Construction of periodic solutions of nonlinear wave equations with Dirichlet boundary conditions by the Lindstedt series method

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Abstract. We introduce a renormalized Lindstedt series for the oscillatory solutions of nonlinear wave equations (like the nonlinear Klein-Gordon and sine-Gordon) with Dirichlet boundary conditions, and we prove its convergence via Renormalization Group methods.

Résumé. Nous introduisons une série de Lindstedt rénormalisée pour les solutions oscillatoires d’équations des ondes nonlineaires (comme les équations de Klein-Gordon et sine-Gordon nonlineaires) à conditions de frontière de Dirichlet, et nous démontrons sa convergence avec des méthodes de Group de Rénormalization.

Keywords. Nonlinear wave equation; Periodic solutions; Lindstedt series method; KAM theory; Perturbation theory; Counterterms; Renormalization Group; Diophantine conditions.

1. Introduction

We consider the nonlinear wave equation in \( d = 1 \) given by

\[
\begin{align*}
\left\{ \begin{array}{l}
u_{tt} - u_{xx} + \mu u &= f(u), \\
u(0, t) = u(\pi, t) &= 0.
\end{array} \right.
\end{align*}
\] (1.1)

where \( \mu \) is a positive real parameter (mass) and \( f(u) \) is an odd analytic function (perturbation):

\[
f(u) = \sum_{N=1}^{\infty} f_{2N+1} u^{2N+1}, \quad |f_s| \leq F e^{\xi s},
\] (1.2)

for some real constants \( F \) and \( \xi \). Let us call

\[
M = \inf \{ s \geq 1 : f_s \neq 0 \}.
\] (1.3)
Dirichlet boundary conditions allow us to use as a basis in $L^2([0, \pi])$ the set of functions \( \{e_m\} = \{\sin mx, m \in \mathbb{N}\} \). Equations of the form (1.1) are the nonlinear Klein-Gordon equation
\[
 u_{tt} - u_{xx} + \mu u - \frac{\mu}{3} u^3 = 0, \tag{1.4}
\]
and the sine-Gordon equation
\[
 u_{tt} - u_{xx} + \mu \sin u = 0. \tag{1.5}
\]
If \( f = 0 \) every solution of (1.1) can be written as
\[
 u(x,t) = \sum_{m=0}^{\infty} A_m \cos(\omega_m t + \theta_m) \sin mx, \tag{1.6}
\]
where \( \omega_m = \sqrt{m^2 + \mu} \) and \( \theta_m \) is an arbitrary phase. In particular
\[
 \varepsilon \cos \omega_1 t \sin x \tag{1.7}
\]
is a solution for all values of the real parameter \( \varepsilon \).

If a nonlinear term (perturbation) is added to the wave equation \( f \neq 0 \) one can ask if periodic solutions close to (1.7) exist. This is not a trivial problem because, as we shall see later, small divisors appear of the form \(-\omega_1^2 n^2 + \omega_m^2\), with \( m \geq 2 \), and they can be arbitrarily small when \( n \) and \( m \) are large. We look for which conditions the periodic solution (1.7) can be continued for \( f \neq 0 \) into
\[
 u(x,t; \varepsilon) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{i\Omega t + imx} \hat{u}_{n,m}(\varepsilon), \tag{1.8}
\]
possibly with a different frequency \( \Omega \), with \( \Omega = \omega_1 + O(\varepsilon) \). Of course by the symmetry of (1.1) if such a solution exists it verifies
\[
 \hat{u}_{n,m} = -\hat{u}_{n,-m} = \hat{u}_{-n,m} \tag{1.9}
\]
for all \( n, m \in \mathbb{Z} \) (here and henceforth we shorten \( \hat{u}_{n,m}(\varepsilon) \) with \( \hat{u}_{n,m} \)).

Remark 1. For finite dimensional Hamiltonian systems existence of periodic orbits follows from the Lyapunov center theorem [18], which states that, for a Hamiltonian system with an elliptic equilibrium at 0, by denoting with \( \omega_1, \ldots, \omega_n \) the frequencies of the linear oscillations close to 0, if a nonresonance condition holds, namely if \( \omega_1 \) is such that \( \omega_1 \ell - \omega_j \neq 0 \) for all \( \ell \in \mathbb{N} \) and for all \( j \neq 1 \), then the periodic solution of the linearized system with frequency \( \omega_1 \) and small amplitude can be continued into periodic solutions with frequency close to \( \omega_1 \). Existence of periodic orbits in the case in which the frequencies are possibly resonant was proved by the Weinstein theorem [21]. The infinite dimensional case poses difficulties which are qualitatively different with respect to the finite dimensional case; the denominators can be arbitrarily small.
and the problem appears to be related to KAM problems, like the existence of lower dimensional tori in a Hamiltonian system.

We shall prove the following result.

**Theorem 1.** There exists a set $\mathcal{M} \subset \mathbb{R}$ of full Lebesgue measure and for all $\mu \in \mathcal{M}$ there exist $\varepsilon_0 > 0$ and, for all $\varepsilon_* \in (0, \varepsilon_0)$, a set $\mathcal{E} \subset [-\varepsilon_*, \varepsilon_*]$ with complement of relative Lebesgue measure tending to 0 as $\varepsilon_* \to 0$ such that for all $\varepsilon \in \mathcal{E}$ there exists a value $\Omega(\varepsilon)$ such that $u(x, t; \varepsilon)$ in (1.8), with $\Omega = \Omega(\varepsilon)$, is a solution of (1.1), analytic in $(x, t)$ and $2\pi/\Omega(\varepsilon)$-periodic in $t$, with

$$|\Omega(\varepsilon) - \omega_1| \leq C|\varepsilon|^{M-1},$$

$$|u(x, t; \varepsilon) - \varepsilon \cos \Omega(\varepsilon)t \sin x| \leq C|\varepsilon|^M,$$

for a suitable constant $C$.

**Remark 2.** Theorem 1 can be immediately extended to equations of the form

$$u_{tt} + Du = f(u),$$

(1.11)

where $D$ is a self-adjoint strictly positive operator with non-degenerate eigenvalues $\mu_k$ such that $\mu_k \geq ck^\gamma$ with $c, \gamma \geq 1$ positive constants and such that, if $\gamma > 1$,

$$\mu_{k'} - \mu_k \geq c(k'^\gamma - k^\gamma)$$

for all $k' > k \geq 1$, while, if $\gamma = 1$, $\mu_{k'} - \mu_k = c_\ell(1 + O(k^{-\xi}))$ for all $k' - k = \ell \geq 1$ and with $c_\ell, \xi > 0$. One can also study the problem of persistence of an unperturbed periodic solution with any other frequency $\omega_m$, $m \neq 1$, and a result analogous to theorem 1 can be easily proved.

**Remark 3.** Physically (1.1) describes a string subjected to an external force $g(u)$, depending only on the displacement $u$, odd and analytic in $u$, and such that one has $g(0) = 0$ (so that there is no acting force when the string is at rest). Theorem 1 essentially states that, for a full Lebesgue measure set of constants $\mu$, small amplitude periodic solutions of the linear equation survive in presence of nonlinear perturbations for a relatively large set of values for the amplitude.

Inserting (1.8) into (1.1) gives

$$\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{i\Omega n t + imx} \hat{u}_{n,m} [-\Omega^2 n^2 + \omega_m^2] = f(u),$$

(1.12)

which, if we expand

$$f(u) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{i\Omega n t + imx} \hat{f}_{n,m}(u),$$

(1.13)

can be written in Fourier space as

$$\hat{u}_{n,m} [-\Omega^2 n^2 + \omega_m^2] = \hat{f}_{n,m}(u).$$

(1.14)

In order to solve such equations it is convenient to write, for any $|m| \geq 1$,

$$\omega_m^2 = (\omega_m^2 - \nu_m) + \nu_m \equiv \tilde{\omega}_m^2 + \nu_m, \quad \nu_m = \nu_{-m},$$

(1.15)
so that (1.14) can be written as
\[ \hat{u}_{n,m}[-\Omega^2 n^2 + \tilde{\omega}_m^2] = \nu_m\hat{u}_{n,m} + \hat{f}_{n,m}(u). \] (1.16)

We shall call \textit{counterterms} the quantities \( \nu_m \). We write the decomposition (1.15) by explicitly requesting the parity property \( \nu_m = \nu_{-m} \), and we will check at the end that indeed it is possible to find functions \( \nu_m \) with such parity property.

We shall look for periodic solutions with period \( 2\pi/\Omega \), where \( \Omega \equiv \tilde{\omega}_1 \). The analysis will be performed in two steps. First we consider \( \varepsilon \) and \( \tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1} \) as independent parameters and we assume the following Diophantine conditions:
\[ |\tilde{\omega}_1 n \pm \tilde{\omega}_m| \geq C_0|n|^{-\tau} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0, \pm 1\}, \]
\[ |\tilde{\omega}_1 n \pm (\tilde{\omega}_m \pm \tilde{\omega}_{m'})| \geq C_0|n|^{-\tau} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{Z} \setminus \{0, \pm 1\}, \] (1.17)
where \( C_0 \) and \( \tau \) are two positive constants. As usual in literature, we shall call the two conditions in (1.17), respectively, as the first and the second \textit{Mel’nikov conditions}; the frequencies \( \tilde{\omega}_m, |m| \geq 2 \), are called the \textit{normal frequencies}.

We shall prove that there is a choice of the sequence \( \nu(\tilde{\omega}, \varepsilon) = \{\nu_m(\tilde{\omega}, \varepsilon)\}_{|m|\geq 1} \), analytically depending on \( \varepsilon \) (for \( \varepsilon \) small enough), such that there exist \( \hat{u}_{n,m} \equiv \hat{u}_{n,m}(\tilde{\omega}, \varepsilon) \) solving (1.16) which are analytic in \( \varepsilon \) and are the Fourier coefficients of a function \( u(x, t; \tilde{\omega}, \varepsilon) \) verifying the bound (1.10).

The second step consists in proving that there are functions \( \tilde{\omega}_m = \tilde{\omega}_m(\omega, \varepsilon) \), with \( \omega = \{\omega_m\}_{|m|\geq 1} \), solving \( \tilde{\omega}_m^2 + \nu_m(\tilde{\omega}, \varepsilon) = \omega_m^2 \); they are obtained by requesting that there is a set \( \mathcal{M} \) such that for all \( \mu \in \mathcal{M} \) there exists a set \( \mathcal{E} \) such that the functions \( \tilde{\omega}_m(\omega, \varepsilon) \) are well defined and verify (1.17) for \( \varepsilon \in \mathcal{E} \). Then we have to check that both \( \mathcal{M} \) and \( \mathcal{E} \) have nonvanishing measure. Hence \( \hat{u}_{n,m}(\tilde{\omega}(\omega, \varepsilon), \varepsilon) \) is indeed the Fourier transform of a periodic solution of (1.1).

The result above is not new. It was proved (also in the case of periodic boundary conditions) by Craig and Wayne [6], with an extension of the Lyapunov-Schmidt method. Later on Pöschel [19] gave an independent proof, under the only request \( \mu > 0 \), based on the work by Kuksin and himself on the nonlinear Schrödinger equation [17] (see also [16]), by writing the wave equation as an infinite-dimensional Hamiltonian system and extending properly KAM techniques. Finally Bourgain proved existence of periodic solutions for nonlinear wave equations (including (1.1)) in any dimensions, both for Dirichlet and periodic boundary conditions [3]; this requires removing the second Mel’nikov conditions in order to deal with the multiplicities of the normal frequencies.

Quasi-periodic solutions were obtained in [16], [20] and [2] by adding to the wave equation (1.1) a linear term \( V(x, u)u \), containing an external potential depending on parameters, for a relatively large set of the parameter values. The case of \textit{constant non-vanishing} external potential, which was excluded in the previous works, was again obtained in [17] and [19], by performing a normal form transformation which reduces the problem to a perturbation of a linear equation with parameters (in fact the periodic case mentioned above is just a particular case of what is proved in
such works). Finally the work by Bourgain (see [5] and references quoted therein) provides the more complete and general results existing on quasi-periodic solutions for nonlinear PDE equations in one-dimensional case and the only existing ones with small divisors problems in the two-dimensional case.

Our approach is based on expanding the periodic solution as a power series of $\varepsilon$; such power series are the analogues of the Lindstedt series for KAM invariant tori in finite-dimensional Hamiltonian systems. Contrary to what happens in such a case (at least for maximal invariant tori, see [9]), such series are not analytic in the perturbative parameter. However, while $\hat{u}_{n,m}(\tilde{\omega}(\omega, \varepsilon), \varepsilon)$ is not analytic in $\varepsilon$, it turns out that the function $\hat{u}_{n,m}(\tilde{\omega}, \varepsilon)$ (that is the function obtained by keeping fixed the parameters $\tilde{\omega}$, without expliciting the dependence of $\tilde{\omega}$ on $\varepsilon$) is analytic in $\varepsilon$, provided that $\tilde{\omega}$ satisfies the conditions (1.17) and $\varepsilon$ is small enough. By inverting (1.15) one writes $\tilde{\omega}$ as a function of $\omega$ and $\varepsilon \in \mathcal{E}$. The main advantage of such an approach is that it is rather constructive, in the sense that we obtain an explicit representation of the periodic solution on the nonlinear wave equation. As a byproduct, it turns out the analyticity of $\hat{u}_{n,m}(\tilde{\omega}, \varepsilon)$ as a function of $\varepsilon$, at fixed $\tilde{\omega}$; more exactly we get the following result.

