit $\beta \to \infty$ based on RG seems impossible. We can consider a simple model of two *Mattis models* exchanging Cooper pairs between them. Even of this model a Renormalization group analysis of the $\beta \to \infty$ limit is not possible (the flow equations are similar to the one for spinning fermions in the attractive case) but it is possible to perform a sort of mean feld theory, see [M3],[M4], obtaining the equivalent of a BCS theory but the corresponding critical temperature T_c is not exponentially small (see also [CG] for a perturbative third order analysis).

We consider the following functional integral

2.8hhh
$$\mathcal{Z}_{L,\beta,r} = \int P_a(d\psi) P_b(d\psi) e^{-\mathcal{V}_a - \mathcal{V}_b - \mathcal{V}_{ab} - h_r} \ , \tag{20.1}$$

where, calling $2g^2 \equiv g_t$

$$\mathcal{V}_i = -\lambda \frac{1}{(L\beta)^4} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \sum_{\omega,\sigma,\sigma'} \psi^+_{\mathbf{k}_1,\omega,\sigma,i} \psi^-_{\mathbf{k}_2,\omega,\sigma,i} \psi^+_{\mathbf{k}_3,-\omega,\sigma',i} \psi^-_{\mathbf{k}_4,-\omega,\sigma',i} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4)$$

1.2ahhh
$$\begin{aligned} \mathcal{V}_{ab} = &-2[\frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}_1,\omega_1} \psi^+_{\mathbf{k}_1,\omega_1,\frac{1}{2},a} \psi^+_{-\mathbf{k}_1,-\omega_1,-\frac{1}{2},a}][\frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}_2,\omega_2} \psi^-_{-\mathbf{k}_2,-\omega_2,-\frac{1}{2},b} \psi^-_{\mathbf{k}_2,\omega_2,\frac{1}{2},b}] \\ &-2[\frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}_1,\omega_1} \psi^+_{\mathbf{k}_1,\omega_1,\frac{1}{2},b} \psi^+_{-\mathbf{k}_1,-\omega_1,-\frac{1}{2},b}][\frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}_2,\omega_2} \psi^-_{-\mathbf{k}_2,-\omega_2,-\frac{1}{2},a} \psi^-_{\mathbf{k}_2,\omega_2,\frac{1}{2},a}] \end{aligned} \tag{20.2}$$

$$h_r = \frac{1}{L\beta} \sum_{\mathbf{k}} \sum_{\omega,i} [r\psi^-_{\mathbf{k},\omega,\frac{1}{2},i} \psi^-_{-\mathbf{k},-\omega,-\frac{1}{2},i} + r\psi^+_{\mathbf{k},\omega,\frac{1}{2},i} \psi^+_{-\mathbf{k},-\omega,-\frac{1}{2},i}]$$

where $\psi^{\pm}_{\mathbf{k},\omega,\sigma,i}$ is grassman variable describing a fermion with momentum $\mathbf{k}$ and spin $\sigma = \pm 1/2$ associated with the chain $= a, b$, $\mathcal{V}_i$ describes the interaction between fermions belonging to the same chain and $\mathcal{V}_{ab}$ describes the tunelling of Cooper pairs from one chain to another, in the *Barden aproximation*. The term h_r represents the interaction with an external field and the parameter r is real and positive (for fixing ideas). If $g = 0$ the system reduces to two independent Mattis models, and the Schwinger functions have an anomalous behaviour like (13.30).

It is convenient to write the interaction in terms of gaussian variables. We write

$$\mathcal{V}_{ab} = -2[\Delta_a \bar{\Delta}_b + \Delta_b \bar{\Delta}_a]$$

where

$$\Delta_i = \frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}',\omega} \psi^+_{\mathbf{k}',\omega,\frac{1}{2},i} \psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},i} \quad \bar{\Delta}_i = \frac{g}{(\beta L)^{3/2}} \sum_{\mathbf{k}',\omega} \psi^-_{-\mathbf{k}',-\omega,-\frac{1}{2},i} \psi^-_{\mathbf{k}',\omega,\frac{1}{2},i}$$

By using the identity (*Hubbard-Stratanovich transformation*) ($\phi = u + iv$, $\bar{\phi} = u - iv$, $u, v \in R$)

z1hhh
$$e^{2ab} = \frac{1}{2\pi} \int_{R^2} du dv e^{-\frac{1}{2}|\phi|^2} e^{a\phi + b\bar{\phi}} \tag{20.3}$$

we can rewrite the partition function as

$$\mathcal{Z}_{L,\beta,r} = \frac{1}{2\pi} \int_{R^2} du_1 dv_1 e^{-\frac{1}{2}|\phi_1|^2} \frac{1}{2\pi} \int_{R^2} du_2 dv_2 e^{-\frac{1}{2}|\phi_2|^2}$$

z2hhh
$$\int P_a(d\psi) e^{-\mathcal{V}_a} \int P_b(d\psi) e^{-\mathcal{V}_b} e^{-h_r} e^{\phi_1 \Delta_a + \bar{\phi}_1 \bar{\Delta}_b} e^{\phi_2 \Delta_b + \bar{\phi}_2 \bar{\Delta}_a} \tag{20.4}$$

Performing the change of variables

$$(u_i, v_i) \to \sqrt{\beta L}(u_i, v_i)$$

we obtain

$$\mathcal{Z}_{L,\beta,r} = \frac{\beta L}{2\pi}\int_{R^2} du_1 dv_1 e^{-\frac{\beta L}{2}|\phi_1|^2}\frac{\beta L}{2\pi}\int_{R^2} du_2 dv_2 e^{-\frac{\beta L}{2}|\phi_2|^2}$$

$$\int P_a(d\psi)e^{-\mathcal{V}_a}\int P_b(d\psi)e^{-\mathcal{V}_b}e^{g(\phi_1-r/g)\mathcal{D}_a+g(\bar\phi_1-r/g)\bar{\mathcal{D}}_b}e^{g(\phi_2-r/g)\mathcal{D}_b+g(\bar\phi_2-r/g)\bar{\mathcal{D}}_a}$$

where

$$\mathcal{D}_i = \frac{1}{(\beta L)}\sum_{\mathbf{k}',\omega}\psi^+_{\mathbf{k}',\omega,\frac12,i}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},i} \quad \bar{\mathcal{D}}_i = \frac{1}{(\beta L)}\sum_{\mathbf{k}',\omega}\psi^-_{-\mathbf{k}',-\omega,-\frac12,i}\psi^-_{\mathbf{k}',\omega,\frac12,i}$$

After the integration of the Fermi fields, if $\vec v = (u_1, u_2, v_1, v_2)$

$$\mathcal{Z}_{L,\beta,r} = [\frac{\beta L}{2\pi}]^2\int_{R^4} du_1 dv_1 du_2 dv_2 e^{-\frac{\beta L}{2}[(u_1+\frac{r}{g})^2+(u_2+\frac{r}{g})^2+v_1^2+v_2^2]}e^{-\beta L\mathcal{F}^{L,\beta,r}_{\lambda,g}(\vec v)}$$

$$= [\frac{\beta L}{2\pi}]^2\int_{R^4} du_1 dv_1 du_2 dv_2 e^{-\beta L\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)} \tag{20.5}$$

ffhhh

where where

$$e^{-\beta L\mathcal{F}^{L,\beta}_{\lambda,g}(\vec v)} = \int P_a(d\psi)\int P_b(d\psi)e^{-\mathcal{V}_a-\mathcal{V}_b}e^{g\phi_1\mathcal{D}_a+g\bar\phi_1\bar{\mathcal{D}}_b}e^{g\phi_2\mathcal{D}_b+g\bar\phi_2\bar{\mathcal{D}}_a} \tag{20.6}$$

z.zzhhh

The partition function is then written as the (four dimensional) integral of the exponential $e^{-\beta L\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)}$.

If the function

$$\mathcal{H}^{\beta,r}_{\lambda,g}(\vec v) = \lim_{L\to\infty}\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)$$

is two times differentiable and it admits a non degenerate *global minimum* $\vec v^*$ for β large enough (the parameter r is introduced just to remove the possible degeneration) then

$$\lim_{r\to 0}\lim_{L\to\infty}\frac{e^{-\beta L\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)}}{\int du_1 du_2 dv_1 dv_2 e^{-\beta L\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)}} = \delta(\vec v - \vec v^*) \tag{20.7}$$

z.zhhh

If we can prove that $\mathcal{H}^{\beta,r}_{\lambda,g}(\vec v)$ has a global minimum the model is solved; all the Schwinger functions can be computed using (20.7) and, if $\vec v^* \neq 0$, there is a *spontaneous gap generation.*

So the problem is reduced to the computation of $\mathcal{H}^{\beta,r}_{\lambda,g}(\vec v)$ and to the determination of its global minimum. However $\mathcal{H}^{\beta,r}_{\lambda,g}(\vec v)$ is given by the Grassmanian integral (20.6) which is not quadratic in the grassman variables and it is non trivial to compute, especially in the $\lambda >> g_0^t$ case. One has to take into account the interaction $\mathcal{V}_a + \mathcal{V}_b$ which is responsible in the $g_0^t = 0$ case of the Luttinger liquid behaviour of the model.

Let us assume that, given $\vec v^*$, the function $\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)$ is differentiable in a small neighborood of $\vec v^*$ (uniformly in L, β) and

$$\frac{\partial\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)}{\partial u_i}|_{\vec v=\vec v^*} = 0 \quad \frac{\partial\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)}{\partial v_i}|_{\vec v=\vec v^*} = 0 \tag{20.8}$$

strhhh

This means that $\vec v^*$ is an *extremal point* for $\mathcal{H}^{L,\beta,r}_{\lambda,g}(\vec v)$. An extremal point satisfys the following *extremality equations*:

$$u_1 + \frac{r}{g} - g\frac{1}{L\beta}\sum_{\mathbf{k}',\omega}[<\psi^+_{\mathbf{k}',\omega,\frac12,a}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},a}> + <\psi^-_{-\mathbf{k}',-\omega,\frac{-1}{2},b}\psi^-_{\mathbf{k}',\omega,\frac12,b}>] = 0$$

$$u_2 + \frac{r}{g} - g\frac{1}{L\beta}\sum_{\mathbf{k}',\omega}[<\psi^+_{\mathbf{k}',\omega,\frac12,b}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},b}> + <\psi^-_{-\mathbf{k}',-\omega,\frac{-1}{2},a}\psi^-_{\mathbf{k}',\omega,\frac12,a}>] = 0$$

$$v_1 + ig\frac{1}{L\beta}\sum_{\mathbf{k}',\omega}[<\psi^+_{\mathbf{k}',\omega,\frac12,a}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},a}> - <\psi^-_{-\mathbf{k}',-\omega,\frac{-1}{2},b}\psi^-_{\mathbf{k}',\omega,\frac12,b}>] = 0 \tag{20.9}$$

nikk

$$v_2 + ig\frac{1}{L\beta}\sum_{\mathbf{k}',\omega}[< \psi^+_{\mathbf{k}',\omega,\frac{1}{2},b}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},b} > - < \psi^-_{-\mathbf{k}',-\omega,\frac{-1}{2},a}\psi^-_{\mathbf{k}',\omega,\frac{1}{2},a} >] = 0$$

where

$$L\beta < \psi^+_{\mathbf{k}',\omega,\frac{1}{2},i}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},i} >=$$

$$\frac{\int P_a(d\psi)e^{-\mathcal{V}_a}\int P_b(d\psi)e^{-\mathcal{V}_b}e^{g\phi_1\mathcal{D}_a+g\bar\phi_1\bar{\mathcal{D}}_b}e^{g\phi_2\mathcal{D}_b+g\bar\phi_2\bar{\mathcal{D}}_a}\psi^+_{\mathbf{k}',\omega,\frac{1}{2},i}\psi^+_{-\mathbf{k}',-\omega,\frac{-1}{2},i}}{\int P_a(d\psi)e^{-\mathcal{V}_a}\int P_b(d\psi)e^{-\mathcal{V}_b}e^{g\phi_1\mathcal{D}_a+g\bar\phi_1\bar{\mathcal{D}}_b}e^{g\phi_2\mathcal{D}_b+g\bar\phi_2\bar{\mathcal{D}}_a}} \tag{20.10}$$

*cc*0000

and a similar one for $< \psi^-_{-\mathbf{k}',-\omega,\frac{-1}{2},i}\psi^-_{\mathbf{k}',\omega,\frac{1}{2},i} >$. One has then to compute the r.h.s. of (20.9); if $\lambda = 0$ such computation is trivial and one obtains, as in BCS theory, that the gap and the critical temperature are exponentially small in $\frac{1}{g^2}$. However the presence of the interaction along the chain, which is responsible of the anomalous behaviour, has a dramatic effect. One could think that the r.h.s. of the self-consistence equation (20.9) is obtained by the one obtained in the $\lambda = 0$ case simply replacing the propagator(3.4) with the Mattis model Schwinger function (see [A1], page 209). This is in fact what is found by a naive first order perturbation theory. However the true result is more complex, as also the gap aquires a critical index. In fact one can compute (20.10) by the techniques describes above and the following result holds, see [M3] and [M4].

THEOREM 12. *There exist an ε such that, if $\lambda \geq 0$, $\lambda, |g| \leq \varepsilon$ the function $\mathcal{H}^{\beta,r}_{\lambda,g}(\vec v)$ defined in (2.10) is differentiable at $u_1 = u_2$, $v_1 = v_2$ and the extremality equations (3.2) are pairwise equal. In particular the l.h.s. of third and the fourth are vanishing while the first and the second are equal to, if $\frac{1}{\beta} \leq K|gu|$, $K < 1$*

$$u + r/g - g^2u\frac{1}{\eta}[(\frac{|gu|}{A})^{-\eta} - 1][a^{-1} + \lambda f(g,\lambda,u)] + g^2u(\frac{|gu|}{A})^{-\eta}\tilde f(g,\lambda,u) = 0 \tag{20.11}$$

ffahhh

where $\eta = \beta_1\lambda + \tilde\eta$, $|\tilde\eta| \leq C\lambda^2$, $|f|, |\tilde f| \leq C$, and C, a, β_1, A are positive constants.

Note that (20.11) is a non-BCS or *anomalous* self consistence equation describing a superconductor whose normal state is a Luttinger liquid; the Luttinger interaction modifies the self-consistence equation for the gap from the BCS-like one to (20.11). Note that λ, g^2 have to be small but there is no restriction on their ratio, in particular it can be $\frac{\lambda}{g^2} >> 1$.

COROLLARY. *There exist ε and $K < 1$ such that, if $\lambda \geq 0$, $\lambda, |g| \leq \varepsilon$ then $\mathcal{H}^{\beta.r}_{\lambda,g}(\vec v)$ admits two extremal points, both if $\frac{\lambda}{g^2} < K$ or $\frac{\lambda}{g^2} > K^{-1}$. In the limit $\beta \to \infty, r \to 0$ they become of the form $(\pm\Delta, \pm\Delta, 0, 0)$. In particular if $\frac{\lambda}{g^2} > K^{-1}$*

$$|g\Delta| = A[\frac{g^2}{a\eta}]^{\frac{1}{\eta}}[1 + O(\lambda) + O(\frac{g^2}{\lambda})]^{\frac{1}{\eta}} \tag{20.12}$$

*ffa*1*hhh*

while if $\frac{\lambda}{g^2} < K$

$$|g\Delta| = Ae^{\frac{-a+O(g)}{g^2}}$$

The above analys says two one dimensional spinning Fermi systems with an *intrachain* interaction given only by forward scattering and an *interchain* interaction expressed by a Cooper pair tunnelling hamiltonian, in the Barden approximation, are such that the two point Schwinger function has a behaviour similar to the Mattis model Schwinger function if $T > T_c$ while for $T \leq T_c$ there is long distance exponential decay releted to the opening of a gap Δ; $T_c \simeq \Delta$ and Δ has the non BCS form given by (20.12) if the intrachain interaction is smaller than the interchain one.

sec.21

21. Bidimensional Fermi liquids

The techniques we have applied to one dimensional fermions are general and can be applied also in $d \geq 2$. In this case much less is known, and there is no till now rigorous costruction of the theory in the $\beta \to \infty$ limit. The study of $d \geq 2$ fermions started in [BG] and in [FT1],[FT2], [FTMR1,2], [DR1,2]. In [BG] and in [FT1], [FT2] a renormalization group analogous to the $d = 1$ case was defined; many new problems appear due to the fact that the singularity (*i.e.* the Fermi surface) are not two points but a circle or a shere. The main result obtained in such papers was the definition of a well defined matematical setting, $n!$ bounds for the perturbative series and the definition of the beta function.

However it appears that even truncating arbitrarely (as there is no proof of the convergence of the Beta function, but only $n!$ bounds) at the second order there are problems; one has infinitely many running coupling constants and: (1) if the interaction is attractive, the flow is not bounded due to the BCS instability, while (2) if it is repulsive due to the Kohn-Luttinger phenomenon it is likely that, except for very particular interactions with special symmetries, the flow is still not bounded. As there is the generation of a gap, the fermionic techniques discussed till here have likely to be supplemented by Cluster expansion techniques (the theory becomes partly *bosonic* due to the appearence of a Goldstone boson).

At the moment the only rigorous construction for a problem of interacting fermions in $d = 2$ is for temperature $T \geq e^{\frac{-k}{|\lambda|}}$ [FMRT2], [DR1] and [DR2]; note that we cannot expect to reach colder region due to the appearence of BCS instability at $T_c = e^{\frac{-a}{|\lambda|}}$ (but $\kappa/c >> 1$, see below; so pehaps fermionic techniques will allow to reach at least $\kappa/c \simeq 1$).

Let us consider a model in $d = 2$ of interacting fermions with Hamiltonian $H = H_0 + \lambda V + \nu N_0$, where H_0 and V are defined by the analogue of (2.2),(2.7) in two dimensions *with an ultraviolet cut-off*. In $d = 2$ the Fermi surface is the circle $k_1^2 + k_2^2 - p_F^2 = E(\mathbf{k})$ and the propagator is given by $\sum_{h=-\infty}^{0} g^{(h)}(\mathbf{x}-\mathbf{y})$ with

mm1
$$g^{(h)}(x-y) = \int dk_0 d\mathbf{k} f_h(k_0^2 + [E(\mathbf{k}) - p_F^2]^2) \frac{e^{ik_0(t-s)+i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + E(\mathbf{k}) - p_F^2} \tag{21.1}$$

Passing to polar coordinates we find

mm2
$$g^{(h)}(x-y) = \int dk_0 d\vartheta \int |\mathbf{k}| d|\mathbf{k}| f_h(k_0^2 + [E(\mathbf{k}) - p_F^2]^2) \frac{e^{ik_0(t-s)+i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + E(\mathbf{k}) - p_F^2} \tag{21.2}$$

and we can introduce another decomposition over the integration in ϑ in the following way. The anolous of radious γ^h around the Fermi surface is divided in *sectors* centered at $\vartheta = \vartheta_r$ and of angular width $\gamma^{h/2}$ (the choice $\gamma^{h/2}$ is not arbitrary, see below). Then $1 = \sum_\omega \chi^{h,\omega}(\vartheta)$, where $\chi^{h,\omega}(\vartheta)$ are compact support functions with support in $\gamma^{h/2-1/2} \leq |\vartheta - \vartheta_\omega| \leq \gamma^{h/2+1/2}$, $\sum_\omega 1 = \gamma^{-h/2}$ and

$$g_\omega^h(\vec{x}-\vec{y}) = e^{i\boldsymbol{\omega} p_F(\mathbf{x}-\mathbf{y})} \bar{g}_\omega^h(\vec{x}-\vec{y})$$

with

bnbn
$$\bar{g}_\omega^h(\vec{x}-\vec{y}) = \int dk_0 d\vartheta \chi_\omega^h(\vartheta) \int k dk f_h(k_0^2 + [E(\mathbf{k}) - p_F^2]^2) \frac{e^{ik_0(t-s)+i[(\mathbf{k}-\boldsymbol{\omega} p_F)(\mathbf{x}-\mathbf{y})}}{-ik_0 + E(\mathbf{k}) - p_F^2} \tag{21.3}$$

which is bounded by

oop
$$|g_\omega^h(\vec{x}-\vec{y})| \leq \gamma^{3h/2} \frac{C_N}{1 + [\gamma^{(h)}|t-s| + \gamma^{(h)}|(\mathbf{x}-\mathbf{y})_r| + \gamma^{h/2}|(\mathbf{x}-\mathbf{y})_t|]^N} \tag{21.4}$$

where $(\mathbf{x}-\mathbf{y})_r = |\mathbf{x}-\mathbf{y}| \cos\vartheta_\omega$ and $(x-y)_t = |\mathbf{x}-\mathbf{y}| \sin\vartheta_\omega$. As in $d = 1$ one can write

oppo
$$\psi_x = \sum_h \sum_\omega e^{i\boldsymbol{\omega} p_F \mathbf{x}} \psi_{\omega,\vec{x}}^{(h)} \tag{21.5}$$

where $\psi^{(h)}_{\omega,\vec{x}}$ has propagator given by $\bar{g}^h_\omega(\vec{x}-\vec{y})$. The difference with respect to the $d=1$ case is that $\sum_\omega = \gamma^{-h/2}$. We write a tree expansion as in the preceding section and we write the truncated expectation as sum over anchored trees times determinants; the Gram-Hadamard inequality can be applied as there is always a finite number of kind of fermions (on the contrary, if like in [BG] one considers a continuous ω variables, one finds technical difficulty for doing the Gram-Hadamard bound). Then we get the following bound for the effective potential; fixed a tree τ and an an anchored tree T we get:
(1) a factor $\gamma^{-(\frac{5}{2})h_v(s_v-1)}$ for the integration over the coordinates, if s_v are the subtrees coming out from the vertex v;
(2) a factor $\gamma^{\frac{3}{2}h_v\tilde{n}_v}$ where $\tilde{n}_v$ are the propagators (in the anchored tree T or in the determinants) in the cluster v and not in any smaller one; calling m^4_v the number of vertices with 4 external lines we get, using (5.32), (5.33), a factor

$$C^n\gamma^{hD}\prod_v \gamma^{(h_v-h_{v'})(\frac{3}{2}(2m^4_v-\frac{n^e_v}{2})-\frac{5}{2}(m^4_v-1))} \tag{21.6}$$

if D is a proper dimension;
(3) we have now to sum over ω, which is the crucial point. In order to perform this sum, suppose that we have a number of vertices v with all the external lines fixed to some scale $h_{v'}$, with n^e_v external lines; then the sum over ω gives

$$\prod_v [\gamma^{\frac{-h_{v'}}{2}(n^e_v-3)}\chi(n^e_v>3)] \tag{21.7}$$

In order to understand this formula one has to note that for each vertex v there are n^e_v sums over v but a)the conservation of momentum on each vertex eliminates one sum b)the vertices are connected by an anchored tree in the truncated expectations; so if v_1, v_2 are two vertices connected by a line l of the the spanning tree, fixing the sector of v_1 of the half-line forming l fixes automatically the half-line line of the vertex v_2 which formes l; c)by geometrical considerations [FMRT1] the fact that the momenta have to stay in an anolous aroun the Fermi surafce of radius $\gamma^{(h)}$ and that the sectors are $O(\gamma^{h/2})$ cancels another sum.

However in general the external lines are not all on the same scale and we need a bit more complicated argument. One can do an iterative argument for summing over ω; let we consider the end-points (assume only four fields interactions for simplicity). In general the scales of the external lines are different; let we fix all of the them equal to the largest one. By the above argument we get a factor (all the lines are fixed to have the same scale):

$$\prod_v \gamma^{-(\frac{1}{2})(h_v-h_{v'})m^4_v} \tag{21.8}$$

Now we have to sum on the lines of the vertices whose scale was not the largest one. We contract all the minimal clusters in points, and we iterate the above argument; the lines external to the minimal clusters v were fixed to a sector of width $\gamma^{h_v/2}$; so summing on the sectors of these lines (fixing all of them to the smallest scale) gives a factor $\gamma^{-(\frac{1}{2})(h_v-h_{v'})}$ and at the end we get

$$\prod_v \gamma^{(\frac{1}{2})(h_v-h_{v'})(n^e_v-3)}\chi(n^e_v>3) \tag{21.9}$$

Putting togheter all terms we get

$$\prod_v \gamma^{(h_v-h_{v'})(\frac{3}{2}(2m^4_v-\frac{n^e_v}{2})-\frac{5}{2}(m^4_v-1)-\frac{1}{2}m^4_v+(\frac{1}{2})(n^e_v-3)\chi(n^e_v>3)} \tag{21.10}$$

which gives

$$\prod_v \gamma^{(h_v-h_{v'})[\frac{-3}{4}n^e_v+\frac{5}{2}+(\frac{1}{2})(n^e_v-3)\chi(n^e_v>3)]} \tag{21.11}$$

From the above formula we see that the power counting is exactly the same than the $d = 1$ case *i.e.* the dimension of the cluster with two external lines is -1 and the one with 4 is 0. Then if one can restrict to sum to $|P_v| \geq 4$ the series for the effective potential would be convergent (the above argument works really for trees which, for any v $20 \geq |P_v| \geq 4$, see [DR1]; in fact the sector sums done like above produces a constant $K^{|P_v|}$ which should develop a factorial. For v with $|P_v| \geq 20$ one uses that the dimension is very negative. For this technical point, see [DR1]).

To renormalize the above theory one uses a definition very similar to the one for $d = 1$ fermions. If we allow logaritmic divergences, we have only to renormalize at the first order the clusters with two external lines (logarithmic divergences give a factor in the bounds $C^n \lambda^n h_\beta^n \simeq C^2 \lambda^n (\log \beta)^n$ which allow to get convergence for $T \geq e^{\frac{-k}{|\lambda|}}$, with $kC \leq 1$).

The definition of localization is the same as in the $d = 1$ case (note that, by the conservation of momenta the ω index of external lines of the clusters with two external lines are the same)

klmb
$$\mathcal{L} \int d\mathbf{k} dk_0 \psi^{+,\leq h}_{k,\omega} \psi^{-,\leq h}_{k,\omega} \mathcal{W}^{(h)}(k_0, \mathbf{k}) = \int d\mathbf{k} dk_0 \psi^{+,\leq h}_{k,\omega} \psi^{-,\leq h}_{k,\omega} \mathcal{W}^{(h)}(0, \boldsymbol{\omega} p_F) \tag{21.12}$$

Note that the theory is rotation invariant so that $\mathcal{W}^{(h)}(0, \boldsymbol{\omega} p_F)$ is in fact independent from ω.

There is however a difference with respect to the $d = 1$ case (see [DR2]). The effect of $\mathcal{R}$ gives

lmbb1
$$\mathcal{R} \int d\mathbf{k} dk_0 \psi^{+,\leq h}_{k,\omega} \psi^{-,\leq h}_{k,\omega} \mathcal{W}^{(h)}(k_0, \mathbf{k}) = \int d\mathbf{k} dk_0 \psi^{+,\leq h}_{k,\omega} \psi^{-,\leq h}_{k,\omega} [(\mathbf{k} - \boldsymbol{\omega} p_F) \partial_{\mathbf{k}} \mathcal{W}^{(h)} + k_0 \partial_{k_0} \mathcal{W}^{(h)}] \tag{21.13}$$

Let us fix a reference frame in which the axis 1 is directed as $\boldsymbol{\omega}$ and 2 is ortogonal; then $\mathbf{k} = k_1, k_2$ and $(1, 0)$ is a radial vector while $0, 1$ is a tangential vector. Then we can write the above equation as, if $\mathbf{k} - \omega p_F = \mathbf{k}'$ ($\mathbf{k}'$ is the momentum measured from the Fermi surface)

klknn
$$k'_1 \int dt \partial_{k_1} W^{(h)} + \int_0^1 dt k'_2 \partial_{k_2} W^{(h)} \, , \tag{21.14}$$

where $k'_1 = O(\gamma^{(h)})$, $k'_2 = O(\gamma^{h/2})$. The first addend gives a factor $\gamma^{h_{v'} - h_v}$ which is the right factor to leave only a logaritmic divergence; however the second addend gives a factor

pomb
$$\gamma^{\frac{h_{v'} - h_v}{2}} \gamma^{-h_v/2} \tag{21.15}$$

which is not the correct one to have only logarithmic divergences. To solve this problem in [DR2] it is used the following argument:

1) One can write the renormalized cluster as

pncs
$$\int_0^1 dt \mathcal{W}'(t) = \mathcal{W}'(0) + \int_0^1 dt (1 - t) \mathcal{W}''(t) \tag{21.16}$$

The second factor has in any case the right dimensional factor $\gamma^{h_{v'} - h_v}$; the first gives problems taking the tangential component of $\mathbf{k}$. Let us fix a reference frame as above. Then we can write the first addend in the above equation as

$$k'_1 \partial_{k_1} \mathcal{W}^{(h)}(\boldsymbol{\omega} p_F) + k'_2 \partial_{k_2} \mathcal{W}^{(h)}(\boldsymbol{\omega} p_F)$$

But

pipo
$$\partial_{k_2} \mathcal{W}^{(h)}(\boldsymbol{\omega} p_F) = \partial_\rho \mathcal{W}^{(h)}(\boldsymbol{\omega} p_F) \frac{\partial \rho}{\partial k_2}|_{\boldsymbol{\omega} p_F} = 0 \tag{21.17}$$

as $\boldsymbol{\omega} p_F = (p_F, 0)$ in this reference frame.

2) We have seen that the fact that an half-line is contracted with another one in the spanning tree has the effect that of the two sums over ω for any half-line a priori necessary only one has to be done; the contraction of two half-line eliminates a sum over ω. One can simply "extract a proper loop line", which means to develop a bit the determinant in the formula for the truncated expectations to extract a propagator. This do no produce factorial and allows to make a sum over ω less, so gaining $\gamma^{h_v/2}$, see [DR2].

app.A1

Appendix A1. Graphs, diagrams and trees

p.A1.1 **A1.1.** *Graphs.* Given a set V with n elements, we shall call *graph* τ on V a couple (V, E), where E is a subset of unordered pairs of elements in V; we shall write $V = V(\tau)$ and $E = E(\tau)$ and shall call *points* the elements of $V(\tau)$ and *lines* the elements of $E(\tau)$. We shall denote by $|V(\tau)|$ and by $|E(\tau)|$ the number of elements in $V(\tau)$ and in $E(\tau)$, respectively; of course $|V(\tau)| = n$. We shall write also $\ell \in \tau$ for $\ell \in E(\tau)$. See Fig. A1.

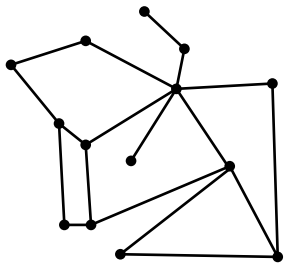


FIG. A1. A graph τ with 14 points and 18 lines.

If a line ℓ connects two points $v, w \in V(\tau)$ we shall write also $\ell = (vw)$: we say that the line ℓ is *incident* with the points v and w. Two points $v, w \in V(\tau)$ are *adjacent* if $(vw) \in E(\tau)$, while two lines are *adjacent* if they are incident with the same point.

Given a point $v \in V(\tau)$ we define *degree* of the point v the number $d(v)$ of lines incident with v; a point such that $d(v) = 1$ is called an *endpoint*. Of course

A1.1
$$\sum_{v \in V(\tau)} d(v) = 2|E(\tau)| \ . \tag{A1.1}$$

A subgraph τ' of τ is a couple (V', E') with $V' = V(\tau') \subset V(\tau)$ and $E' = E(\tau)$ a subset of lines (vw) in $E(\tau)$ with $v, w \in V(\tau')$; we shall write $\tau' \subset \tau$.

A graph τ is *connected* if for any $v, w \in \tau$ there exist $p \in \mathbb{N}$ and p points $v_1, \dots, v_p$, with $v_1 = v$ and $v_p = w$, such that v_j and v_{j+1} are adjacent for each $j = 1, \dots, p-1$: in such a case we say that the lines $(v_1 v_2), \dots, (v_{p-1} v_p)$ form a path $\mathcal{P}$ on τ connecting the point v with the point w. We shall say also that $\mathcal{P}$ *crosses* or *intersects* the points $v_1, \dots, v_p$. See Fig. A2. A graph is *disconnected* if it is not connected.

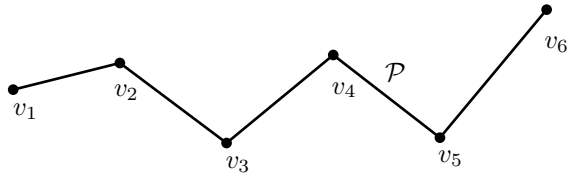


FIG. A2. A path $\mathcal{P}$ connecting v_1 with v_6.

A graph is *acyclic* if it has no cycle (or *loop*), *i.e.* if for any two points $v, w \in V(\tau)$ there is only one path connecting them.

p.A1.2 **A1.2.** *Trees.* A *tree graph* (or *tree* tout court) τ is a connected acyclic graph. If $|V(\tau)| = n$ we say that τ is a tree with n points.

Given a *tree* one has

A1.2
$$|E(\tau)| = |V(\tau)| - 1 \, . \tag{A1.2}$$

Note that given a tree τ any subgraph (subtree) of τ is still connected and acyclic: so any subtree is a tree.

A *rooted tree* is a tree with a distinguished point v_0. A rooted tree can be seen as a partially ordered set of points connected by lines. The partial ordering relation can be denoted by $\preceq$: we shall say that $v \prec w$ if there is a path $\mathcal{P}$ connecting w with v_0 and v is crossed by $\mathcal{P}$. We can also superpose an arrow on each line pointing towards v_0: we say that the lines of the tree are *oriented*; by extension also the tree is said to be oriented. In the following (and in all the paper) by trees we shall mean rooted trees. See Fig. A3.

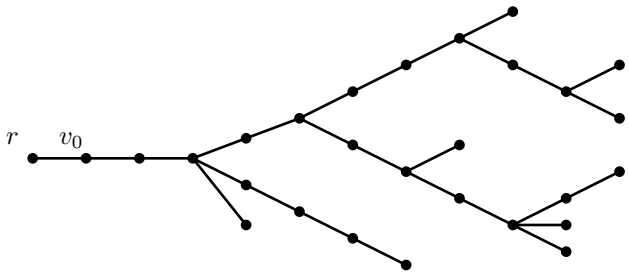


FIG. A3. A rooted tree of order 9 with 27 vertices.

We shall call also *vertices* the points in $V(\tau)$. The point v_0 is called the first vertex of τ. To identify the first vertex v_0 we can draw an extra point r and an extra oriented line ℓ connecting v_0 with r. We shall call r the *root* of τ and ℓ the *root line*. Such a line is added to the lines in $E(\tau)$, while the root is not considered a vertex. With such a convention, (A2.2) has to be replaced with

A1.3
$$|E(\tau)| = |V(\tau)| = n \, . \tag{A1.3}$$

Note also that in this way (A1.1) becomes

A1.4
$$\sum_{v \in V(\tau)} d(v) = 2|E(\tau)| - 1 \, . \tag{A1.4}$$

Given a vertex $v \in V(\tau)$ we denote by v' the node immediately preceeding v, *i.e.* the vertex $\prec v$ such that $(v'v) \in E(\tau)$. We say that the line $\ell = (v'v)$ *exits from* v and *enters* v'. Note that the vertex v' is uniquely defined, as the ordering relation implies a bijective correspondence between lines and vertices: given a vertex there is one and only one line exiting from it.

For any vertex there are $s_v \geq 0$ exiting lines: one has $s_v = 0$ if v is an endpoint. We define the *order* of a tree as the number of its endpoints. We call *trivial* a vertex v with $s_v = 1$ and *nontrivial* a vertex v either with $s_v \geq 2$ or with $s_v = 0$ (this means that the endpoints are counted as nontrivial vertices). Denote by

$V_{\rm f}(\tau)$ the set of endpoints in τ, by $V_{\rm t}(\tau)$ the set of trivial vertices in τ and by $V_{\rm nt}(\tau)$ the set of nontrivial vertices in τ: of course $V(\tau) = V_{\rm t}(\tau) \cup V_{\rm nt}(\tau)$ and

A1.5
$$V_{\rm f}(\tau) = \{v \in V_{\rm nt}(\tau) : s_v = 0\} \, . \tag{A1.5}$$

By the notation $v \notin V_{\rm f}(\tau)$ we mean $v \in V(\tau) \setminus V_{\rm f}(\tau)$.

Given a vertex $v \in V(\tau)$ the subgraph (V', E') with

A1.6
$$\begin{aligned} V' &= \{w \in V(\tau) : w \succeq v\} \, , \\ E' &= \{\ell \in E(\tau) : \ell = (w'w) : w \succ v\} \, , \end{aligned} \tag{A1.6}$$

is a rooted subtree with root v'.

The just defined trees are sometimes called *unlabeled trees*, in order to distinguish them from the "labeled trees" (to be defined).

The unlabeled trees are identified if superposable up to a continuous deformations of the lines on the plane such that the endpoints coincide: in such a case we say that they are *equivalent*. In Fig. A4 two unequivalent unlabeled trees of order $n = 3$ are drawn. Note that the indices used to identify the vertices $v \notin V_{\rm f}(\tau)$ play no rôle.

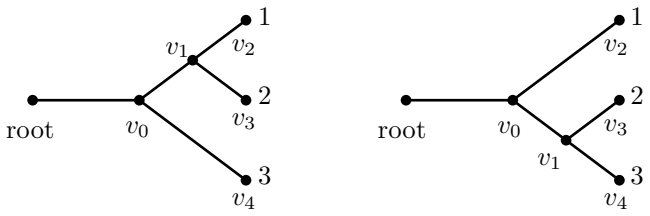


FIG. A4. Two unequivalent unlabeled trees of order 3.

The notions which will be used will be that of unlabeled tree and, mostly, that of labeled tree.