**Theorem 2.** Let us consider the equation

$$\hat{u}_{n,m} [-\tilde{\omega}_1^2 n^2 + \tilde{\omega}_m^2] = \hat{f}_{n,m}(u),$$

(1.18)

and assume that $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m| \geq 1}$ verifies the Diophantine conditions

$$|\tilde{\omega}_n \pm \tilde{\omega}_m| \geq C_0|n|^{-\tau} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0, \pm 1\},$$

$$|\tilde{\omega}_n \pm (\tilde{\omega}_{m'} \pm \tilde{\omega}_m)| \geq C_0|n|^{-\tau} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{Z} \setminus \{0, \pm 1\},$$

(1.19)

where $C_0$ and $\tau$ are two positive constants. There exists $\varepsilon_0 > 0$ and for $|\varepsilon| < \varepsilon_0$ a function $\nu = \nu(\tilde{\omega}, \varepsilon)$, analytic in $\varepsilon$, such that $u(x, t; \tilde{\omega}, \varepsilon)$ is a solution of (1.16) with $\nu_m = \nu_m(\tilde{\omega}, \varepsilon)$, analytic in $(x, t, \varepsilon)$ and $2\pi/\tilde{\omega}_1$-periodic in $t$.

As far as we know, the analiticity in $\varepsilon$ at fixed $\tilde{\omega}$ was not pointed out elsewhere; in a paper by Bourgain [4] something similar is obtained.

Our analysis of the Lindstedt series for $\hat{u}_{n,m}(\tilde{\omega}, \varepsilon)$ is based on Renormalization Group methods similar to the ones applied to the problem of the convergence of the series of classical KAM tori (see [7], [8] and [11]). In the latter problem one has to exploit a number of partial cancellations in the perturbative expansion which at the end ensure the analiticity of the series, while in the case of Lindstedt series for PDE such cancellations are absent. On the contrary here one has to perform a suitable resummation of the formal perturbative expansion, which, at the end, implies non-analyticity of the solution; for similar results along the same direction see also [9] and [10]. Our method is based on techniques which were developed for the proof of the convergence of perturbative series for quantities of interest in quantum field theory or statistical mechanics.

Finally we remark that we are excluding the interesting case $\mu = 0$, namely the nonlinear vibrating string equation, but we believe that our method could be used to...
prove a similar statement also for the μ = 0 case, for which no results are present in the literature at the moment.

2. Recursive relations

By (1.9), we can rewrite (1.16) as

\[
\hat{u}_{n,m} \left[ -\omega_1^2 n^2 + \omega_m^2 \right] = \nu_m^{(a)} \hat{u}_{n,m} + \nu_m^{(b)} \hat{u}_{n,-m} + \hat{f}_{n,m}(u),
\]

(2.1)

with \( \nu_m^{(a)} - \nu_m^{(b)} = \nu_m \).

We shall look for \( \hat{u}_{n,m} \) in the form of a power series expansion in \( \varepsilon \) and in the parameters \( \nu_m^{(c)} \), with \( c = a, b \) and \( |m| \geq 1 \),

\[
\hat{u}_{n,m} = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\mathbf{k}=(k_1^{(a)}, k_1^{(b)}, k_2^{(a)}, \ldots)} \hat{u}_{n,m} \prod_{m'=2}^{\infty} \prod_{c=a,b} (\nu_m^{(c)})^{k_{m'}^{(c)}},
\]

(2.2)

with \( k \geq 1 \) and \( k_{m'}^{(c)} \geq 0 \) for all \( m' \geq 2 \) and \( c = a, b \). We set also \( |\mathbf{k}| = k_1^{(a)} + k_1^{(b)} + k_2^{(a)} + k_2^{(b)} + \ldots \); of course we are using the symmetry property in (1.15) to restrict the dependence only on the positive labels \( m' \). We shall refer to (2.3) as the Lindstedt series of the periodic solution, for the manifest analogy with the case of finite-dimensional Hamiltonian systems.

Remark 4. We shall see that for fixed \( k \) the vector \( \mathbf{k} \) can have only a finite number \( (\leq k) \) of components different from zero, which gives sense to the sum appearing in (2.2).

For \( |m| > 1 \) one has \( \hat{u}_{n,m}^{(1,0)} = 0 \), while for \( m = \pm 1 \) one has

\[
\hat{u}_{n,m}^{(1,0)} = \delta_{n,\pm1} \frac{m}{4\omega_m^2},
\]

(2.3)

this simply follows from the fact that we are looking for a solution which, to first order in \( \varepsilon \), reduces to (1.7).

Moreover, for \( m = \pm 1 \) and \( n = \pm 1 \), we shall choose

\[
\hat{u}_{n,m}^{(k,\mathbf{k})} = 0 \quad \forall (k, \mathbf{k}) \neq (1, \mathbf{0}).
\]

(2.4)

By inserting (2.2) into (1.16) we obtain for \( (k, \mathbf{k}) \neq (1, \mathbf{0}) \) the recursive equations

\[
\hat{u}_{n,m}^{(k,\mathbf{k})} = \frac{1}{-\omega_1^2 n^2 + \omega_m^2} \left[ \left( 1 - \delta_{k_m^{(a)},0} \right) \hat{u}_{n,m}^{(k,k_m^{(a)},\mathbf{k})} + \left( 1 - \delta_{k_m^{(b)},0} \right) \hat{u}_{n,-m}^{(k,k_m^{(b)},\mathbf{k})} \right]
\]

\[
+ \sum_{N=1}^{\infty} \sum_{k_1+k_2+\ldots+k_{2N+1}=k} \sum_{\ell_1+\ell_2+\ldots+\ell_{2N+1}=m} \prod_{j=1}^{2N} \hat{u}_{n_{j},m_{j}}^{(k_j,\mathbf{k})},
\]

(2.5)

which hold both for \( |m| > 1 \) and for \( |m| = 1 \) with \( |n| \neq 1 \). Here \( \delta_{m'}^{(c)} \) is the vector \( \mathbf{k} \) with components \( k_{m'}^{(c)} = 0 \) for all \( (m', c') \neq (m, c) \) and with \( k_m^{(c)} = 1 \).
In the case $m = 1$ and $|n| = 1$ we obtain from (1.16), (1.12) and (2.3)

$$0 = (-\tilde{\omega}^2 + \tilde{\omega}^2) \frac{1}{4i} =$$

$$\nu_1 = \sum_{N=1}^{\infty} f_{2N+1} \sum_{n_1+n_2+\ldots+n_{2N+1} = n} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2N+1},m_{2N+1}},$$

which will be regarded as an equation for fixing $\nu_1$, as a suitable function of $\epsilon$:

$$\nu_1 = -4i \sum_{N=1}^{\infty} f_{2N+1} \sum_{n_1+n_2+\ldots+n_{2N+1} = n} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2N+1},m_{2N+1}}.$$  \hfill (2.6)

It is important to realize that, thanks to the parity property (1.9), even if in principle (2.7) represent two equations (one for $n = 1$ and one for $n = -1$), so that compatibility problems could arise, in fact the right hand side of (2.7) depends only on the modulus of $n$: this means that the value of $\nu_1$ computed if $n = -1$ is the same as the one computed if $n = 1$, hence (2.7) represents only one equation and it fixes $\nu_1$.

**Remark 5.** If one computes $\nu_{-1}$ (corresponding to $m = -1$), one finds $\nu_{-1} = \nu_1$ (simply because for $m = -1$ both terms in the right hand side of (2.6) change sign), so that the parity property $\nu_m = \nu_{-m}$ is immediately seen to be satisfied for $m = 1$.

**Remark 6.** Note also that the quantity $\tilde{\omega}_1 n \pm \tilde{\omega}_m$ has modulus bounded by a constant for all $(n, m) \neq (\pm 1, \pm 1)$ such that either $|n| = 1$ or $|m| = 1$.

### 3. Tree representation of the Lindstedt series

Note that (2.5) expresses $\hat{u}_{n,m}^{(k,k)}$ in terms of coefficients $\hat{u}_{n',m'}^{(k',k')}$ with $k' + |k'| < k + |k|$, so that, by iterating (2.5), at the end one can express $\hat{u}_{n,m}^{(k,k)}$, for all $(k,k) \neq (1,0)$, in terms of the only coefficients $\hat{u}_{n,0}^{(1,0)} \pm 1, \pm 1$, as given by (2.3).

It is immediate to verify that, by such an iteration procedure, $\hat{u}_{n,m}^{(k,k)}$ can be written for all $(k,k) \neq (0,0)$ as sum over (rooted) trees in the following way

$$\hat{u}_{n,m}^{(k,k)} = \sum_{\theta \in \Theta_{n,m}^{(k,k)}} \text{Val}(\theta),$$ \hfill (3.1)

where, if, given a tree $\theta$, we denote with $L(\theta)$, $V(\theta)$ and $E(\theta)$ the set of lines, nodes and end-points, respectively, of $\theta$ one has

$$\text{Val}(\theta) = \left( \prod_{\ell \in L(\theta)} g_{\ell} \right) \left( \prod_{v \in V(\theta)} \eta_v \right) \left( \prod_{v \in E(\theta)} \frac{\varepsilon_m v}{4i} \right).$$ \hfill (3.2)
and \( \Theta_{n,m}^{(k,l)} \) is the set of all the possible trees defined according to the following rules (see Figure 3.1).

1. To each end-point \( v \in E(\theta) \) one associates the mode label \((n_v, m_v)\), with \( m_v = \pm 1 \) and \( n_v = \pm 1 \), such that

\[
\sum_{v \in E(\theta)} n_v = n \quad \text{and} \quad \sum_{v \in E(\theta)} m_v = m,
\]

and the factor \( \varepsilon m_v / 4i \).

2. To each line \( \ell \in L(\theta) \) one associates the propagator

\[
ge_{\ell} \equiv g(\vec{\omega}_1 n_{\ell}, m_{\ell}) = \begin{cases} 
\frac{1}{-\omega_1^2 n_{\ell}^2 + \omega_m^2}, & (n_{\ell}, m_{\ell}) \neq (\pm 1, \pm 1), \\
1, & (n_{\ell}, m_{\ell}) = (\pm 1, \pm 1),
\end{cases}
\]

with momentum \((n_{\ell}, m_{\ell})\), where one has

\[
n_{\ell} = \sum_{w \in E_{\ell}} n_w \quad \text{and} \quad m_{\ell} = \sum_{w \in E_{\ell}} m_w,
\]

if we denote by \( E_{\ell} \) the set of end-points of the subtree with root line \( \ell \). Only the lines coming out from the end-points can have momentum \((n_{\ell}, m_{\ell}) = (\pm 1, \pm 1)\).

3. For each node \( v \in V(\theta) \) one has \( s_v \) entering lines. If \( s_v = 1 \) and the momentum of the exiting line \( \ell \) is \((n_{\ell}, m_{\ell})\), the momentum of the entering line can be only \((n_{\ell}, \pm m_{\ell})\).

One associates to such a node \( v \) a factor \( \eta_v = \nu_{(c,m)}^{(c)} \), where \( c = a \) if the momentum of the entering line is \((n_{\ell}, m_{\ell})\) and \( c = b \) if the momentum of the entering line is \((n_{\ell}, -m_{\ell})\); we call \( \nu \)-vertex a node of this kind. If \( s_v > 1 \) one associates to the node a factor \( \eta_v = f_{s_v} \); in particular \( s_v \) is odd and if \( s_v > 1 \) then \( s_v \geq M \).

4. One has the constraints

\[
|E(\theta)| = \sum_{v \in E(\theta)} |n_v| = \sum_{v \in E(\theta)} |m_v| = k,
\]

\[
\sum_{m=1}^{\infty} \sum_{c=a,b} k_{(c)}^{(c)} = |k|,
\]

and the number of vertices \( v \) with \( s_v = 1 \) such that a factor \( \nu_{(c,m)}^{(c)} \) is associated to them is \( k_{(c)}^{(c)} \).