A (rooted) labeled tree can be obtained from an unlabeled tree by assigning labels h_v to its vertices $v \in V(\tau)$ in the following way. A label $h \le 0$ is associated to the root. If $\mathcal{T}_{h,n}$ denotes the corresponding set of labeled trees of order n (*i.e.* with n endpoints), we introduce a set of vertical lines, labelled by an integer assuming values in $[h, 2]$, such that each vertex $v \in V(\tau)$ is contained in a vertical line $h' \in [h, 2]$ (this will be always possible, as the lines can be continously deformed): then we set $h_v = h'$. The label h_v will be called the *frequency* or the *scale* of the vertex v. By construction $h_v > h$ for all $v \in V(\tau)$ and $h_v > h + 1$ for all $v \in V_{\rm f}(\tau)$. Moreover if $v \prec w$ then $h_v < h_w$.

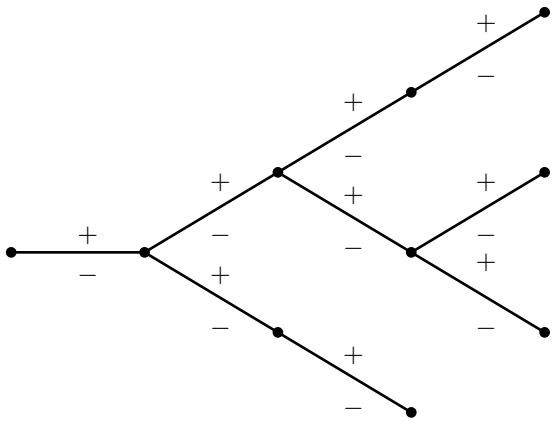
The number of trees is controlled through the following result.

LEMMA A1. *The number of (rooted) unlabeled trees with n points is bounded by C^n for some constant C.*

Proof. The number of (rooted) unlabeled trees is bounded by the number of one-dimensional random walks W with $2n$ steps. This can be proved as follows.

We can imagine to move along the tree by remaining to the left of the lines and starting from the root line. We move forward until an endpoint is reached: in this case we turn backwards until we meet a nontrivial vertex; then we turn once more forward and so on, until we come back to the root line. See Fig. A5: $+$ means that we move from left to right along the line, while $-$ means that we move from right to left.

Each time we move forward along a line we associate to it a sign $+$, while we associate to it a sign $-$ when we move backwards. So the tree can be characterized by a collections of $2n$ signs $\pm$ which define a walk $W = \{\pm \pm \dots \pm\}$. Note that not all one-dimensional random walks with $2n$ steps correspond to unlabeled



$$W = \{+++\,+--+\,+-+--\,++--\}$$

FIG. A5. A rooted tree and the corresponding walk W.

trees: we call *compatible* the random walks for which this happens. For instance the first sign is always a $+$ and the last one is always a $-$: moreover the overall number of $+$ signs has to be equal to the overall number of signs $-$: note that the correspondence between unlabeled trees and one-dimensional compatible random walks is 1-to-1. By neglecting all the constraints we can bound the number of collections of $2n$ signs, hence the number of unlabeled trees with n nodes, by 2^{2n}, that is the overall number of random walks with $2n$ steps. So we can choose $C = 4$ and the assertion follows. ■

Given a tree with n vertices one has, as it is straighforward to check,

A1.7
$$\begin{aligned} & 1 \le |V_{\rm f}(\tau)| \le \begin{cases} n-1\,, & \text{if } n \ge 2\,, \\ 1\,, & \text{if } n = 1\,, \end{cases} \\ & |V_{\rm nt}(\tau)| \le 2|V_{\rm f}(\tau)| - 1\,. \end{aligned} \tag{A1.7}$$

The number of labeled tree in $\mathcal{T}_{h,n}$ can not be bounded uniformly in h: there are at most $2n-1$ nontrivial vertices, by (A1.7), but once they has been fixed, one can add many trivial vertices between them, and the number of possible insertions goes to infinity for $h \to \infty$. Nevertheless we have the following result about labeled trees.

LEMMA A3. *Let $\mathcal{T}_{h,n}$ be the number of labeled trees of order n and with scale h assigned to the root. If $\gamma > 1$ and $\alpha > 0$, then*

A1.8
$$\sum_{\tau \in \mathcal{T}_{h,n}} \prod_{v \notin V_{\rm f}(\tau)} \gamma^{-\alpha(h_v - h_{v'})} \le C_2^n\,, \tag{A1.8}$$

for some constant C_2.

Proof. Let us denote by $\mathcal{T}^*_{h,n}$ the set of labeled trees of order n having only nontrivial vertices, and by τ^* any element in $\mathcal{T}^*_{h,n}$: of course τ^* will have $n-1$ (nontrivial) vertices. A labeled tree τ of order n can be imagined as formed from a tree τ^* of order n, by inserting trivial vertices between them: the number of inserted vertices automatically determines the values of the scale labels.

Fixed a tree τ, so that the corresponding tree τ^* is determined, we can write

A1.9
$$\prod_{v \notin V_{\rm f}(\tau)} \gamma^{-\alpha(h_v - h_{v'})} = \prod_{v \in V_{\rm nt}(\tau^*) \setminus V_{\rm f}(\tau^*)} \gamma^{-\alpha(h_v - h_{v'})}\,, \tag{A1.9}$$

where, for v seen as a vertex of τ^*, v' denotes the vertex in τ^* immediately preceeding v. The tree τ can be obtained by inserting $h_v - h_{v'}$ trivial vertices between $v \in \tau^*$ and $v' \in \tau^*$. Then we have

A1.10
$$\sum_{\tau \in \mathcal{T}_{h,n}} \prod_{v \notin V_{\rm f}(\tau)} \gamma^{-\alpha(h_v - h_{v'})} = \sum_{\tau^* \in \mathcal{T}^*_{h,n}} \prod_{v \in V(\tau^*) \setminus V_{\rm f}(\tau^*)} \gamma^{-\alpha(h_v - h_{v'})} \ . \tag{A1.10}$$

Denote by $\mathcal{T}^*_n$ the set of unlabeled trees of order n having only nontrivial vertices. Then

A1.11
$$\sum_{\tau^* \in \mathcal{T}^*_{h,n}} = \sum_{\tau^* \in \mathcal{T}^*_n} \sum_{\{h_v\}_{v \in \tau^*}} \ , \tag{A1.11}$$

so that

A1.12
$$\begin{aligned} \sum_{\tau^* \in \mathcal{T}^*_{h,n}} \prod_{v \in V(\tau^*) \setminus V_{\rm f}(\tau^*)} \gamma^{-\alpha(h_v - h_{v'})} &= \sum_{\tau^* \in \mathcal{T}^*_n} \sum_{\{h_v\}_{v \in \tau^*}} \prod_{v \in V(\tau^*) \setminus V_{\rm f}(\tau^*)} \gamma^{-\alpha(h_v - h_{v'})} \\ &\le \sum_{\tau^* \in \mathcal{T}^*_n} \left(\frac{1}{\gamma^\alpha - 1} \right)^n \le C^n \ , \end{aligned} \tag{A1.12}$$

where we used $|V(\tau^*)| = |V_{\rm nt}(\tau)| \le 2n$ (see (A1.7)), so that the number of elements in $\mathcal{T}^*_n$ is bounded by C^{2n}, for a constant C (see Lemma A1); moreover in performing the sum over the scales we neglected all constraints except that $h_v - h_{v'} \ge 1$.

p.A1.3 **A1.3.** *Feynman diagrams.* A graph can be imagined as formed by giving n points $v_1, \dots, v_n$ with $d_{v_1}, \dots, d_{v_n}$ outcoming lines, respectively, and contractring (some of) such lines between themselves. We can also associate to each line a sign $\sigma = \pm 1$ and allow only contractions such that a line with a sign $+$ is contracted with a line with a sign $-$.

In particular we can consider points with 2 or 4 outcoming lines: in the first case there is one line with a sign $+$ and one line with a sign $-$, while in the second one there are two lines with a sign $+$ and two lines with a sign $-$. We denote by n_2 the number of points v with $d_v = 2$ and by n_4 the number of points v with $d_v = 4$: of course $n = n_2 + n_4$.

The points can have also a *structure*: when $d_v = 4$ the point v is formed by two disjoint points connected through an ondulated lines, while when $d_v = 2$ the point can be characterized by an extra label. We shall call *graph elements* the points with structure.

We shall consider only graphs of the above type which are connected: such graphs will be called *Feynman diagrams* and will be denoted by Γ. Note that if all the lines are contracted then for each $v \in \Gamma$ one has $d(v) = d_v$, while of we allow to some lines to remain uncontracted then $d(v) \le d_v$: in such a case the uncontracted lines are called the *external lines of the diagram*.

The number of Feynman diagrams is controlled through the following result.

LEMMA A4. *Consider a Feynman diagram formed with n graph elements $v_1, \dots, v_n$ such that $d_{v_j} \in \{2, 4\}$ $\forall j = 1, \dots, n$, and with $2p$ uncontracted lines (p with the sign $+$ and p with the sign $-$). Then the number of Feynman graphs is bounded by $C^n(2n)!$ uniformly in p.*

Proof. A generic Feynman graph can be obtained in the following way.

First construct a tree graph between the n graph elements: such a tree will be formed by contracting $2(n-1)$ lines. The number of trees which can be obtained in this way is bounded by $C^n n!$ (by Lemma A2).

Then contract all the remaining $4n - 2p - 2(n-1) = 2(n-p+1)$ lines (one has to exclude the p lines which have to be left uncontracted), by using that only lines with opposite signs can be contracted between themselves. Of course among the $2(n-p+1)$ lines there are $n-p+1$ lines with a sign $+$ and $n-p+1$ lines with a sign $-$: therefore such lines can be contracted in $(n-p+1)!$ possible ways, so that the number of diagrams which can be obtained starting from a fixed tree between the graph elements is bounded by $n!$ uniformly in p.

By collecting together the two bounds the assertion follows. ■

app.A2

Appendix A2. Discrete versus continuum

p.A2.1 **A2.1.** *Discrete derivatives* Given a function $\hat{F}(\mathbf{k})$ with $\mathbf{k} = (k, k_0) \in \mathcal{D}_{L,\beta}$, we set $\partial_{\mathbf{k}} = (\partial_k, \partial_{k_0})$, where

A2.1
$$\partial_k F(k, k_0) = \frac{F(k + \Delta k, k_0) - F(k, k_0)}{\Delta k} , \qquad \Delta k = \frac{2\pi}{L} \tag{A2.1}$$

and, analogously,

A2.2
$$\partial_{k_0} F(k, k_0) = \frac{F(k, k_0 + \Delta k_0) - F(k, k_0)}{\Delta k_0} , \qquad \Delta k_0 = \frac{2\pi}{\beta} . \tag{A2.2}$$

Note that, if

A2.3
$$F(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}(\mathbf{k}) , \tag{A2.3}$$

then

A2.4
$$\sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot\mathbf{x}} \partial_k \hat{F}(\mathbf{k}) = \left(\frac{e^{-i\Delta k x} - 1}{\Delta k} \right) \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}(\mathbf{k}) , \tag{A2.4}$$

so that, for $|x| \le L/2$,

A2.5
$$\begin{aligned} |xF(\mathbf{x})| &\le C \left| \left(\frac{e^{-i\Delta k x} - 1}{\Delta k} \right) F(\mathbf{x}) \right| \\ &\le C \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \left| e^{-i\mathbf{k}\cdot\mathbf{x}} \partial_k \hat{F}(\mathbf{k}) \right| \le C \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \left| \partial_k \hat{F}(\mathbf{k}) \right| , \end{aligned} \tag{A2.5}$$

where C denotes some constant.

app.A3

Appendix A3. Truncated expectations and Gram-Hadamard inequality

p.A3.1 **A3.1.** *Truncated expectations and graphic representations.* Given a Grassman algebra as in (4.1) and an integration measure like (4.10) we define a *simple expectation* as in (4.12). Then

A3.1
$$g_\alpha = \mathcal{E}(\psi^-_\alpha \psi^+_\alpha) . \tag{A3.1}$$

Given a monomial

A3.2
$$X(\psi) \equiv \tilde{\psi}_B = \prod_{\alpha \in B} \psi^{\sigma_\alpha}_\alpha , \tag{A3.2}$$

where B is a subset of A and $\sigma_\alpha \in \{\pm\}$, the expectation $\mathcal{E}(\tilde{\psi}_B)$ can be graphically represented in the following way.

Represent the indices $\alpha \in B$ as points on the plane. With each ψ^+_α, $\alpha \in B$, we associate a line exiting from α, while with each ψ^-_α, $\alpha \in B$, we associate a line entering α. Let $\mathcal{T}$ be the set of graphs obtained by contracting such lines in all possible ways so that only lines with opposite σ_α are contracted: given $\alpha, \beta \in B$, denote by $(\alpha\beta)$ the line joining α and β and by τ an element of $\mathcal{T}$, *i.e.* a graph in $\mathcal{T}$.

Then we can easily verify that

A3.2a
$$\mathcal{E}(\tilde{\psi}_B) = \sum_{\tau \in \mathcal{T}} \prod_{(\alpha\beta) \in \tau} (-1)^{\pi_\tau} g_\alpha \delta_{\alpha,\beta} , \tag{A3.3}$$

which is the *Wick rule* stated in §4.1: here π_τ is a sign which depends on the graph τ (see (4.20)).

Then define the *truncated expecation*

A3.3
$$\mathcal{E}^T\left(\tilde\psi_{B_1},\dots,\tilde\psi_{B_p};n_1,\dots,n_p\right)\ , \tag{A3.4}$$

with $B_j\subset A$ for any j, as in (4.13).

One easily check that, if X_j are analytic functions of the Grassman variables (each depending on an even number of variables, for simplicity, so that no change of sign intervenes in permuting the order of the X_j), then

A3.3a
$$\begin{aligned}
(1)\quad & \mathcal{E}^T(X_1,X_2)=\mathcal{E}(X_1X_2)-\mathcal{E}(X_1)\mathcal{E}(X_2)\ ,\\
(2)\quad & \mathcal{E}^T(X_1,X_2,X_3)=\mathcal{E}(X_1X_2X_3)-\mathcal{E}(X_1X_2)\mathcal{E}(X_3)-\mathcal{E}(X_1X_3)\mathcal{E}(X_2)\\
&\quad-\mathcal{E}(X_2X_3)\mathcal{E}(X_1)+2\mathcal{E}(X_1)\mathcal{E}(X_2)\mathcal{E}(X_3)\ ,\\
(3)\quad & \mathcal{E}^T(X_1,X_2,X_3,X_4)=\mathcal{E}(X_1X_2X_3X_4)-\mathcal{E}(X_1X_2X_3)\mathcal{E}(X_4)-\mathcal{E}(X_1X_2X_4)\mathcal{E}(X_3)\\
&\quad-\mathcal{E}(X_1X_3X_4)\mathcal{E}(X_2)-\mathcal{E}(X_2X_3X_4)\mathcal{E}(X_1)\\
&\quad-\mathcal{E}(X_1X_2)\mathcal{E}(X_3X_4)-\mathcal{E}(X_1X_3)\mathcal{E}(X_2X_4)-\mathcal{E}(X_1X_4)\mathcal{E}(X_2X_3)\\
&\quad+2\mathcal{E}(X_1X_2)\mathcal{E}(X_3)\mathcal{E}(X_4)+2\mathcal{E}(X_1X_3)\mathcal{E}(X_2)\mathcal{E}(X_4)+2\mathcal{E}(X_1X_4)\mathcal{E}(X_2)\mathcal{E}(X_3)\\
&\quad+2\mathcal{E}(X_2X_3)\mathcal{E}(X_1)\mathcal{E}(X_2)+2\mathcal{E}(X_2X_4)\mathcal{E}(X_1)\mathcal{E}(X_3)+2\mathcal{E}(X_3X_4)\mathcal{E}(X_1)\mathcal{E}(X_2)\\
&\quad-6\mathcal{E}(X_1)\mathcal{E}(X_2)\mathcal{E}(X_3)\mathcal{E}(X_4)\ ,
\end{aligned} \tag{A3.5}$$

and so on. One can always write the truncated expectations in terms of simple expectations: it is easy to check that in general one has

A3.3b
$$\mathcal{E}^T(X_1,\dots,X_s)=\sum_{p=1}^{s}\sum_{Y_1,\dots,Y_p}(-1)^\pi\mathcal{E}(Y_1\dots Y_p)\ , \tag{A3.6}$$

where
(1) the sum is over all the possible partitions of $\{1,\dots,s\}$ into p subsets such that $\cup_{j=1}^s X_j=\cup_{k=1}^p Y_p$ and each Y_k is the union of sets X_j,
(2) π is the parity leading to $\{Y_1,\dots,Y_p\}$ with respect to the initial ordering.

Also the truncated expectation (A3.4) can be graphically represented. Draw in the plane n_1 boxes $G_{11},\dots,G_{1n_1}$, such that each of them contains all points representing the indices belonging to B_1, n_2 boxes $G_{21},\dots,G_{2n_2}$, such that each of them contains all points representing the indices belonging to B_2, and so on: we call *clusters* such boxes (for obvious reasons, if one recalls the definition of clusters given in §5.2). Then consider all possible graphs τ obtained by contracting as before all the lines emerging from the points in such a way that no line is left uncontracted and with the property that if the clusters were considered as points then τ would be connected. If we denote the lines as before we have

A3.4
$$\mathcal{E}^T\left(\tilde\psi_{B_1},\dots,\tilde\psi_{B_p};n_1,\dots,n_p\right)=\sum_{\tau\in\mathcal{T}_0}\prod_{(\alpha\beta)\in\tau}(-1)^{\pi_\tau}g_\alpha\delta_{\alpha,\beta}\ , \tag{A3.7}$$

where $\mathcal{T}_0$ denotes the set of all graphs obtained following the just given prescription; again π_τ is a sign depending on τ.

If $A=A_1\cup A_2$, with $A_1\cap A_2=\emptyset$, we can define $\mathcal{E}_1$ and $\mathcal{E}_2$ as the expectations defined as $\mathcal{E}$ in (4.10) and (4.12), with the difference that we have the constraint $a\in A_1$ and $a\in A_2$, respectively.

Then if each field $\psi_\alpha^{\sigma_\alpha}$ appearing in the products $\tilde\psi_{B_j}$ is replaced by

A3.5
$$\psi_\alpha^{\sigma_\alpha}\to\psi_{\alpha_1}^{\sigma_{\alpha_1}}+\psi_{\alpha_2}^{\sigma_{\alpha_2}}\ , \tag{A3.8}$$

with $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$, we can consider

A3.6
$$\mathcal{E}_2^T \left(\tilde{\psi}_{B_1}, \dots, \tilde{\psi}_{B_p}; n_1, \dots, n_p \right) , \tag{A3.9}$$

where $\mathcal{E}_2^T$ denotes the truncated expectation corresponding to the simple expectation $\mathcal{E}_2$.

Consider for simplicity the case $n_j = 1 \; \forall j$; by (4.18) this is not restrictive. We have for (A3.9) the following graphic representation. For each B_j write $B_j = B_{j1} \cup B_{j2}$, with $B_{j1} \cap B_{j2} = \emptyset$. Fixed the sets $B_{11}, \dots, B_{p1}$, define $B = \cup_{j=1}^p B_{j1}$ and $\mathcal{T}_0(B)$ as the set of graphs obtained by contracting the lines emerging from the points contained inside the boxes corresponding to $B_{21}, \dots, B_{2p}$. Then

A3.7
$$\mathcal{E}_2^T \left(\tilde{\psi}_{B_1}, \dots, \tilde{\psi}_{B_p}; n_1, \dots, n_p \right) = \sum_B \tilde{\psi}_B \sum_{\tau \in \mathcal{T}_0(B)} \prod_{(\alpha\beta) \in \tau} (-1)^{\pi_\tau} g_\alpha \delta_{\alpha,\beta} \; . \tag{A3.10}$$

If $A = A_1 \cup A_2 \cup \ldots \cup A_N$ for some $N \in \mathbb{N}$, the above procedure can be iterated (in the obvious way).

p.A3.2 **A3.2.** *Proof of (4.43).* Given s set of indices $P_1, \dots, P_s$, consider the quantity (4.38). Define

A3.8
$$P_j^{\pm} = \{ f \in P_j \; : \; \sigma(f) = \pm \} \tag{A3.11}$$

and set $f = (j, i)$ for $f \in P_j^{\pm}$, with $i = 1, \dots, |P_j^{\pm}|$. Note that $\sum_{j=1}^s |P_j^+| = \sum_{j=1}^s |P_j^-|$, otherwise (4.38) is vanishing.

Define also

A3.9
$$P(\mathrm{d}\psi) = \prod_{j=1}^s \left(\prod_{f \in P_j^+} \mathrm{d}\psi^+_{\mathbf{x}(f)} \right) \left(\prod_{f \in P_j^-} \mathrm{d}\psi^-_{\mathbf{x}(f)} \right) \tag{A3.12}$$
$$\left(\psi^+, \Gamma \psi^- \right) = \sum_{j,j'=1}^s \sum_{i=1}^j \sum_{i'=1}^{j'} \psi^+_{(j',i')} \Gamma_{(j,i),(j',i')} \psi^-_{(j,i)} \; ,$$

where, if

A3.10
$$n = \sum_{j=1}^s \left| P_j^+ \right| = \sum_{j=1}^s \left| P_j^- \right| \; , \tag{A3.13}$$

then Γ is the $n \times n$ matrix with entries

A3.11
$$\Gamma_{(j,i),(j',i')} = g(\mathbf{x}(j,i) - \mathbf{x}(j',i')) \; . \tag{A3.14}$$

Then one has

A3.12
$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\psi}(P_j) \right) = \det \Gamma = \int P(\mathrm{d}\psi) \, \exp \left[- \left(\psi^+, \Gamma \psi^- \right) \right] \; , \tag{A3.15}$$

which is known as *Berezin integral*, [?].

Setting $X \equiv \{1, \dots, s\}$ and

A3.13
$$\overline{V}_{jj'} = \sum_{i=1}^{|P_j^-|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} \Gamma_{(j,i),(j',i')} \psi^-_{(j,i)} \; , \tag{A3.16}$$

write

A3.14
$$V(X) = \sum_{j,j' \in X} \overline{V}_{jj'} = \sum_{j \le j'} V_{jj'} \; , \tag{A3.17}$$

so defining the quantity $V_{jj'}$ as

A3.14a
$$V_{jj'} = \begin{cases} \overline{V}_{jj'} \,, & \text{if } j = j' \,, \\ \overline{V}_{jj'} + \overline{V}_{j'j} \,, & \text{if } j < j' \,. \end{cases} \tag{A3.18}$$

Then (A3.15) can be written as

A3.13b
$$\mathcal{E}\left(\prod_{j=1}^{s} \tilde{\psi}(P_j)\right) = \int P(\mathrm{d}\psi)\, e^{-V(X)} \,. \tag{A3.19}$$

We want to express (A3.17) in terms of the following quantities. Define

A3.15
$$W_X(X_1, \dots, X_r; t_1, \dots, t_r) = \sum_{\ell} \prod_{k=1}^{r} t_k(\ell)\, V_\ell \,, \tag{A3.20}$$

where
(1) X_k are subsets of X with $|X_k| = k$, inductively defined as

A3.16
$$\begin{cases} X_1 = \{1\} \,, \\ X_{k+1} \supset X_k \,, \end{cases} \tag{A3.21}$$

(2) $\ell = (jj')$ is a pair of elements $j, j' \in X$ and the sum in (A3.20) is over all the possible pairings (jj'),
(3) the functions $t_k(\ell)$ are defined as

A3.17
$$t_k(\ell) = \begin{cases} t_k \,, & \text{if } \ell \sim \partial X_k \,, \\ 0 \,, & \text{otherwise} \quad , \end{cases} \tag{A3.22}$$

where $\ell \sim X_k$ means that $\ell = (jj')$ "intesects the boundary" of X_k, *i.e.* it connects a point belonging to some P_j with $j \in X_k$ to a point contained inside some $P_{j'}$ with $j' \notin X_k$. See Fig. A6.

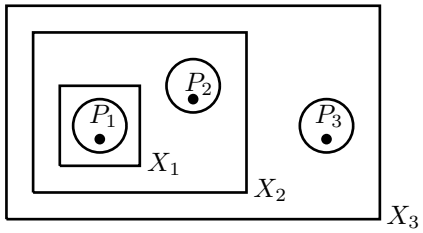


FIG. A6. The sets X_k for k=1,2,3. One has X_1={1}, X_2={1,2} and X_3={1,2,3}.

One has

A3.18
$$W_X(X_1; t_1) = \sum_{j=2}^{s} t_1 V_{1j} + V_{11} + \sum_{1<j'\le j} V_{j'j} = (1-t_1)\,[V(X_1) + V(X \setminus X_1)] + t_1 V(X) \tag{A3.23}$$

so that

A3.19
$$\begin{aligned} e^{-V(X)} &= \int_0^1 \mathrm{d}t_1 \left[\frac{\partial}{\partial t_1}\, e^{-W_X(X_1;t_1)}\right] + e^{-W_X(X_1;0)} \\ &= -\sum_{\ell_1 \sim \partial X_1} V_{\ell_1} \int_0^1 \mathrm{d}t_1\, e^{-W_X(X_1;t_1)} + e^{-W_X(X_1;0)} \,. \end{aligned} \tag{A3.24}$$

If we define $X_2 \equiv X_1 \cup \ell_1$, *i.e.* $X_2 = \{1, \text{point connected by } \ell_1 \text{ with } 1\}$, then

A3.20
$$\begin{aligned} W_X(X_1, X_2; t_1, t_2) &= t_2 W_X(X_1; t_1) + (1 - t_2)\Big[\sum_{j=2}^{s} t_1 V_{1j} + V_{11} \\ &\quad + \sum_{1<j'\le j} V_{j'j} - \sum_{j=3}^{s} (\, t_1 V_{1j} + V_{2j}\,)\Big] \\ &= (1 - t_2)\,[W_{X_2}(X_1; t_1) + V(X \setminus X_2)] + t_2 W_X(X_1; t_1)\ , \end{aligned} \tag{A3.25}$$

so that

A3.21
$$\begin{aligned} e^{-W_X(X_1;t_1)} &= \int_0^1 \mathrm{d}t_2 \left[\frac{\partial}{\partial t_2}\, e^{-W_X(X_1,X_2;t_1,t_2)}\right] + e^{-W_X(X_1,X_2;t_1,0)} \\ &= -\sum_{\ell_2 \sim \partial X_2} V_{\ell_2} \int_0^1 \mathrm{d}t_2\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)} + e^{-W_X(X_1,X_2;t_1,0)}\ . \end{aligned} \tag{A3.26}$$

Therefore

A3.22
$$\begin{aligned} e^{-V(X)} &= \sum_{\ell_1 \sim \partial X_1} \sum_{\ell_2 \sim \partial X_2} \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2\, (-1)^2\, V_{\ell_1}\, V_{\ell_2}\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)} \\ &\quad + \sum_{\ell_1 \sim \partial X_1} \int_0^1 \mathrm{d}t_1\, (-1)\, V_{\ell_1}\, e^{-W_X(X_1,X_2;t_1,0)} + e^{-W_X(X_1;0)}\ , \end{aligned} \tag{A3.27}$$

and, iterating $s - 1$ times,

A3.23
$$\begin{aligned} e^{-V(X)} = \sum_{r=0}^{s-1} &\sum_{\ell_1 \sim \partial X_1} \cdots \sum_{\ell_r \sim \partial X_r} \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_r\, (-1)^r\, V_{\ell_1} \ldots V_{\ell_r} \\ &\left(\prod_{k=1}^{r-1} t_1(\ell_{k+1}) \ldots t_k(\ell_{k+1})\right) e^{-W_X(X_1,\ldots,X_{r+1};t_1,\ldots,t_r,0)}\ , \end{aligned} \tag{A3.28}$$

where the factors which are meaningless have to be set equal to 1 and for $r = s - 1$ one has

A3.24
$$W_X(X_1, \ldots, X_s; t_1, \ldots, t_{s-1}, 0) = W_X(X_1, \ldots, X_{s-1}; t_1, \ldots, t_{s-1})\ . \tag{A3.29}$$

One can easily check that

A3.25
$$W_X(X_1, \ldots, X_r; t_1, \ldots, t_{r-1}, 0) = W_{X_r}(X_1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1}) + V(X \setminus X_r)\ . \tag{A3.30}$$

Let us introduce a tree graph T between the sets $X_1, \ldots, X_r$, such that
(1) for each $k = 1, \ldots, r$, it is "anchored" to some point (j, i), *i.e.* it contains a line incident with (j, i), where $j \in X_k$ and $i \in \{1, \ldots, |P_j^{\pm}|\}$,
(2) each line $\ell \in T$ intersects at least one boundary ∂X_k,
(3) the lines $\ell_1, \ell_2, \ldots$ are ordered so that $\ell_1 \sim \partial X_1, \ell_2 \sim \partial X_2, \ldots$,
(4) for each $\ell \in T$ one defines two indices $n(\ell)$ and $n'(\ell)$ such that

A3.27
$$\begin{aligned} n(\ell) &= \max\{k : \ell \sim \partial X_k\}\ , \\ n'(\ell) &= \min\{k : \ell \sim \partial X_k\}\ . \end{aligned} \tag{A3.31}$$

We shall call T an *anchored tree*.

Then we can rewrite (A3.28) as

A3.28
$$e^{-V(X)} = \sum_{r=1}^{s} \sum_{X_r \subset X} \sum_{X_2 \dots X_{r-1}} \sum_{T \text{ on } X_r} (-1)^{r-1} \prod_{\ell \in T} V_\ell \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1,\dots,X_{r-1};t_1,\dots,t_{r-1})} \, e^{-V(X \setminus X_r)} \tag{A3.32}$$

where " T on X_r " means that T is an anchored tree for the clusters P_j such that $j \in X_r$.

Define

A3.29
$$K(X_r) = \sum_{X_2 \dots X_{r-1}} \sum_{T \text{ on } X_r} \prod_{\ell \in T} V_\ell \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1,\dots,X_{r-1};t_1,\dots,t_{r-1})} \, , \tag{A3.33}$$

so that (A3.32) becomes

A3.30
$$e^{-V(X)} = \sum_{\substack{Y \subset X \\ Y \ni \{1\}}} (-1)^{|Y|-1} \, K(Y) \, e^{-V(X \setminus Y)} \, , \tag{A3.34}$$

and, iterating,

A3.31
$$e^{-V(X)} = \sum_{Q_1,\dots,Q_m} (-1)^{|X|} \, (-1)^m \prod_{q=1}^{m} K(Q_q) \, . \tag{A3.35}$$

Note that the constraint $\{1\} \in Y$ in (A3.34) would yield a constraint like $\{1\} \in Q_1$, $\min\{k : k \in X \setminus Q_1\} \in Q_2$ and so on in (A3.35), but, as a rearrangment of the sets Q_q inside the partition $\{Q_1, \dots, Q_m\}$ does not change (A3.35) because the Grassman fields $\psi^\pm$ appear always in pairs, we can forget such a constraint.

Therefore, by (A3.19) and (A3.35), one has (recall also the first of (A3.12))

A3.32
$$\mathcal{E}\left(\prod_{j=1}^{s} \tilde{\psi}(P_j) \right) = \int P(\mathrm{d}\psi) \sum_{Q_1,\dots,Q_m} (-1)^s \, (-1)^m \prod_{q=1}^{m} K(Q_q) \, . \tag{A3.36}$$

In (A3.33) we can sum first over the trees T, then over the sets X_k,

A3.33
$$\sum_{X_2 \dots X_{r-1}} \sum_{T \text{ on } X_r} = \sum_{T \text{ on } X_r} \sum_{\substack{X_2 \dots X_{r-1} \\ \text{fixed } T}} \, , \tag{A3.37}$$

where "fixed T" recalls that the sets $X_2, \dots, X_r$ have to be compatible with the tree T.

Moreover we can write, by (A3.20),

A3.34
$$W_{X_r}(X_1, \dots, X_{r-1}; t_1, \dots, t_{r-1}) = \sum_{\ell \in X_r} t_1(\ell) \dots t_{r-1}(\ell) V_\ell = \sum_{\ell \in X_r} t_{n'(\ell)} \dots t_{n(\ell)-1} V_\ell \tag{A3.38}$$

and set in (A3.33)

A3.35
$$\frac{\prod_{k=1}^{r-1} t_r(\ell)}{t_{n(\ell)}} = t_{n'(\ell)} \dots t_{n(\ell)-1} \, , \tag{A3.39}$$

so obtaining

A3.36
$$K(X_r) = \sum_{T \text{ on } X_r} \sum_{\substack{X_2 \dots X_{r-1} \\ \text{fixed } T}} \prod_{\ell \in T} V_\ell \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{l-1} \prod_{\ell \in T} \left(t_{n'(\ell)} \dots t_{n(\ell)-1} \right) e^{-\sum_{\ell \in X_r} t_{n'(\ell)} \dots t_{n(\ell)-1} V_\ell} \, . \tag{A3.40}$$

We can reorder the integration measure $P(\mathrm{d}\psi)$ in (A3.12) as

A3.37
$$P(\mathrm{d}\psi) = \prod_{j=1}^{s} \left(\prod_{i=1}^{|P_j^-|} \mathrm{d}\psi_{(j,i)}^- \right) \left(\prod_{i'=1}^{|P_{j'}^+|} \mathrm{d}\psi_{(j',i')}^- \right)$$
$$= (-1)^\sigma \prod_{q=1}^{m} \left(\prod_{i=1}^{|Q_q^-|} \mathrm{d}\psi_i^{(q)-} \right) \left(\prod_{i'=1}^{|Q_q^+|} \mathrm{d}\psi_{i'}^{(q)+} \right) = \overline{P}(\mathrm{d}\psi) \; , \tag{A3.41}$$

where
(i) $\psi_{(j,i)}^-$ and $\psi_{(j,i)}^+$ correspond to indices $f \in P_j$, while $\psi_i^{(q)-}$ and $\psi_{i'}^{(q)+}$ corresponds to indices (q,i) and (q,i') in $Q_q = Q_q^+ \cup Q_q^-$,
(ii) $\sum_{q=1}^m |Q_j^-| = \sum_{q=1}^m |Q_j^+|$,
(iii) σ is the parity of the permutaion leading the Grassman fields $\psi^\pm$ from the initial ordering (left hand side) to the final one (right hand side).

The simple expectations can be expressed in terms of truncated expectations through the relation

A3.38
$$\mathcal{E}\left(\prod_{j=1}^{s} \tilde{\psi}(P_j) \right) = \sum_{Q_1,\dots,Q_m} (-1)^\pi \mathcal{E}^T \left(\tilde{\psi}(Q_1), \dots, \tilde{\psi}(Q_m) \right) \; , \tag{A3.42}$$

where
(1) the sum is over all the possible partitions of $\{1,\dots,s\}$ into m subsets $Q_1,\dots,Q_m$ such that each Q_k, $k=1,\dots,m$ is the union of sets P_j and $\cup_{j=1}^s P_j = \cup_{q=1}^m Q_q$,
(2) π is the parity leading to $\{Q_1,\dots,Q_m\}$ with respect to the initial ordering.

It is easy to realize that the parity σ in (A3.41) is equal to the parity π in (A3.42), if the sets $Q_1,\dots,Q_m$ are chosen in the same way (*i.e.* if the sets Q_q in (A3.41) are the same sets Q_q as in (A3.42)).

Therefore, by comparing (A3.42) with (A3.36) (by taking into account also (A3.40) and (A3.41)), we find the following expression for the truncated expectations:

A3.39
$$\mathcal{E}^T \left(\tilde{\psi}(Q_1), \dots, \tilde{\psi}(Q_m) \right) = (-1)^{m+1} \int \overline{P}(\mathrm{d}\psi) \sum_{T \text{ on } X_m} \sum_{\substack{X_2 \dots X_{m-1} \\ \text{fixed } T}} \prod_{\ell \in T} V_\ell$$
$$\int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{m-1} \prod_{\ell \in T} \left(t_{n'(\ell)} \dots t_{n(\ell)-1} \right) e^{-\sum_{\ell \in X} t_{n'(\ell)} \dots t_{n(\ell)-1} V(\ell)} \; . \tag{A3.43}$$

A remarkable property of (A3.43) is the following result.

LEMMA A5. *In (A3.43) one has*

A3.40
$$\sum_{\substack{X_2 \dots X_{m-1} \\ \text{fixed } T}} \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{m-1} \prod_{\ell \in T} \left(t_{n'(\ell)} \dots t_{n(\ell)-1} \right) = 1 \; , \tag{A3.44}$$

for any anchored tree T. As in (A3.44)

A3.41
$$\mathrm{d}P_T(\mathbf{t}) \equiv \sum_{\substack{X_2 \dots X_{p-1} \\ \text{fixed } T}} \prod_{\ell \in T} \left(t_{n'(\ell)} \dots t_{n(\ell)-1} \right) \prod_{q=1}^{m-1} \mathrm{d}t_q \tag{A3.45}$$

is positive and σ-additive, it can be interpreted as a probability measure in the variable $\mathbf{t} = (t_1,\dots,t_{m-1})$.

Proof. Let us denote by b_k the number of lines $\ell \in T$ exiting from points $\mathbf{x}(j,i)$, with $j \in X_k$. By construction the parameter t_k inside the integral in the left hand side of (A3.44) appears to the power $b_k - 1$, as all the

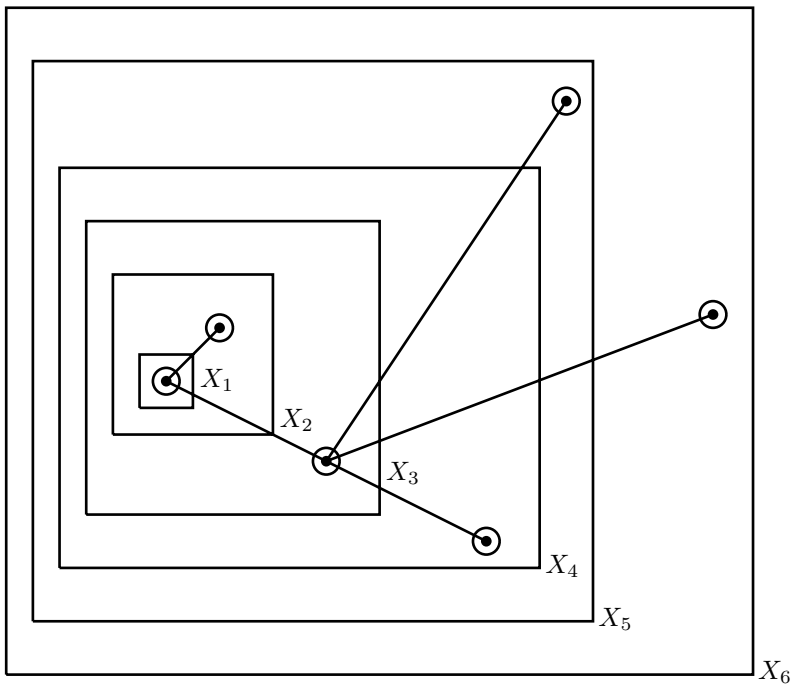


FIG. A7. The sets X_1,...,X_6, the (anchored) tree T and the lines belonging to T.

lines intersecting ∂X_k contribute to t_k, except the one connecting X_k with the point whose union with X_k gives the set X_{k+1} (this is clear by using the notations introduced after (A3.20)). See Fig. A7.