Remark 7. It was showed in [1] that by assuming the first Diophantine condition in (1.17) with \( \tau = 1 \) then the proof of the existence of periodic solution becomes quite simpler. This can be understood in the formalism described here by noting that if \( \tau = 1 \) each propagator \( g_{\ell} \) is bounded by a constant and hence each tree is bounded by \( C^k \); for an explicit discussion see [12]. Of course assuming \( \tau = 1 \) one can prove the
Figure 3.1. A tree $\theta$ belonging to the set $\Theta_{\omega, M}^{k, k}$ for the model (1.1) with $M = 3$. The end-points are represented by black bullets, while the $\nu$-vertices are represented as grey bullets.

existence of periodic solutions only for a zero measure set of values of $\varepsilon$. In the case $\tau > 1$ it is not true that the propagators are bounded by a constant; there is instead a small divisor problem and a multiscale analysis is necessary: to such a case the rest of the paper is devoted.

4. Multiscale decomposition: clusters and self-energy graphs

We assume the Diophantine conditions (1.17). We introduce a multiscale decomposition of the propagator. Let $\chi(x)$ be a $C^\infty$ non-increasing function such that $\chi(x) = 0$ if $|x| \geq C_0$ and $\chi(x) = 1$ if $|x| \leq 2C_0$ ($C_0$ is the same constant appearing in (1.17)), and let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1} x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$; such functions realize a smooth partition of the unity as

$$1 = \chi_{-1}(x) + \sum_{h=0}^{\infty} \chi_h(x) = \sum_{h=-1}^{\infty} \chi_h(x).$$

Note that if $\chi_h(x) \neq 0$ for $h \geq 0$ one has $2^{-h-1}C_0 \leq |x| \leq 2^{-h+1}C_0$, while if $\chi_{-1}(x) \neq 0$ one has $|x| \geq C_0$.

We write the propagator as sum of propagators on single scales in the following way:

$$g(\tilde{\omega}_1 n, m) = \sum_{h=-1}^{\infty} \chi_h(\tilde{\omega}_1 n - \tilde{\omega}_m) = \sum_{h=-1}^{\infty} g^{(h)}(\tilde{\omega}_1 n, m).$$

Note that we can bound $|g^{(h)}(\tilde{\omega}_1 n, m)| \leq 2^{-h+1}C_0$. 

9
This means that we can attach a scale label \( h \geq -1 \) to each line \( \ell \in L(\theta) \), which is the scale of the propagator which is associated to \( \ell \). We can denote with \( \Theta_{n,m}^{(k,k)} \) the set of trees which differ from the previous ones simply because the lines carry also the scale labels. The set \( \Theta_{n,m}^{(k,k)} \) is defined according to the rules (1) ÷ (4) of Section 3, by changing item (2) into

\( h_{\ell} = \begin{cases} 0, 1, 2, 3, \ldots, & \text{if } (n_\ell, m_\ell) \neq (\pm 1, \pm 1), \\ -1, & \text{if } (n_\ell, m_\ell) = (\pm 1, \pm 1), \end{cases} \)  

(4.3)

and the propagator

\[
\begin{align*}
g_{\ell}^{(h_{\ell})} &\equiv g_{n_\ell, m_\ell}^{(n_\ell)}(\bar{\omega}_{1, n_\ell}, m_\ell) = \\
&= \begin{cases} \chi_{h_{\ell}}(|\bar{\omega}_1 n_\ell| - \bar{\omega}_{m_\ell}) &\text{if } (n_\ell, m_\ell) \neq (\pm 1, \pm 1), \\ 1, &\text{if } (n_\ell, m_\ell) = (\pm 1, \pm 1), \end{cases}
\end{align*}
\]

(4.4)

with momentum \((n_\ell, m_\ell)\).

Looking at the scale labels we identify the connected cluster \( T \) of nodes which are linked by a continuous path of lines with the same scale label \( h_T \) or a lower one and which are maximal; we shall say that the cluster has scale \( h_T \). We shall denote with \( V(T) \) and \( E(T) \) the set of and the set of end-points, respectively, which are contained inside the cluster \( T \), and with \( L(T) \) the set of lines connecting them.

Therefore an inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. The value of a tree can be written then as

\[
\text{Val}(\theta) = \left( \prod_{\ell \in L(\theta)} g_{\ell}^{(h_{\ell})} \right) \left( \prod_{v \in V(\theta)} \eta_v \right) \left( \prod_{v \in E(\theta)} \frac{\varepsilon m_v}{4i} \right),
\]

(4.5)

so that (3.1) is replaced with

\[
\hat{u}_{n,m}^{(k,k)} = \sum_{\theta \in \Theta_{n,m}^{(k,k)}} \text{Val}(\theta),
\]

(4.6)

with the new definition

\[
\text{Val}(\theta) = \left( \prod_{\ell \in L(\theta)} g_{\ell}^{(h_{\ell})} \right) \left( \prod_{v \in V(\theta)} \eta_v \right) \left( \prod_{v \in E(\theta)} \frac{\varepsilon m_v}{4i} \right).
\]

(4.7)

for the tree value \( \text{Val}(\theta) \).

Each cluster \( T \) has an arbitrary number of lines entering it (incoming lines), but only one or zero line coming from it (outcoming line); we shall denote the latter (when it exists) with \( \ell_T^1 \). We shall call external lines of the cluster \( T \) the lines which either enter or come out from \( T \), and we shall denote by \( h_T^{(\ell)} \) the maximum among the scales
of the external lines of $T$. Define $K(T)$ as the number of end-points contained inside $T$, so that $K(T) = |E(T)|$.

If a cluster has only one entering line $\ell^3_T$ and $(n, m)$ is the momentum of such a line, for any line $\ell \in L(T)$ one can write $(n_\ell, m_\ell) = (n^0_\ell, m^0_\ell) + \eta_\ell(n, m)$, where $\eta_\ell = 1$ if the line $\ell$ is along the path connecting the external lines of $T$ and $\eta_\ell = 0$ otherwise.

The clusters with only one incoming line $\ell^2_T$ such that one has $n_{\ell^1_T} = n_{\ell^2_T}$ and $m_{\ell^1_T} = \pm m_{\ell^2_T}$ will be called self-energy graphs or resonances (the first name is usual in quantum field theory, the second one was introduced by Eliasson in his basic paper [7]). In such a case we shall call a resonant line the line $\ell^1_T$ we shall refer to its momentum as the momentum of the self-energy graph.

The value of the self-energy graph $T$ is defined as

$$V_{h^1 n, m} = \left( \prod_{\ell \in T} g^{(n_\ell)}_{\ell} \right) \left( \prod_{v \in V(T)} \eta_v \right) \left( \prod_{v \in E(T)} \frac{\varepsilon m_v}{4i} \right),$$

where $h = h^{(e)}_T$ is the maximum between the scales of the two external lines of $T$ (they can differ at most by a unit), and one has

$$n(T) \equiv \sum_{v \in E(T)} n_v = 0, \quad m(T) \equiv \sum_{v \in E(T)} m_v \in \{0, 2m\},$$

by definition of self-energy graph; one has $c = a$ when $m(T) = 0$ and $c = b$ when $m(T) = 2m$.

Remark 8. Note that, if $|m| = 1$ and $|n| \neq 1$ the scale of the external lines is such that $h \leq h_0$, where $h_0$ is a suitable constant (see remark 6): one can take $h_0 = 0$ for $C_0$ small enough. In particular this implies that no self-energy graph with $m = 1$ is possible; we shall repeatedly use such a property in the following.

Given a tree $\theta$, we shall denote by $N_h(\theta)$ the number of lines with scale $h$, and by $C_h(\theta)$ the number of clusters with scale $h$. We will get immediately the following bound, for $\theta \in \Theta^{(k,k)}_{n,m}$ and assuming $|\nu_m^{(c)}| \leq C|\varepsilon|$, we will get the following bound

$$|\text{Val}(\theta)| \leq |\varepsilon|^k |\kappa|^{k} D^{k+\kappa} \prod_{h=0}^{\infty} 2^{h N_h(\theta)},$$

if $D$ is a suitable constant. Note that one has $|n_\ell|, |m_\ell| \leq k$ for all trees $\theta \in \Theta^{(k,k)}_{n,m}$ and for all lines $\ell \in L(\theta)$. The following result is proved in Appendix A1.

Lemma 1. For any tree $\theta \in \Theta^{(k,k)}_{n,m}$ and for all $h \geq 0$ one has

$$N_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M_h^\varepsilon(\theta),$$
where \( K(\theta) = \sum_{v \in E(\theta)} |n_v| \), while \( S_h(\theta) \) is the number of self-energy graphs \( T \) in \( \theta \) with \( h_T^{(c)} = h \) and \( M^c_\nu(\theta) \) is the number of \( \nu \)-vertices in \( \theta \) such that the maximum scale of the two external lines is \( h \).

Let us consider a tree with no self-energy graphs and with no \( \nu \)-vertices. Then by (4.10) and (4.12) we get

\[
|\text{Val}(\theta)| \leq |\varepsilon|^k D^k \prod_{h=0}^{\infty} 2^{4k2^{-(h-2)/\tau}} \leq \tilde{D}^k |\varepsilon|^k,
\]

for a suitable constant \( \tilde{D} \), and we have used that one has \( k = K(\theta) \); see (3.6). On the other hand the bound for a generic tree with self-energy graphs is quite bad; this is not just a technical problem, as one can easily identify trees whose value is essentially given by \( C_k |\varepsilon|^k \), for some positive constants \( C \) and \( \alpha \). We have then to define a different expansion, as it will be shown in the following section.

5. The renormalized expansion

We have seen that the expansion envisaged in the previous sections can be written as sum over trees and some of such trees (the trees containing self-energy graphs) cannot be bounded by \( C_k |\varepsilon|^k \), as they are of order of \( C_k |\varepsilon|^k k! \). In this section we will set up a different expansion, which, by choosing in a proper way the parameters \( \nu_m, m > 1 \), can be written as sum over trees in which all of them can be bounded by \( C_k |\varepsilon|^k \).

We introduce a localization operator acting on the self-energy graphs in the following way, for \( h \geq h_0 \) and \( m > 1 \),

\[
\mathcal{L}V^c_T(\omega n, m) = V^c_T(\text{sgn}(n) \tilde{\omega}_m, m),
\]

and we define the regularization operator \( R = 1 - \mathcal{L} \) as

\[
RV^c_T(\omega n, m) = V^c_T(\omega n, m) - V^c_T(\text{sgn}(n) \tilde{\omega}_m, m).
\]

We shall define, for \( m > 1 \) and \( h \geq h_0 \), the running coupling constants

\[
2^{-h} \nu_{h,m}^{(c)} = \nu_{m}^{(c)} + \frac{1}{2} \sum_{\sigma = \pm 1} \sum_{T \in T_{<c}^{(m)}} V^c_T(\sigma \tilde{\omega}_m, m),
\]

where \( c = a, b \), and \( T_{<c}^{(m)} \) denotes the set of self-energy graphs \( T \) with \( m_{T_1}^{(a)} = m_{T_2}^{(a)} \) and \( h_T < h \), while \( T_{<h}^{(b)} \) denotes the set of self-energy graphs \( T \) with \( m_{T_1}^{(b)} = -m_{T_2}^{(b)} \), and \( h_T < h \). We are using that, by the parity properties (1.9) and (1.15), the self-energy graph values in (5.3) do not depend on \( \tau \): we have introduced the “average” on \( \sigma \) just to stress such an independence and obtain a formula which manifestly depends only on \( m \). We shall set also \( \nu_{m}^{(c)} = \nu_{1,m}^{(c)} \). Recall that we are assuming that we can take \( h_0 = 0 \) (see remark 6); so in the following we shall set \( h_0 = 0 \).
Then one obtains

\[ \hat{u}_{n,m} = \sum_{k=1}^{\infty} \sum_{k} \sum_{\theta \in \Theta_{n,m}^{(k,k)^{\mathcal{R}}}} \text{Val}(\theta), \]  

(5.4)

where the set \( \Theta_{n,m}^{(k,k)^{\mathcal{R}}} \) is the set of renormalized trees, which are defined as in the previous section (see rules (1), (2'), (3) and (4)) except that the following rules are added:

(5) to each self-energy graph (with \(|m| > 1\)) the \( \mathcal{R} \) operation is applied;

(6) there are nodes \( v \) with \( s_v = 1 \) such that, if the momenta of the external lines are, respectively, \((n, m)\) and \((n, \pm m)\), with \( m > 1 \), and \( h \) is the maximal scale of the external lines (they can differ at most by 1), then a factor \( 2^{-h} \nu_{h,m}^{(c)} \), \( c = a, b \) is associated to \( v \) (where \( a \) corresponds to the sign + and \( b \) to the sign -).

We call regularized self-energy graphs the self-energy graphs on which \( \mathcal{R} \) applies, and we denote with \( S(\theta) \) the set of regularized self-energy graphs contained in \( \theta \). We still call \( \nu \)-vertices the nodes \( v \) with \( s_v = 1 \).

In the following it will be useful to define also the renormalized self-energy graphs, which are defined as the self-energy graphs except that the same items (1) and (2) as for the renormalized trees apply to the self-energy graphs and nodes contained inside.