Then

A3.41a
$$\prod_{\ell\in T}\left(t_{n'(\ell)}\dots t_{n(\ell)-1}\right)=\prod_{k=1}^{m-2}t_k^{b_k-1}\,, \tag{A3.46}$$

and in (A3.44) one has $m-1$ independent integrations

A3.41b
$$\int_0^1 \mathrm{d}t_{m-1}\prod_{k=1}^{m-2}\left(\int_0^1 \mathrm{d}t_k\, t_k^{b_k-1}\right)=\prod_{k=1}^{m-2}\frac{1}{b_k}\,, \tag{A3.47}$$

which is a well defined expression as $b_k\geq 1$ for $k=1,\dots,m-2$. Moreover we can write

A3.41c
$$\sum_{\substack{X_2\dots X_{m-1}\\ \text{fixed } T}}=\sum_{\substack{X_2\\ \text{fixed } X_1}}\ \sum_{\substack{X_3\\ \text{fixed } X_1,X_2}}\dots\sum_{\substack{X_{m-1}\\ \text{fixed } X_1,\dots,X_{m-2}}}\,, \tag{A3.48}$$

where the number of possible choices in summing over X_k, once $X_1,\dots,X_{k-1}$ have been fixed, is exactly b_{k-1}: if b_{k-1} lines exit from X_{k-1} then X_k is obtained by adding to X_{k-1} one of the b_{k-1} points connected to X_{k-1} through one of the lines of the tree. Then

A3.41d
$$\sum_{\substack{X_2\dots X_{m-1}\\ \text{fixed } T}}1=b_1\dots b_{m-2}\,, \tag{A3.49}$$

and, at the end,

A3.41e
$$\sum_{\substack{X_2\dots X_{m-1}\\ \text{fixed } T}}\int_0^1 \mathrm{d}t_1\dots\int_0^1 \mathrm{d}t_{m-1}\prod_{\ell\in T}\left(t_{n'(\ell)}\dots t_{n(\ell)-1}\right)=\prod_{k=1}^{m-2}\frac{b_k}{b_k}\,, \tag{A3.50}$$

which yields (A3.44). ■

Set

A3.42
$$V(\mathbf{t}) \equiv \sum_{\ell \in X} t_{n'(\ell)} \dots t_{n(\ell)-1} V_\ell \ , \tag{A3.51}$$

so that, in (A3.43), we can rewrite

A3.43
$$\prod_{\ell \in T} V_\ell = \prod_{(i,j)} (\overline{V}_{ij} + \overline{V}_{ji}) \tag{A3.52}$$

and use the definition (A3.45) to obtain

A3.43a
$$\mathcal{E}^T \left(\tilde{\psi}(Q_1), \dots, \tilde{\psi}(Q_m) \right) = (-1)^{m+1} \int \overline{P}(\mathrm{d}\psi) \sum_{T \text{ on } \{x_i^{(q)}\}} \prod_{(jj') \in T} (\overline{V}_{jj'} + \overline{V}_{jj'}) \int \mathrm{d}P_T(\mathbf{t}) \, e^{-V(\mathbf{t})} \ , \tag{A3.53}$$

where $\sum_{T \text{ on } \{x_i^{(q)}\}}$ denotes the sum over the trees on X, seen as a sum over the trees anchored on some point $x_i^{(q)}$, $q = 1, \dots, m$ and $i = 1, \dots, |Q_q|$.

If we integrate the Grassman fields appearing in the product

A3.46
$$\prod_{(jj') \in T} (\overline{V}_{jj'} + \overline{V}_{jj'}) \tag{A3.54}$$

in (A3.53), we obtain

A3.47
$$\mathcal{E}^T \left(\tilde{\psi}(Q_1), \dots, \tilde{\psi}(Q_m) \right) = (-1)^{m+1} \sum_{T \text{ on} \{x_i^{(q)}\}} \prod_{\ell \in T} g_\ell \int \overline{P}^*(\mathrm{d}\psi) \int \mathrm{d}P_T(\mathbf{t}) \, e^{-V(\mathbf{t})} \ , \tag{A3.55}$$

where $\overline{P}^*(\mathrm{d}\psi)$ means that the Grassman fields which are left to integrate are the ones not appearing in (A3.54).

The term

A3.48
$$\int \overline{P}^*(\mathrm{d}\psi) \int \mathrm{d}P_T(\mathbf{t}) \, e^{-V(\mathbf{t})} \tag{A3.56}$$

in (A3.55) is the determinant of a suitable matrix $G^T(\mathbf{t})$ with elements

A3.49
$$G^T_{(j,i)(j',i')} = t_{n'(jj')} \dots t_{n(jj')-1} g \left(\mathbf{x}(j,i) - \mathbf{x}(j',i') \right) \ . \tag{A3.57}$$

So (4.43) is proven, with $t_{j,j'} = t_{n'(jj')} \dots t_{n(jj')-1}$.

p.A3.3 **A3.3.** *Estimates for the truncated expectations.* The following results holds.

LEMMA A6. *Given m set of indices $Q_1, \dots, Q_m$ such that*

A3.101
$$\{\mathbf{x}(f) \ : \ f \in Q_q\} = \{\mathbf{x}_1^{(q)}, \dots, \mathbf{x}_{|Q_q|}^{(q)}\} \ , \qquad q = 1, \dots, m \ , \tag{A3.58}$$

and $\sum_{q=1}^m |Q_q^+| = \sum_{q=1}^m |Q_q^-| = n$, then the number of trees T anchored on $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ is bounded by

A3.102
$$\sum_{T \text{ on } \{x_i^{(q)}\}} 1 \leq m! \, C^n \ , \tag{A3.59}$$

for some constant C.

Proof. The proof goes through the following steps.
(1) First suppose that each set Q_q is a point: we shall see at the end what happens if the sets contain several points. We can write

A3.103
$$\sum_{T \text{ on } \{x_i^{(q)}\}} 1 = \sum_{\{d_q\}} \sum_{\substack{T \\ \text{fixed } \{d_q\}}} 1 \,, \tag{A3.60}$$

where in the right hand side the first sum is over all the possible configurations $\{d_q\}$, if we denote by d_q the number of lines emerging from (*i.e.* entering or exiting from) $Q_q \equiv q$, while the second sum is over all the trees compatible with a fixed configuration $\{d_q\}$.
(2) The second sum in the right hand side of (A3.60) can be exactly computed and it gives

A3.104
$$\sum_{\substack{T \\ \text{fixed } \{d_q\}}} 1 = \frac{(m-2)!}{(d_1-1)! \dots (d_m-1)!} \,. \tag{A3.61}$$

In fact, by definition of T, there are at least 2 points (which we can call 1 and m) such that there is only one line emerging from them: then $d_1 = d_m = 1$. The line emerging from 1 can reach one of the other $m-2$ points: we call 2 the point it reaches. Then there are $d_2 - 1$ lines emerging from 2 leading the first one to one of the other $m-3$ points, the second one to one of the other $m-4$ points, ..., the (d_2-1)-th one to one of the other $m - d_2 - 1$ points; moreover if we permute between themselves the $d_2 - 1$ lines ther is no change in the above discussion. Therefore so far we have obtained

A3.105
$$\frac{(m-2)}{(d_1-1)!} \cdot \frac{(m-3)\,(m-4) \dots (m-d_2-1)}{(d_2-1)!} \tag{A3.62}$$

possible contributions. By iterating until the m-th point is reached we find (A3.61).
(3) The first sum in (A3.60) can be bounded by

A3.106
$$\sum_{\{d_q\}} 1 \le C^m \,, \tag{A3.63}$$

where one can choose $C = 2$. In fact one has two constraints $\sum_{q=1}^m d_q = 2(m-1)$ and $1 \le d_q \le m-1 \ \forall$ $i = 1, \dots, m$, as the tree T has $m-1$ lines, each lines emerge from two points and each point is connected with no less than 1 point and no more than with all the others. Then, if we set $M = 2(m-1)$ and ignore for simplicity the second constraint on $\{d_q\}$, we have

A3.107
$$\begin{aligned}
\sum_{\{d_q\}} 1 &\le \int_0^M \mathrm{d}x_1 \int_0^{M-x_1} \mathrm{d}x_2 \dots \int_0^{M-\sum_{q=1}^{m-1} 1} \mathrm{d}x_m \\
&\le \int_0^M \mathrm{d}x_1 \dots \int_0^{M-\sum_{q=1}^{m-2} 1} \mathrm{d}x_{m-1} \Big(M - \sum_{q=1}^{m-1} 1 \Big) \\
&\le \int_0^M \mathrm{d}x_1 \dots \int_0^{M-\sum_{q=1}^{m-3} 1} \mathrm{d}x_{m-2} \, \frac{1}{2!} \Big(M - \sum_{q=1}^{m-2} 1 \Big)^2 \\
&\le \int_0^M \mathrm{d}x_1 \dots \int_0^{M-\sum_{q=1}^{m-4} 1} \mathrm{d}x_{m-3} \, \frac{1}{3!} \Big(M - \sum_{q=1}^{m-3} 1 \Big)^3 \\
&\le \frac{1}{m!} \, M^m = \frac{1}{m!} \, [2(m-1)]^m \le 2^m \, \frac{m^m}{m!} \,,
\end{aligned} \tag{A3.64}$$

and as $e^{-m} \leq m^m/m! \leq 1$, then (A3.63) immediately follows with $C = 2$.
(4) As $1/(d_q - 1)! \leq 1$, by using (A3.61) and (A3.63), we see that (A3.58) follows with $C = 2$.
(5) Now we take into account that, for each $q = 1, \dots, m$, Q_q is a collection of points. Then (A3.61) has to be replaced with

A3.108
$$\sum_{T \text{ on } \{x_i^{(q)}\}} 1 = \sum_{\{d_q\}} \sum_{\substack{\text{anchored } T \\ \text{fixed } \{d_q\}}} 1 \, . \tag{A3.65}$$

Fixed T on $\mathcal{Q}$, the number of anchored trees is

A3.109
$$\prod_{q=1}^{m} \frac{|Q_q|!}{(|Q_q| - d_q)!} \, , \tag{A3.66}$$

as we have to consider the $|Q_q|!$ permutations of the $|Q_q|$ elements of the set Q_q and divide by the $(|Q_q| - d_q)!$ permutations of the elements of Q_q which no line emerges from. So, by using that $[(d_q - 1)!]^{-1} \leq [d_q!]^{-1} 2^{d_q}$ and $\prod_{q=1}^{m} 2^{d_q} = 2^{2(m-1)} \leq 4^m$, we obtain

A3.110
$$\sum_{\substack{\text{anchored } T \\ \text{fixed } \{d_q\}}} 1 \leq \tilde{C}^{2(n+m)} \, (m-2)! \, , \tag{A3.67}$$

where one can take $\tilde{C} = 2^2$.
(6) From the previous bounds one has

A3.111
$$\sum_{\{d_q\}} \sum_{\substack{\text{anchored } T \\ \text{fixed } \{d_q\}}} 1 \leq m! \, C^n \, , \tag{A3.68}$$

where one can take $C = 2^5$, if $2n = \sum_{q=1}^{m} |Q_q|$. Then the proof of the Lemma is complete. ■

LEMMA A7. *In (A3.57) one has $|g(\mathbf{x} - \mathbf{y})| \leq C_0$ for some constant C_0, then the term (A3.56) is bounded by*

A3.112
$$\left| \int \overline{P}^*(\mathrm{d}\psi) \int P_T(d\mathbf{t}) \, e^{-V(\mathbf{t})} \right| \equiv \left| \det G^T \right| \leq (C_0 C)^{n-m+1} \, , \tag{A3.69}$$

for some constant C.

Proof. As the entries of the matrix G^T are given by (A3.57), we try to write

A3.113
$$t_{n'(jj')} \dots t_{n(jj')-1} g\left(\mathbf{x}(j,i) - \mathbf{x}(j',i')\right) = (\mathbf{u}_j \otimes A(\mathbf{x}(j,i) - \cdot), \mathbf{u}_{j'} \otimes B(\mathbf{x}(j',i') - \cdot)) \equiv (\mathbf{f}_\alpha, \mathbf{g}_\beta) \, , \tag{A3.70}$$

where $(\cdot, \cdot)$ denotes the inner product, *i.e.*

A3.114
$$(\mathbf{u}_j \otimes A(\mathbf{x}(j,i) - \cdot), \mathbf{u}_{j'} \otimes B(\mathbf{x}(j',i') - \cdot)) = \mathbf{u}_j \cdot \mathbf{u}_{j'} \int \mathrm{d}\mathbf{y} \overline{A(\mathbf{x}(j,i) - \mathbf{y})} \, B(\mathbf{x}(j',i') - \mathbf{y}) \tag{A3.71}$$

and the vectors $\mathbf{f}_\alpha$ and $\mathbf{g}_\beta$, with $\alpha, \beta = 1, \dots, n$ are implicitly defined by (A3.70).

The reason why to rewrite (A3.7) as in (A3.70) is that then we can apply the Gram-Hadamard inequality, [Ga], in order to bound the determinant of the matrix with entries

A3.115
$$M_{\alpha,\beta} = (\mathbf{f}_\alpha, \mathbf{g}_\beta) \tag{A3.72}$$

as

A3.116
$$|\det M| \leq \prod_{\alpha=1}^{n} \|\mathbf{f}_\alpha\| \, \|\mathbf{g}_\alpha\| \, , \tag{A3.73}$$

so that, if

A3.117
$$\max_{\alpha=1,\dots,m}\{\|\mathbf{f}_\alpha\|\}\le C_0^a\ ,\qquad \max_{\alpha=1,\dots,m}\{\|\mathbf{g}_\alpha\|\}\le C_0^b\ ,\qquad a+b=1\ , \tag{A3.74}$$

then (A3.69) follows. The bound (A3.73) is a standard result: a proof is given in §A3.4 just for completeness.

So we are left with verifying that (A3.70) is possible and that the bounds (A3.74) hold.

We can define a family of vectors in $\mathbb{R}^m$ inductively as

A3.118
$$\begin{cases}\mathbf{u}_1=\mathbf{v}_1\ ,\\ \mathbf{u}_j=t_{j-1}\mathbf{u}_{j-1}+\mathbf{v}_j\sqrt{1-t_{j-1}^2}\ ,\quad j=2,\dots,m\ ,\end{cases} \tag{A3.75}$$

where $\{\mathbf{v}_i\}_{i=1}^m$ is an orthonormal basis and the sets X_k have been relabeled so that $X_1=\{1\}$, $X_2=\{1,2\}$, …, $X_m=\{1,2,\dots,m\}$, hence

A3.119
$$t_{n'(jj')}\dots t_{n(jj')-1}=tj\dots t_{j'-1} \tag{A3.76}$$

for a line (jj').

By the definitions (A3.75) one has

A3.120
$$\mathbf{u}_j\cdot\mathbf{u}_{j'}=t_j\dots t_{j'-1}\ . \tag{A3.77}$$

Therefore if we define the vectors $A(\mathbf{x}(j,i)-\mathbf{y})$ and $B(\mathbf{x}(j',i')-\mathbf{y})$ so that

A3.121
$$g\left(\mathbf{x}(j,i)-\mathbf{x}(j',i')\right)=\left(A(\mathbf{x}(j,i)-\cdot),B(\mathbf{x}(j',i')-\cdot)\right)=\int\mathrm{d}\mathbf{y}\,\overline{A(\mathbf{x}(j,i)-\mathbf{y})}\,B(\mathbf{x}(j',i')-\mathbf{y})\ , \tag{A3.78}$$

and, simultaneously,

A3.122
$$\begin{aligned}&\left(A(\mathbf{x}(j,i)-\cdot)\,,A(\mathbf{x}(j',i')-\cdot)\right)<\infty\ ,\\ &\left(B(\mathbf{x}(j,i)-\cdot)\,,B(\mathbf{x}(j',i')-\cdot)\right)<\infty\ ,\end{aligned} \tag{A3.79}$$

we can apply (A3.70) and (A3.74). Then the proof of the lemma is complete. ■

How to define the vectors $A(\mathbf{x}(j,i)-\mathbf{y})$ and $B(\mathbf{x}(j',i')-\mathbf{y})$ depends on the problem one has to study. For instance for propagators $g(\mathbf{x})=g_\omega^{(h)}(\mathbf{x})$ such that

A3.123
$$g_\omega^{(h)}(\mathbf{x})=\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}\cdot\mathbf{x}}\hat g_\omega^{(h)}(\mathbf{k})\ ,\qquad \hat g_\omega^{(h)}(\mathbf{k})=\frac{f_h(\mathbf{k}')}{-ik_0+E(k)}\ , \tag{A3.80}$$

a possible definition for such vectors, in terms of their Fourier transforms, is

A3.124
$$\hat A(k)=\frac{\sqrt{f_h(\mathbf{k}')}}{k_0^2+E^2(k)}\ ,\qquad\qquad \hat B(k)=-\sqrt{f_h(\mathbf{k}')}\,(ik_0+E(k))\ , \tag{A3.81}$$

so that $|g(\mathbf{x}-\mathbf{y})|\le C_0$, with $C_0=C_N\gamma^h$ (see (5.27)).

p.A3.4 **A3.4.** *Gram-Hadamard inequality.* Let $\mathbf{x}_1,\dots,\mathbf{x}_m$ be m vectors of an Euclidean space E of finite dimension n. We define the *Gram determinant* as

A3.125
$$\Gamma(\mathbf{x}_1,\dots,\mathbf{x}_m)\equiv\det\Gamma=\det\begin{pmatrix}(\mathbf{x}_1,\mathbf{x}_1)&\dots&(\mathbf{x}_1,\mathbf{x}_m)\\ \dots&\dots&\dots\\ (\mathbf{x}_m,\mathbf{x}_1)&\dots&(\mathbf{x}_m,\mathbf{x}_m)\end{pmatrix}\ , \tag{A3.82}$$

where $(\cdot,\cdot)$ denotes the inner product in E. The following results hold.