The crucial point is now that, if \( |\nu_{h,m}^{(c)}| \leq C |\varepsilon| \) for some constant \( C \), then each renormalized tree \( \theta \) admit a good bound. First of all note that a bound like (4.12) holds also for a renormalized tree \( \theta \), namely one has the following result (proved in Appendix A2).

**Lemma 2.** For any tree \( \theta \in \Theta_{n,m}^{(k,k)^{\mathcal{R}}} \) one has

\[ N_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M^c_h(\theta), \]  

(5.5)

where the notations are as in lemma 1.

Therefore, by using lemma 2, we obtain the following result (which is proved in Appendix A3), which, under the assumption that the running coupling constants remain bounded of order \( \varepsilon \), allows us to prove the convergence of the Linstedt series as a function of \( \varepsilon \) and the parameters \( \nu_{m',h',c}^{(c)} \), with \( c = a, b \) and \(|m'| > 1\).

**Lemma 3.** Assume that for all \(|m| > 1\) and all \( h \geq 0 \) there exist a constant \( C \) such that one has \( |\nu_{h,m}^{(c)}| \leq C|\varepsilon| \), with \( c = a, b \). Then there exists \( \varepsilon_0 > 0 \) such that for all \(|\varepsilon| < \varepsilon_0 \) and for all \((n, m) \neq (\pm 1, \pm 1) \) one has

\[ |\hat{u}_{n,m}| \leq D_0D|n| + |m| |\varepsilon|(|n| + |m|)/2, \]  

(5.6)

where \( D \) and \( D_0 \) are positive constants. Moreover \( \hat{u}_{n,m} \) depend analytically on \( \varepsilon \) and on all the parameters \( \nu_{m',h',c}^{(c)} \), with \( c = a, b \) and \(|m'| > 1\).

**Remark 9.** One should compare the above power series expansion with the analogous Lindstedt series for invariant KAM tori in quasi-integrable Hamiltonian system (see
The quantities \( \nu_{h,m}^{(c)} \) for \( h + 1 \geq 0 \) and \( |m| > 1 \), verify the recursive relations

\[
\nu_{h+1,m}^{(c)} = 2\nu_{h,m}^{(c)} + \beta_{h,m}^{(c)}(\tilde{\omega}, \epsilon, \{\nu_{h',m'}^{(c)}\}),
\]

where, by defining \( T_{h}^{(c)} \) as the set of self-energy graphs in \( T_{h+1}^{(c)} \) which are on scale \( h \), the beta function

\[
\beta_{h,m}^{(c)} \equiv \beta_{h,m}^{(c)}(\tilde{\omega}, \epsilon, \{\nu_{h',m'}^{(c)}\}) = 2^{h+1} \frac{1}{2} \sum_{\sigma = \pm 1} \sum_{T \in T_{h}^{(c)}} V_{T}^{h+1}(\sigma \tilde{\omega}_m, m),
\]

depends only on the scales \( h' \leq h \).

**Remark 10.** The validity of (5.7) can be checked from (5.3) by noting that the dependence of the value \( V_{T}^{h+1}(\pm \tilde{\omega}_1 n, m) \) of a self-energy graph \( T \) on \( h \) is only through the constraint that the scale \( h_T \) of \( T \) has to be such that \( h_T < h \). Then, subtracting (5.3) from the equivalent expression for \( h + 1 \), one finds (5.7).

In order to obtain a bound on the beta function, hence on the running coupling constants, we need to bound \( V_{T}^{h+1}(\pm \tilde{\omega}_m, m) \) for \( T \in T_{h}^{(c)} \). First of all we note that a bound like that of lemma 2 applies also to the renormalized self-energy graphs.

To see this we first need to introduce some auxiliary notations.

We enlarge the set of trees by allowing more general values for the mode labels of the end-points. More precisely we define \( \Theta_{n,m}^{(k,h)} \) as the set \( \Theta_{n,m}^{(k,h)} \) introduced after (5.4), but by changing item (1) into the following one:

(1') We divide the set \( E(\theta) \) of end-points into three sets \( E(\theta), E_1(\theta) \) and \( E_0(\theta) \), where \( E(\theta) \) is defined as before, \( E_1(\theta) \) is formed by end-points \( v \) with mode label \((n_v, m_v) \in \mathbb{Z} \times \mathbb{Z} \setminus \{\pm, \pm\} \), and \( E_0(\theta) \) is either the empty set or a single end-point \( v_0 \) which has labels \((\overline{n}, \overline{m}, m) \), where \((\overline{n}, \overline{m}, m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) and \( \overline{m} = \pm \overline{\omega}_m / \overline{\omega}_1 \) (so that \( \overline{m} \) is not an integer). The factors corresponding to the end-points in \( E_1(\theta) \cup E_0(\theta) \) are just 1.

Then we have the following generalization of lemma 2 (proved in Appendix A4).

**Lemma 4.** For any tree \( \theta \in \Theta_{n,m}^{(k,h)} \) one has

\[
N_h(\theta) \leq 4\tilde{K}(\theta) 2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M_h^c(\theta),
\]

where

\[
\tilde{K}(\theta) = \begin{cases} 
\sum_{v \in E(\theta)} |n_v| & \text{if } E_0(\theta) = \emptyset, \\
\sum_{v \in E_1(\theta)} |n_v| + |\overline{n}| & \text{if } E_0(\theta) \neq \emptyset,
\end{cases}
\]

(5.10)
while the other notations are as in lemma 1.

By using the just proved result we can show that the bound of lemma 2 extends also to self-energy graphs, as we anticipated, so obtaining the following result (see Appendix A5 for the proof). The idea is that one can deal with the self-energy graphs by imagining them as they were trees, but to do this one has allow more general values for the mode labels of the end-points: the definition of the set $\theta \in \Theta_{n,m}^{(h,k,R)}$ takes into account exactly such a generalization.

**Lemma 5.** Consider $T \in \mathcal{T}_h^{(c)}$. If $N_{h'}(T)$ is the number of lines internal to $T$ with scale $h' \leq h$ and $C_{h'}(T)$ is the number of clusters on scale $h'$ contained in $T$, then one has

$$N_{h'}(T) \leq 4K(T)2^{(2-h')/\tau} - C_{h'}(T) + S_{h'}(T) + M_{h'}^{(c)}(T), \tag{5.11}$$

where $K(T) = \sum_{v \in V(T)} |n_v|$, while $S_{h'}(T)$ and $M_{h'}^{(c)}(T)$ are, respectively, the number of self-energy graphs $T'$ with $h'^{(c)} = h'$ contained in $T$ and the number of vertices contained in $T$.

Then in Appendix A6 we prove the following bound for $V_T^{h+1}(\pm \tilde{\omega}_m,m)$. The main point is that, thanks to lemma 5, we can essentially reason for the self-energy graphs as for the trees.

**Lemma 6.** Assume that for all $|m| > 1$ and all $h \geq 0$ there exist a constant $C$ such that one has $|\nu_{h,m}^{(c)}| \leq C|\epsilon|$, with $c = a, b$. Then for all $h \geq 0$ and for all $T \in \mathcal{T}_h^{(c)}$ one has

$$|V_T^{h+1}(\pm \tilde{\omega}_m,m)| \leq B_0 B^K(T)|\epsilon|^{2(h-1)/\tau}/2|\epsilon|^{K(T)/2}, \tag{5.12}$$

where $K(T) = \sum_{v \in V(T)} |n_v|$, for some positive constants $B_0$ and $B$.

Note that in each contribution to $\nu_{h,m}^{(c)}$ containing a $\nu_{h',m}^{(c)}$, there are at least $M - 1$ end-points; in fact the self-energy graphs (with an incoming external line carrying momentum $(n,m)$) with only $\nu$-vertices are such that all the propagators have argument given by $\tilde{\omega}_1 n - \tilde{\omega}_m$; hence when the $\mathcal{L}$ operation is applied (consisting in replacing $\tilde{\omega}_1 n$ with $\pm \tilde{\omega}_m$) they are vanishing by the support properties of the propagators. This, together with the inequality $M \geq 3$, implies that we can bound in (A2.2)

$$|\nu_{h,m}^{(c)}| \leq C|\epsilon|^{M-1} \leq C|\epsilon|^2, \tag{5.13}$$

which follows from the very definition for the contributions arising from self-energy graphs containing only one node (hence on scale $h = -1$), and from the bound (A3.2) for the contribution containing at least two nodes (so that one line can be on scale $h \geq 0$).

### 6. The choice of the counterterms

In this section we show that it is possible to choose $\nu^{(c)} = \nu^{(c)}_{\nu,m} |m| \geq 1$ such that $\nu^{(c)}$ is analytic in $\epsilon$ for all $|m| \geq 1$ and $c = a, b$, and, for a suitable positive constant $C$,
one has $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$ for all $h \geq 0$ and for all $|m| > 1$. This and lemma 3 will prove theorem 2.

For any sequence $a \equiv \{a_m\}_{|m| \geq 1}$ we introduce the norm

$$
\|a\|_{\infty} = \sup_{|m| \geq 1} |a_m|,
$$

and the seminorm

$$
|a|_{\infty} = \sup_{|m| > 1} |a_m|,
$$

which will be useful in order to bound the counterterms.

We prove the following lemma in Appendix A7.

**Lemma 7.** There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there is a family of intervals $I_{c,m}^{(k)}$, $h \geq 0$, $|m| > 1$, $c = a, b$, such that $I_{c,m}^{(k+1)} \subset I_{c,m}^{(k)}$, $|I_{c,m}^{(k)}| \leq 2|\varepsilon|(|\sqrt{2}|^{-(k+1)}$ and, if $\nu_{m}^{(c)} \in I_{c,m}^{(k)}$, then

$$
|\nu_{h}^{(c)}| \leq C|\varepsilon|, \quad h \geq h > 0,
$$

for some positive constant $C$. Finally one has $\nu_{k,-m}^{(c)} = \nu_{h,m}^{(c)}$, $c = a, b$, for all $h \geq h > 0$ and for all $m > 1$.

It will be useful to really construct the $\nu_{h,m}^{(c)}$ by a contraction method. By iterating (5.7) we find, for $m > 1$

$$
\nu_{h,m}^{(c)} = 2^{h+1}\left(\nu_{m}^{(c)} + \sum_{k=1}^{h} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})\right), \quad (6.4)
$$

where $\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})$ depends on $\nu_{k',m'}$ with $k' \leq k - 1$. If we put $h = \bar{h}$ in (6.4) we get the following identity

$$
\nu_{m}^{(c)} = -\sum_{k=1}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\}) + 2^{-\bar{h}-1}\nu_{k',m'}^{(c)} \quad (6.5)
$$

and (6.4), (6.5) are equivalent, for $\bar{h} > h \geq 0$, to

$$
\nu_{h,m}^{(c)} = -2^{h+1}\left(\sum_{k=h}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})\right) + 2^{\bar{h}-h}\nu_{k',m'}^{(c)}. \quad (6.6)
$$

The sequences $\{\nu_{h,m}^{(c)}\}_{|m| > 1}$, $h > h \geq 0$, parametrized by $\{\nu_{h,m}^{(c)}\}_{|m| > 1}$ such that $|\nu_{h}^{(c)}| \leq C|\varepsilon|$, can be obtained as the limit as $q \to \infty$ of the sequences $\{\nu_{h,m}^{(c)(q)}\}$, $q \geq 0$, defined recursively as

$$
\nu_{h,m}^{(c)(0)} = 0,
$$

$$
\nu_{m}^{(c)(q)} = -\sum_{k=h}^{\bar{h}-1} 2^{-k-2}\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c)(q-1)}\}) + 2^{\bar{h}-h}\nu_{k',m'}^{(c)}. \quad (6.7)
$$
In fact, it is easy to show inductively that, if \( \epsilon \) is small enough, \( |\nu_h^{(q)}|_\infty \leq C|\epsilon|^2 \), so that (6.7) is meaningful, and
\[
\max_{0 \leq h \leq h'} |\nu_h^{(q)} - \nu_h^{(q-1)}|_\infty \leq (C|\epsilon|)^q.
\] (6.8)

For \( q = 1 \) this is true as \( \nu_h^{(c)(0)} = 0 \); for \( q > 1 \) it follows trivially by the fact that
\[
\beta_k^{(c)}(\tilde{\omega}, \epsilon, \nu_k^{(c)(q-1)}) - \beta_k^{(c)}(\tilde{\omega}, \epsilon, \nu_k^{(c)(q-2)})
\]
can be written as a sum of terms in which there are at least one \( \nu \)-vertex, with a difference \( \nu_{h'}^{(c)}(q-1) - \nu_{h'}^{(c)}(q-2) \), with \( h' \geq k \), in place of the corresponding \( \nu_{h'}^{(c)} \), and one node carrying an \( \epsilon \). Then \( \nu_h^{(q)} \)
converges as \( q \to \infty \), for \( h < h \leq 1 \), to a limit \( \nu_h \), satisfying the bound \( |\nu_h|_\infty \leq C|\epsilon| \).

Since the solution is unique, it must coincide with the one implied by lemma 7.

The above prescription fixes \( \nu_{h,m}^{(c)} \) for all \( |m| > 1 \). The values \( \nu_{\pm 1}^{(c)} \) are then fixed by inserting the values of \( \nu_{h,m}^{(c)} \) for all \( |m| > 1 \) into the expressions of \( \tilde{u}_{n,m} \) appearing in the right hand side of (2.7).

7. Construction of the perturbed frequencies

In the following it will be convenient to set \( \omega = \{\omega_m\}_{|m| \geq 1} \) and \( \tilde{\omega} = \{\tilde{\omega}_m\}_{|m| \geq 1} \). By the analysis of the previous sections we have found the counterterms \( \{\nu_m(\tilde{\omega}, \epsilon)\}_{|m| \geq 1} \) as functions of \( \epsilon \) and \( \tilde{\omega} \), so proving theorem 2. To prove theorem 1 we have now to invert the relations
\[
\tilde{\omega}_m^2 + \nu_m(\tilde{\omega}, \epsilon) = \omega_m^2.
\] (7.1)

This is given by the following result.

**Proposition 1.** For all \( \mu_0 > 0 \) there exists a full measure set \( \mathcal{M} \subset [0, \mu_0] \) such that for all \( \mu \in \mathcal{M} \) there are \( \xi > 0 \) and a set \( \mathcal{E} \subset [-\epsilon_0, \epsilon_0] \) with complement of relative Lebesgue measure of order \( \epsilon_0^\xi \) such that for all \( \epsilon \in \mathcal{E} \) there exists \( \tilde{\omega} = \tilde{\omega}(\omega, \epsilon) \) solution of (7.1).

We shall show that there exists a sequence of sets \( \{\mathcal{E}^{(p)}\}_{p=0}^\infty \) in \( [-\epsilon_0, \epsilon_0] \), such that \( \mathcal{E}^{(p+1)} \subset \mathcal{E}^{(p)} \), and a sequence of functions \( \{\tilde{\omega}^{(p)}(\omega, \epsilon)\}_{p=0}^\infty \), with each \( \tilde{\omega}^{(p)} \equiv \tilde{\omega}^{(p)}(\omega, \epsilon) \) defined for \( \epsilon \in \mathcal{E}^{(p)} \), such that for all \( \epsilon \in \mathcal{E}^{(p)} \), with
\[
\mathcal{E} = \bigcap_{p=0}^\infty \mathcal{E}^{(p)} = \lim_{p \to \infty} \mathcal{E}^{(p)},
\] (7.2)
there exists the limit
\[
\tilde{\omega}^{(\infty)}(\omega, \epsilon) = \lim_{p \to \infty} \tilde{\omega}^{(p)}(\omega, \epsilon),
\] (7.3)
and it solves (7.1).

To fulfill the program above we shall follow an iterative scheme by setting
\[
\begin{align*}
\tilde{\omega}_m^{(0)} &= \omega_m^2, \\
\tilde{\omega}_m^{(p)} &= \omega_m^2 - \nu_m(\tilde{\omega}^{(p-1)}), \quad p \geq 1,
\end{align*}
\] (7.4)

17
and reducing recursively the set of admissible values of $\varepsilon$.

We impose on $\omega \equiv \bar{\omega}^{(0)}$ the Diophantine conditions

\begin{align*}
|\omega_1 n \pm \omega_m| &\geq C_0 |n|^{-\tau_0} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0, \pm 1\}, \\
|\omega_1 n \pm m| &\geq C_0 |n|^{-\tau_0} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0, \pm 1\}, \\
|\omega_1 n \pm (\omega_{m'} \pm \omega_m)| &\geq C_0 |n|^{-\tau_0} \quad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{Z} \setminus \{0, \pm 1\},
\end{align*}

(7.5)

where $C_0$ and $\tau_0$ are two positive constants. This will imply some restriction on the admissible values of $\mu$, as the following result shows (see Appendix A8 for the proof).

Lemma 8. For all $\mu_0 > 0$ the set of values $\mu \in [0, \mu_0]$ such that (7.5) are satisfied for some positive constant $C_0$ is of full measure provided that one has $\tau_0$ large enough.

The sets $E^{(p)}$ will be defined recursively as

\begin{align*}
E^{(0)} &\equiv (-\varepsilon_0, \varepsilon_0), \\
E^{(p)} &\equiv \left\{ \varepsilon \in E^{(p-1)} : |\bar{\omega}^{(p)}_1 n \pm \bar{\omega}^{(p)}_m| > C_0 |n|^{-\tau}, \right. \\
\left. |\bar{\omega}^{(p)}_1 n \pm (\bar{\omega}^{(p)}_{m'} \pm \bar{\omega}^{(p)}_m)| > C_0 |n|^{-\tau} \right\}, \quad p \geq 1,
\end{align*}

(7.6)

for $\tau > \tau_0$ to be fixed.

In Appendix A9 we prove the following result.

Lemma 9. For all $p \geq 1$ one has

\begin{align*}
\| \bar{\omega}^{(p)}(\omega, \varepsilon) - \bar{\omega}^{(p-1)}(\omega, \varepsilon) \|_\infty &\leq C \varepsilon_0^p \quad \forall \varepsilon \in E^{(p)},
\end{align*}

(7.7)

for some constant $C$.

This implies that there exist a sequence $\{\bar{\omega}^{(p)}\}_{p=0}^\infty$ converging to $\bar{\omega}^{(\infty)}$ for $\varepsilon \in E$.

We have now to show that the set $E$ has positive (large) measure.

It is convenient to introduce a set of variables $\mu(\bar{\omega}, \varepsilon)$ such that

\begin{align*}
\bar{\omega}_m + \mu_m(\bar{\omega}, \varepsilon) &= \omega_m;
\end{align*}

(7.8)

the variables $\mu(\bar{\omega}, \varepsilon)$ and the counterterms are trivially related by

\begin{align*}
\nu_m(\bar{\omega}, \varepsilon) &= \mu_m^2(\bar{\omega}, \varepsilon) + 2 \bar{\omega}_m \mu_m(\bar{\omega}, \varepsilon).
\end{align*}

(7.9)

One can write $\bar{\omega}^{(p)} = \omega - \mu(\bar{\omega}^{(p-1)}, \varepsilon)$, according to (7.4), so that the Diophantine conditions in (1.17) can be written as

\begin{align*}
|\omega_1 n - \mu_1(\bar{\omega}^{(p-1)}, \varepsilon)n| &\equiv (\omega_m - \mu_m(\bar{\omega}^{(p-1)}, \varepsilon)) > C_0 |n|^{-\tau}, \\
|\omega_1 n - \mu_1(\bar{\omega}^{(p-1)}, \varepsilon)n| &\equiv ((\omega_m \pm \omega_{m'}) \pm (\mu_m(\bar{\omega}^{(p-1)}, \varepsilon) + \mu_{m'}(\bar{\omega}^{(p-1)}, \varepsilon))) > C_0 |n|^{-\tau}.
\end{align*}

(7.10)
Suppose that for \( \varepsilon \in \mathcal{E}(p^{-1}) \) the functions \( \nu(\tilde{\omega}^{(p-1)}, \varepsilon) \) are well defined; then define \( \mathcal{I}(p) = \mathcal{I}_1(p) \cup \mathcal{I}_2(p) \cup \mathcal{I}_3(p) \), where \( \mathcal{I}_1(p) \) is the set of values \( \varepsilon \) verifying the conditions

\[
|\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon)n| \pm (\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)) | \leq C_0 |n|^{-\tau}, 
\]

where \( \mathcal{I}_2(p) \) is the set of values \( \varepsilon \) verifying the conditions

\[
|\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon)n| \\
\pm ((\omega_m - \omega_m') \pm (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) - \mu_m'(\tilde{\omega}^{(p-1)}, \varepsilon))) | \leq C_0 |n|^{-\tau},
\]

and \( \mathcal{I}_3(p) \) is the set of values \( \varepsilon \) verifying the conditions

\[
|\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon)n| \\
\pm ((\omega_m + \omega_m') \pm (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) + \mu_m'(\tilde{\omega}^{(p-1)}, \varepsilon)) | \leq C_0 |n|^{-\tau}.
\]

For future convenience we shall call, for \( i = 1, 2, 3 \), also \( \mathcal{I}_i(p)(n) \) the subsets of \( \mathcal{I}_i(p) \) which verify the Diophantine conditions (7.11), (7.12) and (7.13), respectively, for fixed \( n \). We want to bound the measure of the set \( \mathcal{I}(p) \). First we need to know a little better the dependence on \( \varepsilon \) of the counterterms: this is provided by the following results (to be proved in Appendix A10 and in Appendix A11, respectively).

**Lemma 10.** For all \( p \geq 1 \) and for all \( \varepsilon \in \mathcal{E}(p) \) one has

\[
\nu_1(\tilde{\omega}, \varepsilon) = a_M\varepsilon^{M-1} + O(\varepsilon^M), \\
\nu_m(\tilde{\omega}, \varepsilon) = b_M\varepsilon^{M-1} + O(\varepsilon^M), \quad m \geq 2,
\]

for suitable positive constants \( a_M, b_M \) such that \( |na_M - b_M| > 1/4^{M-1} \) for all \( n \in \mathbb{Z} \).

**Lemma 11.** For all \( p \geq 1 \) and for all \( \varepsilon \in \mathcal{E}(p) \) one has

\[
|\hat{\partial}_{\omega_m} \nu_m(\tilde{\omega}^{(p)}, \varepsilon)| \leq C |\varepsilon|^{2(M-1)}, \quad m \geq 2, \\
|\hat{\partial}_{\omega_m} \nu_m(\tilde{\omega}^{(p)}, \varepsilon)| \leq C |\varepsilon|^{M-2} \quad m \geq 2,
\]

where the derivatives are in the sense of Whitney.

Now we can bound the measure of the set we have to exclude.

We start with the estimate of the measure of the set \( \mathcal{I}_1(p) \). When (7.11) is satisfied one must have (by using also the first of (7.5))

\[
C_0 |n|^{-\tau} \leq |\omega_1 n - \omega_m| \\
\leq |\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon)n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\
+ |\mu_1(\tilde{\omega}^{(p-1)}, \varepsilon)n - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\
\leq C_0 |n|^{-\tau} + C |\varepsilon_0| |n|,
\]

19
which implies, for $|n| > 1$ and $\tau > \tau_0 + 1$, 

$$|n| \geq N_0 \equiv \left( \frac{C_0}{2C\bar{C}_0} \right)^{1/(\tau_0 + 1)}, \quad (7.17)$$

and also

$$C_1|m| \leq |\omega_m - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \leq |(\omega_1 - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon))n| + C_0|n|^{-\tau} \leq C_2|n|, \quad (7.18)$$

which implies

$$|m| \leq M_0|n|, \quad M_0 \equiv \frac{C_2}{C_1}. \quad (7.19)$$

Let us consider the function $\mu(\tilde{\omega}^{(p-1)}, \varepsilon)$: we can define a map $t \rightarrow \varepsilon(t)$ such that

$$f(\varepsilon(t)) \equiv \omega_1n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon(t))n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon(t)) = t \frac{C_0}{|n|^{\tau}}, \quad t \in [-1, 1], \quad (7.20)$$

describes the interval defined by (7.11); then one has

$$\int_{\mathcal{T}_1^{(p)}} d\varepsilon = \sum_{|n| \geq N_0} \sum_{|m| \leq M_0|n|} \int_{-1}^{1} dt \left| \frac{d\varepsilon(t)}{dt} \right|. \quad (7.21)$$

We have from (7.20)

$$\frac{df}{dt} = \frac{df}{d\varepsilon} \frac{d\varepsilon}{dt} = \frac{C_0}{|n|^{\tau}}, \quad (7.22)$$

hence

$$\int_{\mathcal{T}_1^{(p)}} d\varepsilon = \sum_{|n| \geq N_0} \sum_{|m| \leq M_0|n|} \frac{C_0}{|n|^{\tau}} \int_{-1}^{1} dt \left| \frac{d\varepsilon(t)}{d\varepsilon(t)} \right|^{-1}. \quad (7.23)$$

In order to perform the derivative in (7.22) we write

$$\frac{d\mu_m}{d\varepsilon} = \partial_{\varepsilon} \mu_m + \sum_{m' \in \mathbb{Z} \setminus \{0\}} \partial_{\omega_m, \mu_m} \partial_{\varepsilon, \tilde{\omega}_{n,m}, \varepsilon^{(p-1)}} \mu_{m'}, \quad (7.24)$$

and the first term is bounded by the first of (7.15), while each term of the sum in (7.24) is bounded through the second of (7.15). Moreover one has

$$\left| \partial_{\omega_m, \mu_m} \right| \leq |\varepsilon|^{\max\{|m| - |m'|, 2(M-1)|} \bar{C}^{\max\{|m| - |m'|, 2(M-1)|}}, \quad (7.25)$$

because $\mu_m$ depends on $\omega$ if there are at least $2(M-1)$ end-points and if $|m'| > |m|$ there must be at least $|m'| - |m|$ end-points.