LEMMA A8. *Given an Euclidean space E and m vectors $\mathbf{x}_1,\dots,\mathbf{x}_m$ in E, the Gram determinant (A3.82) satisfies*

A3.126
$$\Gamma(\mathbf{x}_1,\dots,\mathbf{x}_m)=0\ , \tag{A3.83}$$

if and only if the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ *are linearly independent. If the vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_m$ *are linearly independent then one has*

A3.127
$$\Gamma(\mathbf{x}_1, \ldots, \mathbf{x}_m) > 0 \, . \tag{A3.84}$$

Proof. If the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly dipendent then there esits m coefficients $c_1, \ldots, c_m$ not all vanishing such that the vector $\sum_{j=1}^m c_j \mathbf{x}_j$ is vanishing. By considering its inner product with the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$, we obtain the system

A3.128
$$\begin{array}{ccccccc} \overline{c}_1(\mathbf{x}_1, \mathbf{x}_1) & + & \ldots & + & \overline{c}_m(\mathbf{x}_1, \mathbf{x}_m) & = & 0 \\ \ldots & & \ldots & & \ldots & & \ldots \\ \overline{c}_1(\mathbf{x}_m, \mathbf{x}_1) & + & \ldots & + & \overline{c}_m(\mathbf{x}_m, \mathbf{x}_m) & = & 0 \end{array} \tag{A3.85}$$

which is an homogeneous system admitting a nontrivial solution $\overline{c}_1, \ldots, \overline{c}_m$: therefore the determinant of the matrix of the coefficients is zero, so implying (A3.83).

Vice versa if (A3.83) holds the system (A3.85) admits a nontrivial solution $\overline{c}_1, \ldots, \overline{c}_m$. If we multiply the m equationns defining the system by $c_1, \ldots, c_m$, respectively, then we sum them, we obtain

A3.129
$$\| c_1 \mathbf{x}_1 + \ldots + c_m \mathbf{x}_m \| = 0 \, , \tag{A3.86}$$

where $\| \cdot \|$ is the norm induced by the inner product $(\cdot, \cdot)$. Therefore the vector $\sum_{j=1}^m c_j \mathbf{x}_j$ has to be identically vanishing: as the coefficients $c_1, \ldots, c_m$ are not all vanishing, then the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ have to be linearly independent.

To prove (A3.84) consider a subset $S \subset E$, and set, for any $\mathbf{x} \in E$, $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_N$, where $\mathbf{x}_S \in S$ and $\mathbf{x}_N$ belonging to the orthogonal complement to S. We can write $\mathbf{x}_N$ as $\mathbf{x}_N = \mathbf{x}_1 + \ldots + \mathbf{x}_p$, with $p = n - \dim(S)$; then we have that the vectors

A3.130
$$\det \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & \ldots & (\mathbf{x}_1, \mathbf{x}_p) & \mathbf{x}_1 \\ \ldots & \ldots & \ldots & \ldots \\ (\mathbf{x}_p, \mathbf{x}_1) & \ldots & (\mathbf{x}_p, \mathbf{x}_p) & \mathbf{x}_p \\ (\mathbf{x}, \mathbf{x}_1) & \ldots & (\mathbf{x}, \mathbf{x}_p) & \mathbf{x}_S \end{pmatrix} \tag{A3.87}$$

are identically vanishing. In particular it follows that

A3.131
$$\mathbf{x}_S = -\frac{1}{\det \Gamma} \det \begin{pmatrix} & & & \mathbf{x}_1 \\ & \Gamma & & \ldots \\ & & & \mathbf{x}_p \\ (\mathbf{x}, \mathbf{x}_1) & \ldots & (\mathbf{x}, \mathbf{x}_p) & 0 \end{pmatrix} \, , \tag{A3.88}$$

and, analogoulsy,

A3.132
$$\mathbf{x}_N \equiv \mathbf{x} - \mathbf{x}_S = \frac{1}{\det \Gamma} \det \begin{pmatrix} & & & \mathbf{x}_1 \\ & \Gamma & & \ldots \\ & & & \mathbf{x}_p \\ (\mathbf{x}, \mathbf{x}_1) & \ldots & (\mathbf{x}, \mathbf{x}_p) & \mathbf{x} \end{pmatrix} \quad , \tag{A3.89}$$

so that

A3.133
$$0 \leq h^2 \equiv (\mathbf{x}_N, \mathbf{x}) = \frac{1}{\det \Gamma} \det \begin{pmatrix} & & & (\mathbf{x}_1, \mathbf{x}) \\ & \Gamma & & \ldots \\ & & & (\mathbf{x}_p, \mathbf{x}) \\ (\mathbf{x}, \mathbf{x}_1) & \ldots & (\mathbf{x}, \mathbf{x}_p) & (\mathbf{x}, \mathbf{x}) \end{pmatrix} = \frac{\Gamma(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x})}{\Gamma(\mathbf{x}_1, \ldots, \mathbf{x}_p)} \, . \tag{A3.90}$$

By setting $\mathbf{x} \equiv \mathbf{x}_{p+1}$ and $h^2 = h_p^2$, we can write (A3.90) as

A3.134
$$\frac{\Gamma(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1})}{\Gamma(\mathbf{x}_1, \ldots, \mathbf{x}_p)} = h_p^2 \geq 0 \, , \tag{A3.91}$$

where $\mathbf{x}_1, \dots, \mathbf{x}_p$ are p linearly indepenedent vectors and $\mathbf{x}_{p+1}$ is arbitrary. The sign $=$ in (A3.91) can holds if and only if $\mathbf{x}_{p+1}$ is a linear combination of the vecors $\mathbf{x}_1, \dots, \mathbf{x}_p$ so that if $\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1}$ are linearly independent, then (A3.91) holds with the strict sign, *i.e.*

A3.135
$$\frac{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1})}{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p)} = h_p^2 > 0 \; . \tag{A3.92}$$

As $\Gamma(\mathbf{x}_1) = (\mathbf{x}_1, \mathbf{x}_1) = \|\mathbf{x}_1\|^2 > 0$ for $\mathbf{x}_1 \neq 0$, (A3.92) implies (A3.84). ■

LEMMA A9 (HADAMARD INEQUALITY). *The Gram determinant satisfies the inequality*

A3.136
$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) \leq \Gamma(\mathbf{x}_1) \dots \Gamma(\mathbf{x}_m) \; , \tag{A3.93}$$

where the sign $=$ holds if and only if the vectors are orthogonal to each other.

Proof. By (A3.92) and by using that $(\mathbf{x}_N, \mathbf{x}_N) \leq (\mathbf{x}, \mathbf{x}) = \Gamma(\mathbf{x})$, we have

A1.136a
$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}) \leq \Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) \Gamma(\mathbf{x}) \; , \tag{A3.94}$$

for any vectors $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x} \in E$. By iterating and recalling the arguments above (A3.93) follows. ■

Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be m linearly independet vectors in E, with $m = n$ if $n = \dim(E)$. Let $\{\mathbf{e}_j\}_{j=1}^m$ an orthonormal basis in E: set $\mathbf{x}_{jk} = (\mathbf{e}_j, \mathbf{x}_k)$, so that $\mathbf{x}_k = \sum_{j=1}^m \mathbf{x}_{jk} \mathbf{e}_j$, $k = 1, \dots, m$. Then

A3.137
$$\begin{aligned}
\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) &= \det \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & \dots & (\mathbf{x}_1, \mathbf{x}_m) \\ \dots & \dots & \dots \\ (\mathbf{x}_m, \mathbf{x}_1) & \dots & (\mathbf{x}_m, \mathbf{x}_m) \end{pmatrix} \\
&= \sum_{i_1 \dots i_m} \sum_{j_1 \dots j_m} \det \begin{pmatrix} \overline{x}_{i_1 1} x_{j_1 1} (\mathbf{e}_{i_1}, \mathbf{e}_{j_1}) & \dots & \overline{x}_{i_1 1} x_{j_m m} (\mathbf{e}_{i_1}, \mathbf{e}_{j_m}) \\ \dots & \dots & \dots \\ \overline{x}_{i_m m} x_{j_1 1} (\mathbf{e}_{i_m}, \mathbf{e}_{j_1}) & \dots & \overline{x}_{i_m m} x_{j_m m} (\mathbf{e}_{i_m}, \mathbf{e}_{j_m}) \end{pmatrix} \\
&= \sum_{i_1 \dots i_m} \det \begin{pmatrix} \overline{x}_{i_1 1} x_{i_1 1} & \dots & \overline{x}_{i_1 1} x_{i_m m} \\ \dots & \dots & \dots \\ \overline{x}_{i_m m} x_{i_1 1} & \dots & \overline{x}_{i_m m} x_{i_m m} \end{pmatrix} \\
&= \det \begin{pmatrix} x_{11} & \dots & x_{m1} \\ \dots & \dots & \dots \\ x_{1m} & \dots & x_{mm} \end{pmatrix} \begin{pmatrix} \overline{x}_{11} & \dots & \overline{x}_{1m} \\ \dots & \dots & \dots \\ \overline{x}_{m1} & \dots & \overline{x}_{mm} \end{pmatrix} \\
&= \left| X^T \, \overline{X} \right| = \det \Xi^T \, \det \overline{X} = |\det X|^2 \; ,
\end{aligned} \tag{A3.95}$$

where the matrix X is defined as

A3.138
$$X = \begin{pmatrix} x_{11} & x_{12} & \dots x_{1m} \\ x_{21} & x_{22} & \dots x_{2m} \\ \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots x_{mm} \end{pmatrix} \; . \tag{A3.96}$$

This yields that the Gram determinant (A3.93) can be written as

A3.139
$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) = |\det X|^2 \; , \tag{A3.97}$$

so that from the lemma above the following result follows immediately.

LEMMA A10. *Given m linearly independent vectors of an Euclidean space E, and defined the matrix X through (A3.95), one has*

A3.140
$$|\det X|^2 \equiv |\det(\mathbf{e}_i, \mathbf{x}_j)|^2 \leq \prod_{j=1}^m \|\mathbf{x}_j\|^2 \; , \tag{A3.98}$$

where $(\mathbf{e}_i, \mathbf{x}_j)$ *stands for the matrix with entries* $X_{ij} = (\mathbf{e}_i, \mathbf{x}_j)$.

The lemma above is simply a reformulation of the preceeding Lemma: it implies the following inequality.

THEOREM A1 (GRAM-HADAMARD INEQUALITY). *Let* $\{\mathbf{f}_j\}_{j=1}^m$ *and* $\{\mathbf{g}_j\}_{j=1}^m$ *two families of* m *linearly independent vectors in an Euclidean space* E*, and let* $(\cdot,\cdot)$ *an inner product in* E *and* $\|\cdot\|$ *the norm induced by that inner product. Then*

A3.141
$$|\det(\mathbf{f}_i, \mathbf{g}_j)| \leq \prod_{j=1}^m \|\mathbf{f}_j\| \, \|g_j\| \, , \tag{A3.99}$$

where $(\mathbf{f}_i, \mathbf{g}_j)$ *stands for the* $m \times m$ *matrix with entries* $(\mathbf{f}_i, \mathbf{g}_j)$.

Proof. If $\{\mathbf{g}_j\}_{j=1}^m$ is an orthogonal basis in E (so that $\{\mathbf{e}_j\}_{j=1}^m$, with $\mathbf{e}_j = \|\mathbf{g}_j\|^{-1}\mathbf{g}_j$, is an orthonormal basis) then (A3.98) gives

A3.142
$$|\det(\mathbf{g}_i, \mathbf{x}_j)| = |\det(\mathbf{e}_i, \mathbf{x}_j)| \prod_{j=1}^m \|\mathbf{g}_j\| \leq \prod_{j=1}^m \|\mathbf{g}_j\| \|\mathbf{x}_j\| \, , \tag{A3.100}$$

Now consider the case in which the only conditions on the vectors $\{g_j\}_{j=1}^m$ is that they are linearly independent. Set $\tilde{\mathbf{g}}_j = \|\mathbf{g}_j\|^{-1}\mathbf{g}_j$, so that $\|\tilde{\mathbf{g}}_j\|^2 = 1$, and define inductively the family of vectors

A3.143
$$\begin{aligned} &\tilde{\mathbf{e}}_1 \equiv \tilde{\mathbf{g}}_1 \, , \\ &\tilde{\mathbf{e}}_2 \equiv \frac{\tilde{\mathbf{g}}_2 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)\tilde{\mathbf{g}}_1}{1 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)^2} \, , \end{aligned} \tag{A3.101}$$

and so on, in such a way that one has $(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) = \delta_{i,j}$. The basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$, with $\mathbf{e}_j = \tilde{\mathbf{e}}_j \; \forall j = 1, \ldots, m$ is by construction an orthonormal basis.

If $c_2 = 1 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)^2$, with $0 \leq c_2 \leq 1$, one has

A3.144
$$\tilde{\mathbf{g}}_2 = c_2\tilde{\mathbf{e}}_2 + c_2(\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)\tilde{\mathbf{g}}_1 \, , \tag{A3.102}$$

i.e. $\tilde{\mathbf{g}}_2 \sim c_2\tilde{\mathbf{e}}_2$, if by $\sim$ we mean that, by computing $\det(\tilde{\mathbf{g}}_i, \mathbf{f}_j)$, no difference is made by the fact that one has the vector $\tilde{\mathbf{g}}_2$ instead of $c_2\tilde{\mathbf{e}}_2$: in fact the contributions arising from the remaining part in (A3.100) sum up to zero.

We can reason analogously for the terms with $j = 3, \ldots, m$, and we find $\tilde{\mathbf{g}}_j \sim c_j\tilde{\mathbf{e}}_j$, where $\sim$ is meant as above and the coefficients c_j are such that $0 \leq c_j \leq 1 \; \forall j = 1, \ldots, m$. In conclusion:

A3.145
$$\begin{aligned} |\det(\mathbf{g}_i, \mathbf{f}_j)| &= |\det(\tilde{\mathbf{g}}_i, \mathbf{f}_j)| \prod_{j=1}^m \|\mathbf{g}_j\| = |\det(\mathbf{e}_i, \mathbf{f}_j)| \prod_{j=1}^m c_j \, \|\mathbf{g}_j\| \\ &= \prod_{j=1}^m c_j \, \|\mathbf{g}_j\| \, \|\mathbf{f}_j\| \leq \prod_{j=1}^m \|\mathbf{g}_j\| \, \|\mathbf{f}_j\| \quad , \end{aligned} \tag{A3.103}$$

so that (A3.99) follows. ■

app.A4

Appendix A4. Dimensional bounds

p.A4.1 **A4.1.** *Proof of (5.27).* We call C_N any constant depending on N and C any constant independent on N and denote by $\partial_{\mathbf{k}} = (\partial_k, \partial_{k_0})$ the discrete derivative (see Appendix A2).

One has

A4.1
$$\begin{aligned}\left|g_\omega^{(h)}(\mathbf{x})\right| &= \left|\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{f_h(\mathbf{k}')}{-ik_0+E(k)}\right| \\ &\le \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\frac{f_h(\mathbf{k}')}{|-ik_0+E(k)|} \le C\gamma^{-h}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} f_h(\mathbf{k}') \le C\gamma^{-h}\gamma^{2h} \le C\gamma^h\ .\end{aligned} \tag{A4.1}$$

In the same way one has

A4.2
$$\begin{aligned}\left|\left(\gamma^h|\mathbf{x}|\right)^N g_\omega^{(h)}(\mathbf{x})\right| &= \left|\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\left[\left(\gamma^h\partial_{\mathbf{k}}\right)^N e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\frac{f_h(\mathbf{k}')}{-ik_0+E(k)}\right| \\ &= \left|\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\left(\gamma^h\partial_{\mathbf{k}}\right)^N\frac{f_h(\mathbf{k}')}{-ik_0+E(k)}\right]\right| \\ &\le \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\left|\left(\gamma^h\partial_{\mathbf{k}}\right)^N\frac{f_h(\mathbf{k}')}{-ik_0+E(k)}\right| \le C_N\gamma^{Nh}\gamma^{-(N+1)h}\gamma^{2h} \le C\gamma^h\ ,\end{aligned} \tag{A4.2}$$

so that, by using together the two bounds, one obtains

A4.3
$$\left|1+\left(\gamma^h|\mathbf{x}|\right)^N g_\omega^{(h)}(\mathbf{x})\right| \le C_N\gamma^h\ , \tag{A4.3}$$

so implying (5.27).

p.A4.2 **A4.2.** *Proof of (5.28) for Feynman diagrams.* Consider a Feynman diagram Γ and call τ the tree associated with it. The diagram Γ consists of $n+n_4$ points (here n_4 is the number of endpoints $v\in V_f(\tau)$ with $i_v=4$). After integrating over n_4 variables by using the potentials $v(\mathbf{x}-\mathbf{y})$, we are left with n integrations. As the diagram has to be connected, for any cluster G_v containing s_v subclusters $G_{v_1},\dots,G_{v_{s_v}}$ one has s_v-1 lines on scale h_v assuring the connection between the subclusters: such lines form an anchored tree T_v for the cluster G_v. Of course the union of the anchored trees T_v corresponding to all the clusters G_v, $v\in V(\tau)$, is a tree T for the Feynman diagram Γ. So we can perform a change of coordinates and integrate over the variables $\mathbf{x}_\ell-\mathbf{y}_\ell$, where $x_\ell,\mathbf{y}_\ell$ are the extremes of the lines ℓ such that $\ell\in T_v$ for some $v\in V(\tau)$, *i.e.* they are the points which the line ℓ is incident with. For each line $\ell\in T_v$ we obtain a factor γ^{-2h_v}, by the compact support properties of the propagators g_ℓ. In fact, by using (5.27), one obtains

A4.4
$$\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}\mathrm{d}x_0\left|g_\omega^{(h)}(\mathbf{x})\right| \le C\gamma^{-2h}\gamma^h\ , \tag{A4.4}$$

where the factor γ^h is taken into account by the factor

A4.5
$$\gamma^{h_v n_v^0}\ , \tag{A4.5}$$

(see (5.29)). As for any v there are s_v-1 lines on scale h_v, then (5.28) follows.

p.A4.3 **A4.3.** *Proof of (6.16) for trees.* By using (6.13) one immediately sees that one has to integrate

A4.6
$$\int\mathrm{d}\mathbf{x}(I_{v_0})\left[\prod_{v\in V(\tau)}\sum_{T_v}\left(\prod_{\ell\in T_v} g_\ell\right)\right]\ , \tag{A4.6}$$

where, for each $v\in V(\tau)$, T_v is an anchored tree between the s_v maximal subclusters contained inside G_v

For each line ℓ representing a propagator g_ℓ let us call $\mathbf{x}_\ell$ and $\mathbf{y}_\ell$ its extremes; analogously we can define $\mathbf{x}_\ell$ and $\mathbf{y}_\ell$ the extremes of an ondulated line representing the potential $v(\mathbf{x}-\mathbf{y})$, if $\mathbf{x}_\ell \equiv \mathbf{x}$ and $\mathbf{y}_\ell \equiv \mathbf{y}$.

The overall number of lines is $\sum_{v\in V(\tau)}(s_v-1) = n-1$, by (6.17), while the overall number of integratio variables is $n+n_4$. As the union of all trees T_v and of all ondulated lines representing the two-body interaction potentials assures the connectedness of the clusters G_v, we can make a change of coordinates and integrate over

(1) the $n-1$ differences $\mathbf{x}_\ell - \mathbf{y}_\ell$, such that $g_\ell \in T_v$ for some $v \in V(\tau)$,

(2) over the n_4 differences $\mathbf{x}_\ell - \mathbf{y}_\ell$, such that $v(\mathbf{x}_\ell - \mathbf{y}_\ell)$ represents an ondulated line, and (3) over a fixed variable, say $\mathbf{x}_1$.

The last integrations give simply a constant to the power $n_4 \le 4$, while the first one gives a factor γ^{-2h_v} for each line g_ℓ on scale $h_\ell = h_v$ (see the analogous discussion in §A4.2). The integration over $\mathbf{x}_1$ gives a factor $(L\beta)$.

app.A5

Appendix A5. Diophantine numbers

p.A5.1 **A5.1.** *Definitions.* Given a vector $\boldsymbol{\omega} \in \mathbb{R}^n$, we say that $\boldsymbol{\omega}$ is a *Diophantine vector* if

A5.1
$$|\boldsymbol{\omega}\cdot\mathbf{n}| \ge C_0\,|\mathbf{n}|^{-\tau} \qquad \forall \mathbf{n}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}\,, \tag{A5.1}$$

where $|\mathbf{n}| = |n_1|+\ldots+|n_n|$ and C_0, τ are suitable positive constants, which are called *Diophantine constants.*

If $n=2$ and $\boldsymbol{\omega} = (2\pi, \omega)$ the above inequality can be rewritten as

A5.2
$$\|n\omega\|_{\mathbb{T}} \equiv \sup_{p\in\mathbb{Z}} |n\omega - 2p\pi| \ge C_0|n|^{-\tau} \qquad \forall n\in\mathbb{Z}\setminus\{0\}\,, \tag{A5.2}$$

by renaming the constant C_0. In fact (A5.1) for $n=2$ would give $|n_1\omega + 2n_2\pi| \ge C_0(|n_1|+|n_2|)^{-\tau}$. Of course $|n_1\omega+2n_2\pi|$ can be small only if, say, $|n_1\omega+2n_2\pi| < 1/2$, *i.e.* if n_2 is such that $2n_2\pi$ differs from $n_1\omega$ less than $1/2$: therefore the supremum in the left hand side of (A5.2) is assumed for n_2 such that $a_1 n_1 \le |n_2| \le a_2 n_1$, for some constants a_1 and a_2. So, by redefining the constant C_0, the inequality in (A5.2) follows.

If we write $\boldsymbol{\omega} = (2\pi, \omega)$ we call ω the *rotation number.*

p.A5.2 **A5.2.** *Properties.* The Diophantine vectors are of great interest as they are of full measure in $\mathbb{R}^n$, provided that $\tau > n-1$ in (A5.1); see for instance [G0]. Very likely the results in §8 and §13 could be obtained by relaxing the hypothesis on the rotation number, *e.g.* by imposing the weaker *Bryuno condition*: also in KAM theory the persistence of invariant tori (for flows) and invariant curves (for diffeomorphisms) has been proven under such a condition; see [Rü], [D], [BeG].

Anyway we note that the fact the Diophantine vectors are of full measure make such an extension to more general vectors of secondary importance, unless some explicit questions are asked (as the dependece on the rotation vector of the domain of convergence for the perturbative series, or the optimal condition on the rotation vector, and so on).

app.A6

Appendix A6. Some technical results

p.A6.1 **A6.1.** *Proof of (5.39) and (8.96).* We want to show that, for any constant $\alpha > 0$,

A6.1
$$S(P_{v_0},\tau,n) \equiv \sum_{\tau\in\mathcal{T}_{h,n}} \sum_{\{P_v\}} \left(\prod_{v\notin V_{\rm f}(\tau)} \gamma^{-2\alpha|P_v|} \right) \le C^n\,, \tag{A6.1}$$

for some constant C depending on α.

In (A6.1) we can write

A6.2
$$\prod_{v\notin V_{\rm f}(\tau)} \gamma^{-2\alpha|P_v|} = \prod_{v\notin V_{\rm f}(\tau)} \gamma^{-\alpha|P_v|} \prod_{v\notin V_{\rm f}(\tau)} \gamma^{-\alpha|P_v|} \leq \prod_{v\notin V_{\rm f}(\tau)} \gamma^{-\alpha(h_v-h_{v'})} \prod_{v\notin V_{\rm f}(\tau)} \gamma^{-\alpha|P_v|} \ . \tag{A6.2}$$

and

A6.3
$$\sum_{\{P_v\}} = \prod_{v\in V(\tau)} \sum_{p_v} \sum_{\substack{P_v \\ |P_v|=p_v}} \chi\left(\text{constraint on } \{p_v\}\right) \ , \tag{A6.3}$$

if the χ denotes the constraint

A6.4
$$1 \leq p_v \leq \sum_{j=1}^{s_v} q_{v_j} \ , \tag{A6.4}$$

where $q_{v_j} = |Q_{v_j}|$ and $v_1, \ldots, v_{s_v}$ are the vertices immediately following v along the tree τ (we use the notations (5.25)).

If we neglect the constraint χ and remove also the constraint that P_{v_0} is fixed (*i.e.* we sum over all the possible P_{v_0}), we can bound

A6.5
$$\sum_{\{P_v\}} \left(\prod_{v\in V(\tau)} \gamma^{-\alpha|P_v|} \right) \leq \prod_{v\in V(\tau)} \left[\sum_{p_v} \gamma^{-\alpha p_v} \begin{pmatrix} p_{v_1} + \ldots + p_{v_{s_v}} \\ p_v \end{pmatrix} \right] \ , \tag{A6.5}$$

where we used that

A6.6
$$\prod_{v\in V(\tau)} \sum_{\substack{P_v \\ |P_v|=p_v}} \chi\left(\text{constraint on } \{p_v\}\right) \leq \prod_{v\in V(\tau)} \sum_{\substack{P_v \\ |P_v|=p_v}} 1 \leq \begin{pmatrix} p_{v_1} + \ldots + p_{v_{s_v}} \\ p_v \end{pmatrix} \ . \tag{A6.6}$$

We can write (A6.5) as

A6.7
$$\prod_{v\in V(\tau)} \left[\sum_{p_v} \gamma^{-\alpha p_v} \begin{pmatrix} p_{v_1} + \ldots + p_{v_{s_v}} \\ p_v \end{pmatrix} \right] \equiv \prod_{v\in V(\tau)} \mathcal{I}_v \ , \tag{A6.7}$$

which defines the factors $\mathcal{I}_v$. In particular we have

A6.8
$$\begin{aligned} \mathcal{I}_{v_0} &= \sum_{p_{v_0}} \gamma^{-\alpha p_{v_0}} \begin{pmatrix} p_{v_{01}} + \ldots + p_{v_{0s_{v_0}}} \\ p_{v_0} \end{pmatrix} \\ &= \left(1+\gamma^{-\alpha}\right)^{p_{v_{01}}+\ldots+p_{v_{0s_{v_0}}}} = \prod_{j=1}^{s_{v_0}} \left(1+\gamma^{-\alpha}\right)^{p_{v_{0j}}} \ , \end{aligned} \tag{A6.8}$$

where $v_{01}, \ldots, v_{0s_{v_0}}$ are the vertices immediately following v_0, so that

A6.9
$$\prod_{v\in V(\tau)} \left[\sum_{p_v} \gamma^{-\alpha p_v} \begin{pmatrix} p_{v_1} + \ldots + p_{v_{s_v}} \\ p_v \end{pmatrix} \right] = \Big(\prod_{\substack{v\in V(\tau) \\ v\succ v_0}} \mathcal{I}_v \Big) \prod_{j=1}^{s_{v_0}} \left(1+\gamma^{-\alpha}\right)^{p_{v_{0j}}} \ . \tag{A6.9}$$

If we iterate the procedure we obtain

A6.10
$$\mathcal{I}_{v_0} \prod_{j=1}^{s_{v_0}} \mathcal{I}_{v_{0j}} = \prod_{j=1}^{s_{v_0}} \prod_{j'=1}^{s_{v_j}} \left(1+\gamma^{-\alpha}\left(1+\gamma^{-\alpha}\right)\right)^{p_{v_{jj'}}} \ , \tag{A6.10}$$

where $v_{j1},\dots,v_{js_v}$ are the vertices immediately following v_{0j}. And so on until we reach all the endpoints of the tree τ. If we denote by $\mathcal{P}$ a path (*i.e.* an oriented connected set of lines) from the root to an endpoint we find

A6.11
$$\prod_{v\in V(\tau)} \mathcal{I}_v = \prod_{\mathcal{P}} \left[\left(1+\gamma^{-\alpha}\left(1+\gamma^{-\alpha}\left(1+\gamma^{-\alpha}\left(\dots\right)\right)\right)\right)\right]^4 , \tag{A6.11}$$

where we used that the endpoints have at most four external lines (see (5.19) and the product is over all the possible paths on τ. Then, if we denote by $\ell(\mathcal{P})$ the "lenght" of the path $\mathcal{P}$, *i.e.* number of vertices along the path $\mathcal{P}$, we have

A6.12
$$\prod_{v\in V(\tau)} \mathcal{I}_v = \prod_{\mathcal{P}} \left[\sum_{k=0}^{\ell(\mathcal{P})} \gamma^{-\alpha k}\right]^4 \le \left(\frac{\gamma^\alpha}{\gamma^\alpha-1}\right)^{4n} \equiv C_1^n , \tag{A6.12}$$

where $C_1=\gamma^{4\alpha}\left(\gamma^\alpha-1\right)^4$. By using the results in Appendix A1 one has

A6.13
$$\sum_{\tau\in\mathcal{T}_{h,n}} \prod_{v\notin V_{\rm f}(\tau)} \gamma^{-\alpha(h_v-h_{v'})} \le C_2^n , \tag{A6.13}$$

for some constant C_2, so implying (A6.1) with $C=C_1C_2$, hence (8.96).

p.A6.2 **A6.2.** *Proof of (5.23).* First note that $\mathcal{V}$ is a sum of contributions (see (5.19)) which can be expressed as

A6.14
$$\int d\mathbf{x}(I_v)\, \mathcal{W}^{(1)}(\mathbf{x}(I_v))\, \tilde\psi^{(\le 1)}(I_v) , \tag{A6.14}$$

where, for instance, $\mathbf{x}(I_v)=\{\mathbf{x},\mathbf{y}\}$, $\psi^{(\le1)}(I_v)=\psi^+_{\mathbf{x},\sigma}\psi^+_{\mathbf{y},\sigma'}\psi^-_{\mathbf{y},\sigma'}\psi^-_{\mathbf{x},\sigma}$, $\mathcal{W}^{(1)}(\mathbf{x}(I_v))=v(x-y)$ and

A6.15
$$\int d\mathbf{x}(I_v) = \frac{1}{(2S+1)^2}\sum_{\sigma,\sigma'=\pm S}\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2} dx_0 \sum_{y\in\Lambda}\int_{-\beta/2}^{\beta/2} dy_0 , \tag{A6.15}$$

for $i_v=4$ in the discrete case. So if $h_v=1$ one has (see (5.7) with $h=1$ denoting the ultraviolet scale)

A6.16
$$\begin{aligned}&\frac{1}{n!}\mathcal{E}_1^T\left(\int d\mathbf{x}(I_{v_1})\,\mathcal{W}^{(1)}(\mathbf{x}(I_{v_1}))\tilde\psi^{(\le1)}(I_{v_1}),\dots,\int d\mathbf{x}(I_{v_n})\,\mathcal{W}^{(1)}(\mathbf{x}(I_{v_n}))\tilde\psi^{(\le1)}(I_{v_n})\right)\\ &\quad=\int d\mathbf{x}(I_{v_1})\dots\int d\mathbf{x}(I_{v_n})\,\mathcal{W}^{(1)}(\mathbf{x}(I_{v_1}))\dots\mathcal{W}^{(1)}(\mathbf{x}(I_{v_n}))\frac{1}{n!}\mathcal{E}_1^T\left(\tilde\psi^{(\le1)}(I_{v_1}),\dots,\tilde\psi^{(\le1)}(I_{v_n})\right) ,\end{aligned} \tag{A6.16}$$

which contains an expression like (5.23) with $P_{v_j}=I_{v_j}$ for $j=1,\dots,n$.

If $h_v\le 0$, then one has, by the inductive hypothesis (see (5.22) and (5.23))

A6.17
$$\frac{1}{s_v!}\mathcal{E}^T_{h_v}\left(\mathcal{E}^T_{h_v+1}\left(\tilde\psi^{(\le h_v+1)}(P_{v_{11}}),\dots,\tilde\psi^{(\le h_v+1)}(P_{v_{1s_{v_1}}})\right),\dots,\right) , \tag{A6.17}$$

where $v_1,\dots,v_{s_v}$ are the s_v vertices following v and $v_{j1},\dots,v_{js_{v_j}}$ are the s_{v_j} vertices following v_j, $j=1,\dots,s_v$. Then, by the definitions,

A6.18
$$\begin{aligned}&\mathcal{E}^T_{h_v+1}\left(\tilde\psi^{(\le h_v+1)}(P_{v_{j1}}),\dots\tilde\psi^{(\le h_v+1)}(P_{v_{js_{v_j}}})\right)\\ &\quad=\sum_{Q_{v_{j1}}\subset P_{jv_1}}\dots\sum_{Q_{v_{js_{v_j}}}\subset P_{v_{js_{v_j}}}}\tilde\psi^{(\le h_v)}(Q_{v_{j1}})\dots\tilde\psi^{(\le h_v)}(Q_{v_{js_{v_j}}})\\ &\qquad\mathcal{E}^T_{h_v+1}\left(\tilde\psi^{(h_v+1)}(P_{v_{j1}}\setminus Q_{v_{j1}}),\dots,\tilde\psi^{(h_v+1)}(P_{v_{js_{v_j}}}\setminus P_{v_{js_{v_j}}})\right) ,\end{aligned} \tag{A6.18}$$

where

A6.19
$$\mathcal{E}^T_{h_v+1}\left(\tilde\psi^{(h_v+1)}(P_{v_{j1}}\setminus Q_{v_{j1}}),\dots,\tilde\psi^{(h_v+1)}(P_{v_{js_{v_j}}}\setminus P_{v_{js_{v_j}}})\right) \tag{A6.19}$$

gives a constant (*i.e.* a quantity which does not depend on the fields). Then in (A6.17) one is left with an expression like

A6.20
$$\frac{1}{s_v!}\mathcal{E}^T_{h_v}\left(\tilde\psi^{(\le h_v)}(P_{v_1}),\dots,\tilde\psi^{(\le h_v)}(P_{v_{s_v}})\right)\ , \tag{A6.20}$$

with

A6.21
$$P_{v_j}=\bigcup_{i=1}^{s_{v_j}}Q_{v_{ji}}\ , \tag{A6.21}$$

so that (5.23) is proven.

p.A6.3 **A6.3.** *Proof of (8.91).* Recall the definition of depth of nontrivial vertices given in §8. We call B_D the set of $v\in V^*_{\rm f}(\tau)$ such that the nontrivial vertex immediately preceeding v has depth D.

Given a tree τ define the depth of the tree as

A6.22
$$D_\tau=\max\{D_v : v\in V_{\rm nt}(\tau)\}\ , \tag{A6.22}$$

and set

A6.23
$$\mathcal{B}_D=\bigcup_{p=0}^{D}B_p\ ; \tag{A6.23}$$

by construction $\mathcal{B}_D$ is the collections of all endpoints in $V^*_{\rm f}(\tau)$ contained inside a cluster G_v, for some v with depth $D_v=D$.

We prove by induction on the depth $D\in[0,D_\tau]\cap\mathbb{N}$ the following bound:

A6.24
$$\begin{aligned}&\prod_{v\in V^*_{\rm f}(\tau)\cap\mathcal{B}_D}|\hat\varphi_{m_v}|^{1/2}\\ &\qquad\le\left(\prod_{v\in V^*_{\rm f}\cap\mathcal{B}_D}F_0^{1/2}\right)\left(\prod_{p=0}^{D-1}\prod_{v\in V^*_{\rm f}(\tau)\cap B_p}e^{-\kappa|N_v|/2^{p+2}}\right)\left(\prod_{v\in V^*_{\rm f}(\tau)\cap B_D}e^{-\kappa|N_v|/2^{p+1}}\right)\ ,\end{aligned} \tag{A6.24}$$

where the product in the first parentheses has to be thought as 1 for $D=0$.

For $D=0$, (A6.24) is a trivial identity: it is enough to recall that $|\hat\varphi_m|\le F_0e^{-\kappa m}$ (see (8.6)) and that $N_v=m_v$ if $v\in V_{\rm f}(\tau)$ (see (8.17)).

Suppose that (A6.24) holds for some $D-1$; then we want to show that it holds also for D. In fact, by using that, for any vertex $v\in V(\tau)\setminus V_{\rm f}(\tau)$, one has

A6.25
$$N_v=N_{w_1}+\ldots+N_{w_{s_v}}\ , \tag{A6.25}$$

where $w_1,\dots,w_{s_v}$ are the s_v nontrivial vertices immediately following v: this simply follows from the definition (8.17) and from the fact that if $\tilde v$ is a trivial vertex then $N_{\tilde v}=N_w$, where w is the nontrivial vertex immediately following $\tilde v$.

Then one has

$$\begin{aligned}
\prod_{v\in V_{\rm f}^*(\tau)\cap\mathcal{B}_D}|\hat\varphi_{m_v}|^{1/2} &= \left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_D}|\hat\varphi_{m_v}|^{1/2}\right)\left(\prod_{v\in V_{\rm f}^*(\tau)\cap\mathcal{B}_{D-1}}|\hat\varphi_{m_v}|^{1/2}\right)\\
&\le \left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_D}F_0e^{-\kappa|m_v|/2}\right)\left(\prod_{v\in V_{\rm f}^*\cap\mathcal{B}_{D-1}}F_0^{1/2}\right)\\
&\quad\left(\prod_{p=0}^{D-2}\prod_{v\in V_{\rm f}^*(\tau)\cap B_p}e^{-\kappa|N_v|/2^{p+2}}\right)\left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_{D-1}}e^{-\kappa|N_v|/2^D}\right)\\
&\le \left(\prod_{v\in V_{\rm f}^*\cap\mathcal{B}_D}F_0^{1/2}\right)\left(\prod_{p=0}^{k-1}\prod_{v\in V_{\rm f}^*(\tau)\cap B_p}e^{-\kappa|N_v|/2^{p+2}}\right)\\
&\quad\left[\left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_D}e^{-\kappa|m_v|/2}\right)\left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_{D-1}}e^{-\kappa|N_v|/2^{D+1}}\right)\right]\\
&\le \left(\prod_{v\in V_{\rm f}^*\cap\mathcal{B}_D}F_0^{1/2}\right)\\
&\quad\left(\prod_{p=0}^{k-1}\prod_{v\in V_{\rm f}^*(\tau)\cap B_p}e^{-\kappa|N_v|/2^{p+2}}\right)\left(\prod_{v\in V_{\rm f}^*(\tau)\cap B_D}e^{-\kappa|N_v|/2^{D+1}}\right),
\end{aligned}\tag{A6.26}$$

A6.26

so proving (A6.24). By taking $k = D_\tau$, (A6.23) follows. ■

References

[ADG] A.A. Abrikosov, L.P. Gorkov, I.Y. Dzyaloshinski: *Methods of Quantum Field Theory in Statistical Physics*, Dover Publications, New York, 1965.

[Af] I. Affleck: *Field theory methods and quantum critical phenomena*, Les Houches Summer School on Critical phenomena, Random Systems, Gauge theorys, North Holland (1984), Springer.