At the end we get that the sum in (7.24) is $O(\varepsilon^{2(M-1)|m|})$, and using (7.19) one has

$$f(\varepsilon(t)) = \omega_1n - \omega_m - \alpha M \varepsilon^{M-1}(t) + \beta M n \varepsilon^{M-1}(t) + O(n \varepsilon^{M}(t)) \quad (7.26)$$
where $\alpha_M$ and $\beta_M$ are defined in the statement of lemma 10, hence
\[
\left| \frac{\partial f(\varepsilon(t))}{\partial \varepsilon(t)} \right| \geq c_M |\varepsilon(t)|^{M-2}, \tag{7.27}
\]
for a suitable constant $c_M$, and, for $|n| > 1$ and $\tau > \tau_0 + 1$,
\[
C_1 |n| |\varepsilon(t)|^{M-1} \geq \left| cn^{(1/2)M} |n|^{(0.5M-1)}(t) \right| \geq \left| \omega_1 n - \omega_m \right| - \left| \frac{|C_0|}{|n|^{\tau}} \right| \geq \left| \frac{C_0}{|n|^{\tau}} \right| \tag{7.28}
\]
so that one has
\[
|\varepsilon(t)| > \left( \frac{C_0}{2C_1} \right)^{1/(M-1)} \frac{1}{n^{(\tau_0+1)/(M-1)}}, \tag{7.29}
\]
which, introduced into (7.23), gives together with (7.27),
\[
\int I_1^{(p)} \, d\varepsilon \leq \text{const.} \sum_{|n| \geq N_0} \sum_{|m| \leq M_0 |n|} \frac{C_0}{|n|^\tau} \int_{-1}^{1} |\varepsilon(t)|^{M-2} \, dt \leq \text{const.} \sum_{|n| \geq N_0} \sum_{|m| \leq M_0 |n|} \frac{C_0}{|n|^\tau} \frac{n^{(\tau_0+1)(M-2)/(M-1)}}{\sqrt{C_0 |n|}} \tag{7.30}
\]
with
\[
\tau' = \tau + 1 - \frac{\tau_0 + 1(M-2)}{M-1}, \tag{7.31}
\]
Therefore the Lebesgue measure of the set $I_1^{(p)}$ is bounded by
\[
\text{const.} \sum_{|n| \geq N_0} \sum_{|m| \leq M_0 |n|} \frac{C_0}{|n|^\tau} \leq \text{const.} \sqrt{C_0} \sum_{n=N_0}^{\infty} n^{1-\tau'} \tag{7.32}
\]
\[
\leq \text{const.} \sqrt{C_0} \frac{n^{\tau'-\tau+2}}{\tau_0 + 1} = \text{const.} \sqrt{C_0} \left( \frac{\xi_0}{C_0} \right)^{(\tau'-2)/(1+\tau_0)} \leq \text{const.} \frac{1}{\xi'}, \tag{7.33}
\]
provided that one has
\[
\xi' = \frac{\tau' - \tau_0 - 1}{\tau_0 + 1} = \frac{\tau - \frac{1}{2}(3\tau_0 + 1)}{\tau_0 + 1} > 0, \tag{7.33}
\]
which imposes $\tau > (3\tau_0 + 1)/2$.

Now we discuss how to bound the measure of the set $I_2^{(p)}$. We start by noting that for all $p \geq 0$ one has
\[
\omega_m^{(p)} = \omega_m^2 + \kappa + \sigma_m^{(p)} \quad |\sigma_m^{(p)}| < C \frac{|\xi|}{n^2}, \tag{7.34}
\]
where \( \xi' = 1 \) and \( \kappa = O(\varepsilon^{2(M-1)}) \) is a constant (in \( m \)).

In fact \( \nu_{\ell}^{(p)} \) is given by a sum of values of renormalized self-energy graphs \( T \), and, for each value, the dependence on \( m \) is due to the propagators of lines along the path connecting the external lines of the self-energy graph (the self-energy graphs with no path of lines connecting the external lines contribute to the constant term \( \kappa \)). The propagators of such lines have the form

\[
\chi_k\left( |\tilde{\omega}^{(p)}_{m+\ell} - \tilde{\omega}^{(p)}_m| - \frac{\tilde{\omega}^{(p)}_{m+\ell} - \tilde{\omega}^{(p)}_m}{|\tilde{\omega}^{(p)}_{m+\ell} + \tilde{\omega}^{(p)}_m|} \right),
\]

(7.35)

and the second factor in the denominator is bounded proportionally to \( 2^{-n_\ell} \), while the first is bounded by a constant times \( m_\ell \); hence (7.34) holds with \( \xi = 1 \). As \( m_\ell = m_0^{\ell} + m \) and \( k_T > m_0^{\ell} \) we get

\[
\tilde{\omega}^{(p)}_{m+\ell} - \tilde{\omega}^{(p)}_m = \omega_{m+\ell} - \omega_m + O(\varepsilon(m + \ell)^{-\xi'}) + O(\varepsilon m^{-\xi'})
\]

\[
= \ell + O(\ell m^{-\xi'}) + O(\varepsilon m^{-\xi'}) \leq \ell + O(\ell m^{-\xi'}).
\]

(7.36)

where \( 1 = \min\{\xi, \xi'\} = \min\{1, 2\} \); hence we can bound

\[
|\tilde{\omega}^{(p)}_{m+\ell} - \tilde{\omega}^{(p)}_m - \ell| \leq \frac{2Km_\ell}{m},
\]

(7.37)

with the same constant \( K \) as in (A8.4).

The conditions in (7.12) correspond to several possibilities. If \( n > 0 \), \( m' > m > 0 \) and \( |\tilde{\omega}^{(p)}_{m'}/\tilde{\omega}^{(p)}_{m-1}/\tilde{\omega}^{(p)}_{m}| < 1 \) the discussion proceeds as follows.

When the conditions (7.12) are satisfied, one has

\[
C_0|n|^{-\tau_0} \leq |\omega_1 n - (\omega_{m'} - \omega_m)|
\]

\[
\leq |\omega_1 n - \mu_1(\omega^{(p-1)}_m, \varepsilon)n - (\omega_{m'} - \mu_m^{(p-1)}(\omega^{(p-1)}_m, \varepsilon))|
\]

\[
- (\omega_m + \mu_m^{(p-1)}(\omega^{(p-1)}_m, \varepsilon))|
\]

\[
+ |\mu_1(\omega^{(p-1)}_m, \varepsilon)n - \mu_m^{(p-1)}(\omega^{(p-1)}_m, \varepsilon) + \mu_m(\omega^{(p-1)}_m, \varepsilon)|
\]

\[
\leq C_0|n|^{-\tau} + 3C_{\ell_0}|n|,
\]

(7.38)

which implies

\[
|n| \geq \mathcal{N}_1 \equiv \left( \frac{C_0}{6C_{\ell_0}} \right)^{1/(\tau_0+1)}.
\]

(7.39)

Besides of (7.11) we eliminate also the values \( \varepsilon \) verifying

\[
|\omega_1 n - \mu_1(\omega^{(p-1)}_m, \varepsilon)n - m| \leq C_0|n|^{-\tau},
\]

(7.40)

for \( \tau > \tau_0 + 1 \) and for all \( (n, m) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \). This requires to take off from \( \mathcal{E}^{(p)} \) a subset whose measure is bounded by a constant times \( \varepsilon^{1+\xi}_0 \), as it can easily checked by proceeding as in the proof of the first of (7.11).
We can bound the Lebesgue measure of the set $I_{2}^{(p)}$ by distinguishing, for fixed $(n, \ell)$, with $\ell = m' - m > 0$, the values $m \leq m_{0}$ and $m > m_{0}$, where $m_{0}$ is determined by the request that one has for $m > m_{0}$

$$\frac{2K\mu_{0}\ell}{m^\xi} \leq \frac{C_{0}}{2|n|^\tau},$$

which gives

$$m_{0} = \left(\frac{2\mu_{0}K\ell|n|^\tau}{C_{0}}\right)^{1/\xi}.$$  (7.41)

Therefore for $m > m_{0}$ and $L_{0}$ defined by

$$\frac{C_{1}\ell}{\sqrt{1 + \mu_{0}}} \leq \left|\tilde{\omega}_{m+\ell}^{(p)} - \tilde{\omega}_{m}^{(p)}\right| < \tilde{\omega}_{1}^{(p)}|n| + 1 < C_{2}|n|,$$  (7.42)

where (A8.8) has been used, one has, from (7.39)

$$\left|\omega_{1}n - \mu(\tilde{\omega}_{m}^{(p-1)}, \varepsilon)n - (\tilde{\omega}_{m}^{(p)} - \tilde{\omega}_{m}^{(p)})\right| \geq \left|\omega_{1}n - \mu(\tilde{\omega}_{m}^{(p-1)}, \varepsilon)n - \ell\right| - \frac{2K\mu_{0}\ell}{m^\xi} \geq \frac{C_{0}}{|n|^\tau} - \frac{C_{0}}{2|n|^\tau} \geq \frac{C_{0}}{2|n|^\tau},$$

so that one has to exclude no further value from $\mathcal{E}^{(p-1)}$, provided one takes $C_{0}$ the double of the original value. For $m < m_{0}$ one has to exclude a set of measure bounded by

$$\sum_{|n| \geq N_{1}} \sum_{\ell \leq L_{0}} \sum_{|m| \leq m_{0}} \frac{C_{0}}{|n|^\tau} \leq \text{const.} C_{0} \sum_{n = N_{1}}^{\infty} n^{1-\tau'+(1+\tau)/\xi} \leq \text{const.} C_{0} N_{0}^{-\tau'+1+(1+\tau)/\xi} \leq \text{const.} C_{0} \left(\frac{\varepsilon_{0}}{C_{0}}\right)^{(\tau'-2-\tau)/(1+\tau_{0})}$$

provided that $\tau'$ is such that

$$\xi'' = \frac{\tau' - 2 - 1 + \tau}{1 + \tau_{0}} > 0.$$  (7.45)

Finally we study the measure of the set $I_{3}^{(p)}$. If $n > 0$ and $|\tilde{\omega}_{m}^{(p)} + \tilde{\omega}_{m'}^{(p)} - \tilde{\omega}_{n}| \leq 1$ then one has to sum over $|n| \leq N_{0}$, with $N_{0}$ given by (7.17), while one has $m < \tilde{\omega}_{m}^{(p)} < \tilde{\omega}_{m}^{(p)} + \tilde{\omega}_{n}^{(p)} < \tilde{\omega}_{1}^{(p)}|n| + 1 < (\tilde{\omega}_{1}^{(p)} + 1)|n|$, so that one has to sum only over the $m$’s such that $|m| < (\tilde{\omega}_{1}^{(p)} + 1)|n|$, while $|m'|$ is uniquely determined by the values

23
of $n$ and $m$. Then one can proceed as in the previous case and in the end one excludes a further subset of $E(p-1)$ whose Lebesgue measure admits a bound like (7.26).

By summing together the bounds for $I_1(p)$, for $I_2(p)$ and for $I_3(p)$, then the bound
\[
\text{meas}(I(p)) \leq b\xi_0^{p+1}
\]
follows for all $p \geq 1$.

We can conclude the proof of theorem 1 through the following result (proved in Appendix A11), which shows that the bound (7.47) essentially extends to the union of all $I(p)$ (at the cost of taking a larger constant $B$ instead of $b$).

**Lemma 12.** Define $I(p)$ as the set of values in $E(p)$ verifying (A9.7). Then one has
\[
\text{meas}\left(\bigcup_{p=0}^{\infty} I(p)\right) \leq B\xi_0^{p+1},
\]
for two suitable positive constants $B$ and $\xi'$.

The conclusion is that one has
\[
\text{meas}(E) = 2\xi_0 \left(1 - b\xi_0^p\right),
\]
so that the proof of theorem 1 is complete.

**Appendix A1. Proof of lemma 1**

We prove inductively the bound
\[
N^*_h(\theta) \leq \max\{0, 2K(\theta)2^{(2-h)/\tau} - 1\},
\]
where $N^*_h(\theta)$ is the number of non-resonant lines in $L(\theta)$ on scale $h \geq h'$.

First of all note that for a tree $\theta$ to have a line on scale $h$ the condition $K(\theta) > 2^{(h-1)/\tau}$ is necessary, by the first Diophantine conditions in (1.17). This means that one can have $N^*_h(\theta) \geq 1$ only if $K = K(\theta)$ is such that $K > k_0 \equiv 2^{(h-1)/\tau}$; therefore for values $K \leq k_0$ the bound (A1.1) is satisfied.

If $K = K(\theta) > k_0$, we assume that the bound holds for all trees $\theta'$ with $K(\theta') < K$. Define $E_h = 2^{-1}(2^{(2-h)/\tau})^{-1}$: so we have to prove that $N^*_h(\theta) \leq \max\{0, K(\theta)E_h^{-1} - 1\}$.

Call $\ell$ the root line of $\theta$ and $\ell_1, \ldots, \ell_m$ the $m \geq 0$ lines on scale $\geq h$ which are the closest to $\ell$ (i.e. such that no other line along the paths connecting the lines $\ell_1, \ldots, \ell_m$ to the root line is on scale $\geq h$).

If the root line $\ell$ of $\theta$ is either on scale $< h$ or on scale $\geq h$ and resonant, then
\[
N^*_h(\theta) = \sum_{i=1}^{m} N^*_h(\theta_i),
\]
where $\theta_i$ is the subtree with $\ell_i$ as root line, hence the bound follows by the inductive hypothesis.
If the root line \( \ell \) has scale \( \geq h \) and is non-resonant, then \( \ell_1, \ldots, \ell_m \) are the entering line of a cluster \( T \).

By denoting again with \( \theta_i \) the subtree having \( \ell_i \) as root line, one has
\[
N_k^\ast(\theta) = 1 + \sum_{i=1}^m N_k^\ast(\theta_i), \tag{A1.3}
\]
so that the bound becomes trivial if either \( m = 0 \) or \( m \geq 2 \).

If \( m = 1 \) then one has a cluster \( T \) with two external lines \( \ell \) and \( \ell_1 \), which are both with scales \( \geq h \); then
\[
|\tilde{\omega}_1 n_\ell - \tilde{\omega}_{m\ell}| \leq 2^{-h+1}C_0, \quad |\tilde{\omega}_1 n_{\ell_1} - \tilde{\omega}_{m_{\ell_1}}| \leq 2^{-h+1}C_0, \tag{A1.4}
\]
and \( n_\ell \neq n_{\ell_1} \) as \( T \) is not a self-energy graph. Then, by (A1.4), one has, for suitable \( \eta_\ell, \eta_{\ell_1} \in \{+, -\} \),
\[
2^{-h+2}C_0 \geq |\tilde{\omega}_1(n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}_{m\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}}| \geq C_0|n_\ell - n_{\ell_1}|^{-\tau}, \tag{A1.5}
\]
where the second Diophantine conditions in (1.17) have been used. Hence \( K(\theta) - K(\theta_1) > E_h \), which, inserted into (A1.3) with \( m = 1 \), gives, by using the inductive hypothesis,
\[
N_k^\ast(\theta) = 1 + N_k^\ast(\theta_1) \leq 1 + K(\theta_1)E_h^{-1} - 1 \\
\leq 1 + \left( K(\theta) - E_h \right)E_h^{-1} - 1 \leq K(\theta)E_h^{-1} - 1, \tag{A1.6}
\]
and hence the bound is proved also if the root line is on scale \( \geq h \).

In the same way one proves that, if we denote with \( C_h(\theta) \) the number of clusters on scale \( h \), one has
\[
C_h(\theta) \leq \max \{0, 2K(\theta)2^{(2-h)/\tau} - 1\}. \tag{A1.7}
\]
For a tree to contain a cluster on scale \( h \) it has to contain a fortiori a line on that scale, so that again the bound (A1.7) is trivially satisfied for \( K(\theta) \leq k_0 = 2^{(h-1)/\tau} \).

For \( K(\theta) = K > k_0 \) we can proceed inductively as before. If the node \( v_0 \), which the root line \( \ell \) of \( \theta \) comes out from, is not in a cluster on scale \( h \) the one has
\[
C_h(\theta) = \sum_{i=1}^m C_h(\theta_i), \tag{A1.8}
\]
where \( \theta_1, \ldots, \theta_m \) are the subtrees with root in \( v_0 \); in such a case the bound follows from the inductive hypothesis.

If \( v_0 \) is inside a cluster \( T \) on scale \( h \) then call \( \theta_1, \ldots, \theta_m \) the entering lines of \( T \); then one has
\[
C_h(\theta) = 1 + \sum_{i=1}^m C_h(\theta_i), \tag{A1.9}
\]
and the bound (A1.7) follows again from the inductive hypothesis for either \( m = 0 \) or \( m \geq 2 \). If \( m = 1 \) then one has
\[
|\tilde{\omega}_1 n_\ell - \tilde{\omega}_{m\ell}| \leq 2^{-h+1}C_0, \quad |\tilde{\omega}_1 n_{\ell_1} - \tilde{\omega}_{m_{\ell_1}}| \leq 2^{-h+1}C_0. \tag{A1.10}
\]
where $\ell \in L(T)$ is on scale $h$, while $\ell_1$ is on a scale $h_{\ell_1} > h$ (by definition of cluster). Therefore, if $n_\ell = n_\ell^0$, one has, by the first Diophantine condition in (1.17),

$$2^{-h+1} C_0 \geq |\tilde{\omega}_1 n_\ell| - \tilde{\omega}_{m_\ell} \geq C_0 |n_\ell^0| - \tau,$$

(A1.11)

while, if $n_\ell = n_\ell^0 + n_{\ell_1}$, one has for suitable $\eta_\ell, \eta_{\ell_1}, \in \{+, -\}$,

$$2^{-h+2} C_0 \geq |\tilde{\omega}_1 (n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}}| \geq C_0 |n_\ell - n_{\ell_1}| - \tau \geq C_0 |n_\ell^0| - \tau,$$

(A1.12)

by the second Diophantine conditions in (1.17). So in both cases one has

$$\sum_{v \in V(T)} |n_v| \geq 2^{(h-2)/\tau} > E_h,$$

(A1.13)

so that (A1.9) implies the bound (A1.7) also for $m = 1$. Of course one can bound the number of non-resonant lines on scale $h$ of any tree $\theta$ by $N^*_h(\theta) + C_h(\theta) - C_h(\theta)$, with $N^*_h(\theta) + C_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau}$: therefore (4.12) follows.

Appendix A2. Proof of lemma 2

One has to show inductively that, by defining $N^*_h(\theta)$ as the number of non-resonant lines on scale $h' \geq h$ in the renormalized tree $\theta$, one has

$$N^*_h(\theta) \leq \max \{0, 2K(\theta)2^{(2-h)/\tau} - 1\},$$

(A2.1)

and that an analogous bound holds for the number $C_h(\theta)$ of clusters on scale $h$.

Let us consider a self-energy graph $T$ in $\theta$ which is maximal, i.e. which is not contained in any other resonance. The value of such a self-energy graph is given by the product of the values of the inner maximal self-energy graphs (on which the $\mathcal{R}$ operation is still applied) times the product of propagators corresponding to the remaining lines; there is a chain of propagators and self-energy graphs connecting the external lines such that the momentum $(n_\ell, m_\ell)$ of any of such propagator or self-energy graph has the form

$$(n_\ell^0 + n, m_\ell^0 + m),$$

(A2.2)

where $(n, m)$ is the momentum of the external lines, and $(n_\ell^0, m_\ell^0)$ are implicitly defined; all the other propagators and self-energy graphs are independent of $(n, m)$. Let us consider the action of $\mathcal{R}$ on the maximal self-energy graphs. The action of $\mathcal{R}$ consists in writing the self-energy graph as a sum of terms, in which one propagator $g^{(h_\ell)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 n, m_\ell^0 + m)$ is replaced with $g^{(h_\ell)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 n, m_\ell^0 + m) - g^{(h_\ell)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_m, m_\ell^0 + m)$ or the value of one regularized inner self-energy graph $\mathcal{R}V_{T'}^{(c)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 n, m_\ell^0 + m)$ is replaced with $\mathcal{R}V_{T'}^{(c)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 n, m_\ell^0 + m) - \mathcal{R}V_{T'}^{(c)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_m, m_\ell^0 + m)$.

We can write

$$g^{(h_\ell)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 n, m_\ell^0 + m) - g^{(h_\ell)}(\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_m, m_\ell^0 + m) = (\tilde{\omega}_1 n - \tilde{\omega}_m) \int_0^1 \frac{dt}{\Lambda_t} \frac{\partial}{\partial \Lambda_t} \left[ \frac{1}{(\tilde{\omega}_1 n_\ell^0 + t(\tilde{\omega}_1 n - \tilde{\omega}_m)) - \tilde{\omega}_m} \right.$$

$$\left. - (\tilde{\omega}_1 n_\ell^0 + t(\tilde{\omega}_1 n - \tilde{\omega}_m))^2 + \tilde{\omega}_m^2 \right].$$

(A2.3)
where \( \partial \) denotes the derivative with respect to the argument \( \tilde{\omega}_1 n \).

On the other hand \( \mathcal{L} V^{(c)}_{T'} (\omega_1 n_0 + \omega_1 n, m_0 + m) \) is independent of \( \omega_1 n \) (see (5.1)), so that

\[
\mathcal{R} V^{(c)}_{T'} (\omega_1 n_0 + \omega_1 n, m_0 + m) - \mathcal{R} V^{(c)}_{T'} (\omega_1 n_0 + \omega_m, m_0 + m) = V^{(c)}_{T'} (\omega_1 n_0 + \omega_1 n, m_0 + m) - V^{(c)}_{T'} (\omega_1 n_0 + \omega_m, m_0 + m),
\]

where we can reason as above, writing the inner self-energy graphs as a sum of terms.

Then the proof of lemma 2 proceeds exactly as for lemma 1, but in (A1.4) one has to replace \( \tilde{\omega}_1 n \) with \( \tilde{\omega}_1 n_0 + t(\tilde{\omega}_1 n - \tilde{\omega}_m) \), when \( \ell \) is along a path of lines connecting the external lines of \( T \). But this does not change the proof because when passing from (A1.4) to (A1.5) the terms depending on \( t \) cancel out.

### Appendix A3. Proof of lemma 3

By using the expression (A2.3) and (A2.4) we see that the effect of \( \mathcal{R} \) is to improve by a factor \( 2^{\frac{h}{\tau}} + h \) the bound of the propagator corresponding to the line \( \ell \). In the same way, the difference of propagators in the inner self-energy graph gives the “gain”

\[
2^{-h^{(c)}_{\tau} + h_T} \leq 2^{-h^{(c)}_{\tau} + h_T} 2^{-h^{(c)}_{\tau} + h_T},
\]

as \( h^{(c)}_{\tau} \leq h_T \). At the end (i) the propagators are derived at most one time; (ii) the number of terms so generated is \( \leq k + |k| \); (iii) to each self-energy graph \( T \) a factor \( 2^{-h^{(c)}_{\tau} + h_T} \) is associated.

Assuming that \( |\nu_{h,m}^{(c)}| \leq C \varepsilon \), for any \( \theta \) one obtains, for a suitable constants \( \overline{D} \) and \( \overline{D}_0 \),

\[
|\text{Val}(\theta)| \leq \overline{D}_0 |\theta|^{k+|k|} \overline{D}^{k+|k|} \exp \left[ h \log 2 \left( 4k2^{-(h-2)/\tau} - C_h(T) + S_h(\theta) + M_h(\theta) \right) \right] \prod_{T \in \mathcal{S}(\theta)} 2^{-h^{(c)}_{\tau} + h_T} \prod_{h=0}^{\infty} 2^{-h M_h(\theta)},
\]

where lemma 2 has been used in order to bound the number of lines on scale \( h \), and

\[
\prod_{T \in \mathcal{S}(\theta)} 2^{-h^{(c)}_{\tau} + h_T}
\]

is due to the nontrivial action of the \( \mathcal{R} \) operator on the self-energy graphs, while the factor

\[
\prod_{h=0}^{\infty} 2^{-h M_h(\theta)}
\]
takes into account the $2^{-h}$ factors associated to the $\nu$-vertices contributing a factor $\nu_{h,m}$. Then one has
\[
\left( \prod_{h=0}^{\infty} 2^{hS_h(\theta)} \right) \left( \prod_{T \in S(\theta)} 2^{-h_T} \right) = 1,
\]
\[
\left( \prod_{h=0}^{\infty} 2^{-hC_h(\theta)} \right) \left( \prod_{T \in S(\theta)} 2^{h_T} \right) \leq 1,
\] (A3.5)
so that one finds, for suitable constants $\tilde{D}_0$ and $\tilde{D}_1$,
\[
\sum_{k}: |k| \text{ fixed} \sum_{\theta \in \Theta(k,k)} |\text{Val}(\theta)| \leq \tilde{D}_0 |\epsilon| |k|^{n+m} C^{k+|\bar{k}|},
\] (A3.6)
where $2^{k+|\bar{k}|}$ is a bound on the number of trees in some $\Theta(k,k)$ with fixed $|\bar{k}|$, $4^k$ is a bound on the assignments of the labels $(n_v, m_v)$ for all end-points $v$, and finally $2^{|\bar{k}|}$ is a bound on the labels $c = a, b$ for all self-energy graphs.