[A1] P.W. Anderson: *The theory of superconductivity in high-T_c cuprates*, Princeton Press, Princeton, 1997.

[A2] P.W. Anderson: *Basic principles in solid state physics*, Benjiamin/Commings Publishing Company, 1985.

[Ar] V.I. Arnol'd: Small divisors problems in classical and celestial mechanics, *Russ. Math. Surveys* **18** (1963), No. 3, 85–191.

[AAR] S. Aubry, G. Abramovici, J. Raimbault: Chaotic polaronic and bipolaronic states in the adiabatic Holstein model, *J. Statist. Phys.* **67** (1992), No. 3–4, 675–780.

[AG] Aizenman, Graf: Localization bounds for an electron gas, *J. Phys. A* **31** (1998), No. 32, 6783–6806.

[AM] M. Aizenman, S. Molchanov: Localization at large disorder and at extreme energies: an elementary derivation, *Comm. Math. Phys.* **157** (1993), No. 2, 245–278.

[AxG] F. Axel, D. Gratis: *Beyond quasi-crystals*, Les Houches, 1984, Springer.

[BaM] C. Baesens, R.S. MacKay: Improved proof of existence of chaotic polaronic and bipolaronic states fot the adiabatic Holstein model and generalizations, *Nonlinearity* **7** (1994), 59–84.

[B] R.J. Baxter: One dimensional anysotropic Heisember chain, *Ann. Phys.* **70** (1972), 323–327.

[BLT] J. Belissard, R. Lima, D. Testard: A metal-insulator transition for almost Mathieu model, *Comm. Math. Phys.* **88** (1983), 207–234.

[BG1] G. Benfatto, G. Gallavotti: Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one dimensional systems, *J. Statist. Phys.* **59** (1990), No. 3–4, 541–664.

[BG2] G. Benfatto, G. Gallavotti: *Renormalization Group*, Physics Notes **1**, Princeton University Press, Princeton, 1995.

[BGL] G. Benfatto, G. Gallavotti, J. Lebowitz: Disorder in the 1D Holstein model, *Helv. Phys. Acta* **68** (1995), No. 3, 312–327.

[BeGM] G. Benfatto, G. Gallavotti, V. Mastropietro: Renormalization group and the Fermi surface in the Luttinger model, *Phys. Rev. B* **45** (1992), 5468–5480.

[BGPS] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: Beta functions and Schwinger functions for a many fermions system in one dimension, *Comm. Math. Phys.* **160** (1994), 93–171.

[BGM1] G. Benfatto, G. Gentile, V. Mastropietro: Electrons in a lattice with incommensurate potential *J. Statist. Phys.* **89** (1997), No. 3–4, 655–708.

[BGM2] G. Benfatto, G. Gentile, V. Mastropietro: Peierls instability for the Holstein model with rational density, *J. Statist. Phys.* **92** (1998), No. 5–6, 1071-1113.

[BeM] G. Benfatto, V. Mastropietro: Renormalization Group, hidden symmetries and approximate Ward identities in the XYZ model, I and II, Preprint (1999).

[Be] F.A. Berezin: *The methods of second quantization*, Academic Press, New York, 1966.

[BeG] A. Berretti, G. Gentile: Bryuno function and the standard map, Preprint (1998).

[BP] N.N. Bogolubov, O.S. Parasiuk: *Acta Math.* **97** (1957), 227.

[BM1] F. Bonetto, V. Mastropietro: Beta function and anomaly of the Fermi surface for a d=1 system of interacting fermions in a periodic potential, *Comm. Math. Phys.* **172** (1995), 57–93.

[BM2] F. Bonetto, V. Mastropietro: Filled band Fermi systems *Math. Phys. Electron. J.* **2** (1996), paper 1, 1–43.

[BM3] F. Bonetto, V. Mastropietro: Critical indices in a $d = 1$ filled band Fermi system, *Phys. Rev. B* **56** (1997), No. 3, 1296–1308.

[BM4] F. Bonetto, V. Mastropietro: Critical indices for the Yukawa$_2$ quantum field theory, *Nucl. Phys. B* **497** (1997), 541–554.

[BJ] C. Bourbonnais, D. Jerome: The normal phase of quasi one dimensional superconductors, Preprint (1999) (cond-mat-9903101).

[BC] C. Bourbonnais, L.G. Caron: Renormalization-group approach to quasi-one-dimensional conductors, *Int. J. Mod. Phys. B* **5** (1991), No. 6–7, 1033–1096.

[Br] D. Brydges: *A short course on cluster expansions*, Les Houches summer school on "Critical phenomena, random systems, Gauge theories", Eds. K. Osterwalder & R. Stora, North Holland, Amsterdam, 1984

[C] E.R. Caianiello: Number of Feynman graphs and convergence, *Nuovo Cimento* **3** (1956), No. 1, 223–225.

[CS] J.C. Chaves, I.I. Satjia: Transport properties of one-dimensional interactinf fermions in aperiodic potentials, *Phys. Rev. B* **55** (1997), 14076.

[CF] L. Chierchia, C. Falcolini: A direct proof of a theorem by Kolmogorov in hamiltonian systems, *Ann. Scuola Norm. Super. Pisa Cl. Sci.* **21** (1994), No. 4, 541–593.

[CG] A.H. Castro-Neto, F. Guinea: Superconductivity, Josephson coupling and ordr parameter symmetry in striped cuprates, *Phys. Rev. Lett.* **80**,18 (1998), 4040–4043.

[D] A.M. Davie: The critical function for the semistandard map, *Nonlinearity* **7** (1994), No. 1, 219–229.

[DS] E.I. Dinaburg, Ya.G. Sinai: On the one dimensional Schroedinger equation with a quasiperiodic potential, *Funct. Anal. and its Appl.* **9** (1975), 279–289.

[DR1] M. Disertori, V. Rivasseau: Interacting Fermi liquid in two dimensions at finite temperature, I, Preprint (1999).

[DR2] M. Disertori, V. Rivasseau: Interacting Fermi liquid in two dimensions at finite temperature, II, Preprint (1999).

[DL] I.E. Dzyaloshinskii, A.I. Larkin: Correlation functions for a one-dimensional Fermi system with long-range interaction (Tomonaga model), *Soviet Phys. JETP* **38** (1974), 202–208.

[E1] L.H. Eliasson: Floquet solutions for the one dimensional quasi periodic Schroedinger equation, *Comm. Math. Phys.* **146** (1992), 447–482.

[E2] L.H. Eliasson: Absolutely convergent series expansions for quasi-periodic motions, Report 2–88, 1–31, Department of Mathematics, University of Stockholm (1988) and *Math. Phys. Electron. J.* **2** (1996), paper 4, 1–33.

[F] M.Fabrizio: Role of transverse hopping in a two-coupled chains model *Phis.Rev.B*, 48,21,15838–15859 (1993).

[FMRT1] J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz: A renormalizable field theory: the massive Gross-Neveu model in two dimensions. *Comm. Math. Phys.* **103** (1986), 67–103.

[FMRT2] J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz: An infinite volume expansion for many fermions Freen functions, *Helv. Phys. Acta* **65** (1992), 679–721.

[FST1] J. Feldman, M. Salmhofer, M. Trubowitz: Perturbation theory around nonnested Fermi surface. I. Keeping the Fermi surface fixed, *J. Statist. Phys.* **84** (1996), No. 5–6, 1209–1336.

[FST2] J. Feldman, M. Salmhofer, M. Trubowitz: Regularity of the moving Fermi surface: RPA contributions, *Comm. Pure Appl. Math.* **51** (1998), No. 9–10, 1133–1246.

[FST3] J. Feldman, M. Salmhofer, M. Trubowitz: Regularity of interacting nonspherical Fermi surfaces: the full self-energy, *Comm. Pure Appl. Math.* **52** (1999), No. 3, 273–324.

[FT] J. Feldman, M. Trubowitz: Renormalization on classical mechanics and many body quantum field theory, *J. Anal. Math.* **58** (1992), 213–247.

[FK] H. Frahm, V.E. Korepin: Critical exponents for the one dimensional Hubbard model *Phys. Rev. B* **42** (1990), No. 16, 10553–10565.

[FS] J. Frölich, T. Spencer: Absence of diffusion in the Anderson tight binding model for large disorder and low energy, *Comm. Math. Phys.* **88** (1983), 151–184.

[G0] G. Gallavotti: *The elements of Mechanics*, Springer, New York, 1983.

[G1] G. Gallavotti: Twistless KAM tori, *Comm. Math. Phys.* **164** (1994), 145–156.

[G2] G. Gallavotti: Renormalization group and ultraviolet stability for scalar fields via renormalization group methods *Rev. Mod. Phys.* **57** (1985), 471–562.

[GN] G. Gallavotti, F. Nicoló: Renormalization theory for four dimensional scaler fields, I, *Comm. Math. Phys* **100** (1985), 545–590; II, *Comm. Math. Phys.* **101** (1985), 471–562.

[Ga] F.R. Gantmacher: *The theory of matrices*, Chelsea Publishing, New York, 1960.

[GK1] K. Gawedzki, A. Kupiainen: Gross–Neveu model through convergent perturbative expansions, *Comm. Math. Phys.* **102** (1986), No.1, 1–30.

[GK2] K. Gawedzki, A. Kupiainen: *Asymptotic freedom beyond perturbation theory*, Les Houches Summer School on "Critical phenomena, random systems, Gauge theories" (Les Houches 1984), North Holland, Amsterdam-New York, 1986).

[GM1] G. Gentile, V. Mastropietro: Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications, *Rev. Math. Phys.* **8** (1996), 393–444.

[GM2] G. Gentile, V. Mastropietro: Anderson localization for the Holstein model, Preprint (1999).

[GS] G. Gentile, B. Scoppola: Renormalization group and the ultraviolet problem in the Luttinger model, *Comm. Math. Phys.* **154** (1993), 135–179.

[GiS] T. Giamarchi, H.J. Schultz: Correlation functions of one dimensional quantum systems, *Phys. Rev. B* **39** (1989), 4620–4629.

[Ha] F.D.M. Haldane: Luttinger liquid theory of the one dimensional quantum fluids, *J. Phys. C* **14** (1981), 2585–2609.

[HP] F. Harary, E. Palmer: *Graphical Enumerations*, Academic Press, New York, 1973.

[H] K. Hepp: Proof of the Bogolubov-Parasyuk theorem on renormalization, *Comm. Math. Phys.* **2** (1966), No. 4, 301–326.

[H] T. Holstein: Studies of polaron motion, part 1. The molecular-crystal model. *Ann. Phys.* **8** (1959), 325–342.

[JKM] J.D. Johnson, S. Krinsky, B.M.McCoy: Vertical-Arrow Correlation Length in the Eight-Vertex Model and the Low-Lying Excitations of the $X-Y-Z$ Hamiltonian. Phys. Rev. A **8**, 2526–2547 (1973).

[JM] R.A. Johnson, J. Moser: The rotation number for almost periodic potentials, *Comm. Math. Phys.* **84** (1982), 403–438.

[KL] T. Kennedy, E.H. Lieb: An itinerant electron model with crystalline or magnetic long range order, *Physica A* **138** (1986), 320–358.

[K] W. Kohn: Analytic properties of Bloch waves and Wannier functions, *Phys. Rev.* **114** (1959), No. 4, 809–821.

[Ko] A.N. Kolmogorov: On the preservation of conditionally periodic motions, *Dokl. Akad. Nauk.* **96** (1954), 527–530. English translation published in G. Casati, J. Ford: *Stochastic behavior in classical and quantum Hamiltonians*, Lecture Notes in Physics **93**, Springer, 1979.

[LM] J. L. Lebowitz, N. Macris: Low–temperature phases of itinerant fermions interacting with classical phonons: the static Holstein model, *J. Statist. Phys.* **76** (1994), No. 1–2, 91–123.

[JKM] J.D. Johnson, S. Krinsky, B.M.McCoy: Vertical-Arrow Correlation Length in the Eight-Vertex Model and the Low-Lying Excitations of the $X-Y-Z$ Hamiltonian. Phys. Rev. A **8**, 2526–2547 (1973).

[LP] A. Luther, I.Peschel: Calculation of critical exponents in two dimensions from quantum field theory in one dimension. *Phys. Rev. B* **12**,9, 3908–3917 (1975).

[L] P.A. Lee: Sliding charge density waves, *Nature* **291** (1981), No. 5810, 11–12.

[Le] A. Lesniewski: Effective action for the Yukawa$_2$ quantum field theory, *Comm. Math. Phys.* **108** (1987), 437–467.

[Li] E.H. Lieb: A model for crystallization: a variation of the Hubbard model, *Physica A* **140** (1986), 240–250.

[LM] E.H. Lieb, D. Mattis: *Mathematical Physics in one dimension: exactly soluble models of interacting particles*, Academic Press, New York, 1966.

[LSM] E.H. Lieb, T. Schultz, D. Mattis: Two soluble models of an antiferromagnetic chain, *Ann. Phys.* **16** (1961), 407–466.

[LW] E.H. Lieb, F.Y. Wu: Absence of Mott transition in an exact solution of the short-range, one-band model in one-dimension, *Phys. Rev. Lett.* **20** (1968), 1445–1448.

[LP] A. Luther, I. Peschel: Calculation of critical exponents in two dimension from quantum field theory in one dimension *Phys. Rev. B* **12** (1975), 3908–3917.

[LP] A. Luther, I. Peschel: Calculation of critical exponents in two dimension from quantum field theory in one dimension *Phys. Rev. B* **12** (1975), 3908–3917.

[Lu] J.M. Luttinger: An exactly soluble model of a many fermions system. *J. Math. Phys.* **4** (1963), 1154–1162.

[Mc] B.M. McCoy: Spin Correlation Functions of the $X-Y$ Model. *Phys. Rev.* **173**, 531–541 (1968).

[Ma] G.D. Mahan: Many-particle physics, Plenum Press, New York, 1990.

[M0] V. Mastropietro: Interacting soluble Fermi systems in one dimension, *Nuovo Cim. B* **1** (1994), 304–312.

[M1] V. Mastropietro: Small divisors and anomalous behaviour in the Holstein-Hubbard model, *Comm. Math. Phys.* **201** (1999), 81–115.

[M2] V. Mastropietro: A renormalization group computation of the XYZ correlation functions, *Lett. Math. Phys.* **47** (1999), No. 4, 339–352.

[M3] V. Mastropietro: Anomalous BCS equation for a Luttinger superconductor, *Mod. Phys. Lett* (1999) 13,n.17, 585-597

[M4] V. Mastropietro: Anomalous superconductivity for coupled Luttinger liquids, in print on *Rev. Math. Phys.*

[Ma1] D. Mattis: *The many-body problem: an enciclopedia of exactly soluble models*, World Scientific, Singapore, 1993.

[Ma2] D. Mattis: Band theory of magnetism in metals in the context of band theory of magnetism, *Physics* **1** (1964), 183–193.

[ML] D. Mattis, E. Lieb: Exact solution of a many fermion system and its associated boson field, *J. Math. Phys.* **6** (1965), 304-312.

[MW] B. McCoy, T. Wu: Dynamic mass generation and the Thirring model, *Phys. Lett. B* **87** (1979), No. 1–2, 50–51.

[MCD] W. Metzner, C. Castellani, C. di Castro: Fermi systems with strong forward scattering, *Adv. Phys.* **47** (1998), No. 3, 317–445.

[MD] W. Metzner, C. di Castro: Conservation laws and correlation functions in the Luttinger liquids, *Phys. Rev. B* **47** (1993), No. 24, 16107–16123.

[M] J. Moser: On invariant curves of area preserving mappings of the annulus, *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II* **1962** (1962), 1–20.

[MP] J. Moser, J. Pöschel: An extension of a result by Dinaburg and Sinai on quasi periodic potentials, *Comment. Math. Helv.* **59** (1984), 39–85.

[MPR] J. Magnen, G. Poirot, V. Rivasseau: Ward type identities for the $2d$ Anderson model with weak disorder, *J. Stat. Phys.* **93** (1998), 331.

[NO] J.W. Negele, H. Orland: *Quantum many-particle systems*, Addison-Wesley, New York, 1988.

[OS] M. Ogata, H. Shiba: Bethe-ansatz wave function, momentum distribution, and spin correlation in the one-dimensional strongly correlated Hubbard model, *Phys. Rev B* **41** (1990), no. 4, 2326–2338.

[PF] L. Pastur, A. Figotin: *Spectra of random and almost periodic operators*, Springer, Berlin, 1991.

[P] R.E. Peierls: *Quantum theory of solids*, Clarendon, Oxford, 1955.

[R] V. Rivasseau: *From perturbative to constructive renormalization*, Princeton University Press, Princeton, 1994.

[Rü] H. Rüssmann: Invariant tori in the perturbation theory of weakly non-degenerate integrable Hamiltonian systems, Preprint-Rehie Des Fachbereichs Mathematik der Johannes Gutenberg-Universität Mainz, Nr. 14, 1998.

[Sch] W.M. Schmidt: *Diophantine approximation*, Lecture Notes in Mathematics **785**, Springer, Berlin, 1980.

[SCP] K.J. Schultz, G. Cuniberti, P. Pieri: Fermi liquids and Luttinger liquids, Preprint (1998) (cond-mat-987366).

[S] S.B. Suterland: Two-Dimensional Hydrogen Bonded Crystals. *J. Math. Phys.* **11**, 3183–3186, (1970).

[Sh] R. Shankar: Renormalization group approach to interacting fermions. *Rev. Mod. Phys.* **66** (1994), No. 1, 129–192.

[So] J. Sólyom: The Fermi gas model of one dimensional conductors, *Adv. Phys.* **28** (1979), 201–303.

[Th] W. Thirring: A soluble relativistic field theory, *Ann. Phys.* **3** (1958), 91–112.

[T] E.C. Titchmarsh: *Eigenfunctions expansions associated with second order differential equations*, Clarendon, Oxford, 1955.

[To] S. Tomonaga: Remarks on Bloch's methods of sound waves applied to many fermion systems, *Progr. Theo. Phys.* **5** (1950), 544–569.

[VGM] J. Vidal, D. Mouhanna, T. Giamarchi: Correlated fermions in a one-dimensional quasi-periodic potential, Preprint (1999) (cond-mat-9905080).

[V] J. Voit: One-dimensional Fermi liquids, *Rep. Progr. Phys.* **58** (1995), No. 9, 977–1116.

[W] K.G. Wilson: The renormalization group and critical phenomena, Nobel Prize Lecture, *Rev. Mod. Phys.* **47** (1984), 773–840.

[WF] K.G. Wilson, M. Fisher: Critical exponents in 3.99 dimension, *Phys. Rev. Lett.* **28** (1976), 2240-243.

[YY] C.N. Yang, C.P. Yang: One dimensional chain of anisotropic spin-spin interactions. I. Bethe's hypothesis for ground state in a finite system, *Phys. Rev.* **150** (1966), No. 1, 321–327; II. Properties of the ground-state energy per lattice site for an infinite system. *Phys. Rev.* **150** (1966), No. 1, 327–339.

[Z] W. Zimmermann: Convergence of Bogolubov's method of renormalization in momentum space, *Comm. Math. Phys.* **15** (1969), No. 3, 208–234.