Hence, for fixed $(n, m)$ one has
\[
\sum_{k:|k| \text{ fixed}} \sum_{\theta \in \Theta(k,k)} |\text{Val}(\theta)| \leq D_0 |\epsilon| (|n|+|m|)/2 D^{n+m},
\] (A3.7)
as $k \geq \min\{|n|, |m|\}$ and $|\bar{k}| \geq 0$, so that (5.6) is proved.

**Appendix A4. Proof of lemma 4**

First of all we note that the result stated in lemma 2 still holds, with no change, if we allow any integer value for the mode labels $(n_v, m_v)$ of the end-points. The only difference is that the induction has to be performed on $\tilde{K}(\theta)$, as it is defined in (5.10) for $E_0(\theta) = \emptyset$. So we can assume that the bound (A2.1) holds for any tree $\theta$ with $E_0(\theta) = \emptyset$.

The bound (A2.1), with $\tilde{K}(\theta)$ replacing $K(\theta)$, trivially extends to the trees $\theta$ in $\tilde{\Theta}(k,k) \times \mathbb{R}$ with $E_0(\theta) = \emptyset$: one simply repeats the same proof as given in Appendix A2.

Then we have to show that the bound (5.9) holds for trees $\theta$ with $E_0(\theta) \neq \emptyset$. We mimic the proof of lemma 2 (hence of lemma 1): we prove the bound
\[
N^*_h(\theta) \leq \max\{0, 2\tilde{K}(\theta) 2^{(2-h)/\tau}\},
\] (A4.1)
for all trees $\theta$ with $E_0(\theta) \neq \emptyset$, again by induction on $\tilde{K}(\theta)$.

For any line $\ell \in L(\theta)$ set $\eta_{\ell} = 1$ if the line is along the path connecting $v_0$ to the root and $\eta_{\ell} = 0$ otherwise, and write
\[
n_{\ell} = n_{\ell}^0 + \eta_{\ell} (n + m), \quad m_{\ell} = m_{\ell}^0 + \eta_{\ell} (n + m),
\] (A4.2)
which implicitly defines $n_{\ell}^0$ and $m_{\ell}^0$. 28
Define $k_0 = 2^{(h-1)/\tau}$. One has $N^*_h(\theta) = 0$ for $\tilde{K}(\theta) < k_0$, because if a line $\ell \in L(\theta)$ is indeed on scale $h$ then $|\bar{\omega}_1 n_\ell - \bar{\omega}_m| < C_0 2^{1-h}$, so that (A4.2) and the Diophantine conditions (1.17) imply
\[
\tilde{K}(\theta) \geq |n^0_\ell + \eta_\ell \bar{\pi}| > 2^{(h-1)/\tau} \equiv k_0.
\] (A4.3)

Then, for $K \geq k_0$, we assume that the bound (A4.1) holds for all $\tilde{K}(\theta) = K' < K$, and we show that it follows also for $\tilde{K}(\theta) = K$.

If the root line $\ell$ of $\theta$ is either on scale $< h$ or on scale $\geq h$ and resonant, the bound (A4.1) follows immediately from the bound (A2.1) and from the inductive hypothesis.

The same occurs if the root line is on scale $\geq h$ and non-resonant, and, by calling $\ell_1, \ldots, \ell_m$ the lines on scale $\geq h$ which are the closest to $\ell$, one has $m \geq 2$: in fact in such a case at least $m - 1$ among the subtrees $\theta_1, \ldots, \theta_m$ having $\ell_1, \ldots, \ell_m$, respectively, as root lines have $E(\theta_i) = \emptyset$, so that we can write, by (A2.1) and by the inductive hypothesis,
\[
N^*_h(\theta) = 1 + \sum_{i=1}^m N^*_h(\theta_i) \leq 1 + E_h^{-1} \sum_{i=1}^m \tilde{K}(\theta_i) - (m - 1) \leq E_h \tilde{K}(\theta),
\] (A4.4)
so that (A4.1) follows.

If $m = 0$ then $N^*_h(\theta) = 1$ and $\tilde{K}(\theta) 2^{(2-h)/\tau} \geq 1$ because one must have $\tilde{K}(\theta) \geq k_0$.

So the only non-trivial case is when one has $m = 1$. If this happens $\ell_1$ is, by construction, the root line of a tree $\theta_1$ such that $\tilde{K}(\theta) = \tilde{K}(T) + \tilde{K}(\theta_1)$, where $T$ is the cluster which has $\ell$ and $\ell_1$ as external lines and we have defined
\[
\tilde{K}(T) \equiv \sum_{v \in E(T) \cup E_1(T)} |n_v|,
\] (A4.5)
which satisfies the bound $\tilde{K}(T) \geq |n_{\ell_1} - n_\ell|$.

Moreover, if $E_0(\theta_1) \neq \emptyset$, one has
\[
|\bar{\omega}_1 n^0_\ell + \bar{\omega}_1 \bar{\pi} + \bar{\omega}_m| - |\bar{\omega}_m| \leq 2^{-h} C_0,
\]
\[
|\bar{\omega}_1 n^0_{\ell_1} + \bar{\omega}_1 \bar{\pi} + \bar{\omega}_m| - |\bar{\omega}_m| \leq 2^{-h} C_0,
\] (A4.6)
so that, for suitable $\eta_\ell, \eta_{\ell_1} \in \{-, +\}$, we obtain
\[
2^{-h+2} C_0 \geq |\bar{\omega}_1 (n^0_\ell - n^0_{\ell_1}) + \eta_\ell \bar{\omega}_m + \eta_{\ell_1} \bar{\omega}_m| \geq C_0 |n_\ell - n_{\ell_1}| - \tau \equiv C_0 |n_\ell - n_{\ell_1}| - \tau,
\] (A4.7)
by the second Diophantine conditions in (1.17), as the quantities $\bar{\omega}_1 \bar{\pi} + \bar{\omega}_m$ appearing in (A4.5) cancel out. Therefore one obtains by the inductive hypothesis
\[
N^*_h(\theta) \leq 1 + \tilde{K}(\theta) E_h^{-1} \leq 1 + \tilde{K}(\theta) E_h^{-1} - \tilde{K}(T) E_h^{-1} \leq \tilde{K}(\theta) E_h^{-1},
\] (A4.8)
hence the first bound in (A4.1) is proved.

If $E_0(\theta_1) = \emptyset$, one has by (A2.1)

$$N^*_h(\theta) \leq 1 + \tilde{K}(\theta_1)E_h^{-1} - 1 \leq 1 + \tilde{K}(\theta)E_h^{-1} - 1 \leq \tilde{K}(\theta)E_h^{-1},$$

(A4.9)

and (A4.1) follows also in such a case.

Analogously one can first show that the bound (A1.7), with $\tilde{K}(\theta)$ replacing $K(\theta)$, holds for the trees in $G_{n,m}^{(k,k)}$, hence prove the bound

$$C_h(\theta) \leq \max\{0, 2\tilde{K}(\theta)2^{(2-h')/\tau}\},$$

(A4.10)

for all trees $\theta$ with $E_0(\theta) \neq \emptyset$, so that (5.9) follows.

Appendix A5. Proof of lemma 5

Consider a self-energy graph $T \in T^{(c)}_h$. We can consider the tree $\theta$ obtained from $T$ by adding to it the outcoming line $\ell_1^T$ and replacing the entering line $\ell_2^T$ with a line emerging from an end-point $v_0$ which carries a mode label $(\tilde{\omega}_m, m)$; by construction one has

$$V_h^T(\pm\tilde{\omega}_m, m) = \text{Val}'(\theta),$$

(A5.1)

where Val$(\theta)$ differs from (4.5) as in the first product the line $\ell_1$ coming out from $T$ is missing, and $E_0(\theta) = \{v_0\}$.

We want to prove that one has

$$N^*_h(T) \leq \max\{0, 2K(T)2^{(2-h')/\tau}\},$$

(A5.2)

if $N^*_h(T)$ denotes the number of non-resonant lines on scale $\leq h'$ internal to $T$.

Let $v$ be the node such that $\ell_1^T$ comes out from $v$, and call $\ell_1, \ldots, \ell_{s_v}$ the lines entering $v$; denote with $\theta_1, \ldots, \theta_{s_v}$ the trees which have $\ell_1, \ldots, \ell_{s_v}$ as root lines. By construction one has

$$N^*_h(T) = \sum_{j=1}^{s_v} N^*_h(\theta_j),$$

(A5.3)

and (only) one of the subtrees $\theta_j$, say $\theta_1$, contains the end-point $v_0$. Therefore in (5.7) we can bound

$$\sum_{j=2}^{s_v} N^*_h(T) \leq \sum_{j=2}^{s_v} \max\{0, 2K(\theta_j)2^{(2-h')/\tau} - 1\},$$

(A5.4)

by using (A2.1), and we can bound

$$N^*_h(\theta_1) \leq K(\theta_1)2^{(2-h')/\tau},$$

(A5.5)

by using (A4.1) (with $\pi = 0$).

Analogously one can prove that one has

$$C_h(T) \leq \max\{0, 2K(\theta_j)2^{(2-h')/\tau}\},$$

(A5.6)
if \( C_{h'}(T) \) denotes the number of clusters on scale \( \leq h' \) internal to \( T \).

Then the bound (A4.8) follows.

**Appendix A6. Proof of lemma 6**

By using lemma 5 and the cancellations discussed in Appendix A3 we obtain for all \( T \in \mathcal{T}_h^{(c)} \)

\[
\left| \nu_T^{h+1}(\pm \tilde{\omega}_m, m) \right| \leq C_k |\varepsilon|^{K(T)+K(T)} \prod_{h'=h}^{\infty} \exp \left[ 4k \log 2h'2^{(2-h')/\tau} - C_{h'}(T) + S_{h'}(T) + M_{h'}(T) \right] \bigg( \prod_{T' \subset T} 2^{-h'(c)\tau^{h'} + h'} \bigg) \prod_{h'=h}^{\infty} 2^{-h'M_{h'}(T)} \bigg( \prod_{h'=h}^{\infty} 2^{-h'M_{h'}(T)} \bigg),
\]  

(A6.1)

The main difference with respect to the previous case is that, given a self-energy graph \( T \in \mathcal{T}_h^{(c)} \), there is at least a line \( \ell \in L(T) \) on scale \( h_\ell = h \) and with propagator

\[
\frac{1}{-\tilde{\omega}_T^2(n_\ell^0 + \eta \tilde{\omega}_m)^2 + \tilde{\omega}_m^2 + \eta \tilde{\omega}_m},
\]

where \( \eta \ell = 1 \) if the line \( \ell \) belongs to the path of lines connecting the entering line (carrying a momentum \( (n, m) \)) of \( T \) with the line coming out of \( T \), and \( \eta \ell = 0 \) otherwise. Then by (1.17) one has

\[
C_0 |n_\ell^0|^{-\tau} \leq |\tilde{\omega}_1 n_\ell^0 + \eta \tilde{\omega}_m \pm \tilde{\omega}_m^2 + \eta \tilde{\omega}_m| \leq C_0 2^{-h+1},
\]

(A6.3)

so that \( |n_\ell^0| \geq 2^{(h-1)/\tau} \). On the other hand one has \( |n_\ell^0| \leq K(T) \), hence \( K(T) \geq 2^{(h-1)/\tau} \); so we get the bound (A4.10).

**Appendix A7. Proof of lemma 7**

The proof is done by induction on \( \bar{h} \). Let us define \( J_{c,m}^{(h)} = [-|\varepsilon|, |\varepsilon|] \) and call \( J^{(h)} = \times_{[m] > 1, c = a,b} J_{c,m}^{(h)} \) and \( I^{(h)} = \times_{[m] > 1, c = a,b} I_{c,m}^{(h)} \).

We suppose that there exists \( I^{(h)} \) such that, if \( \nu \) spans \( I^{(h)} \) then \( \nu_{\bar{h}} \) spans \( J^{(h)} \) and \( |\nu_{\bar{h}}(c)| \leq C|\varepsilon| \) for \( \bar{h} \geq h \geq 0 \); we want to show that the same holds for \( \bar{h} + 1 \). Let us call \( \tilde{J}^{(h+1)} \) the interval spanned by \( \{\nu_{h+1,m}^{(c)}\}_{[m] > 1, c = a,b} \) when \( \{\nu_{m}^{(c)}\}_{[m] > 1, c = a,b} \) span \( I^{(h)} \). For any \( \{\nu_{h+1,m}^{(c)}\}_{[m] > 1, c = a,b} \) \( \in I^{(h)} \) one has \( \{\nu_{h+1,m}^{(c)}\}_{[m] > 1, c = a,b} \in [-2|\varepsilon| - C|\varepsilon|^2, 2|\varepsilon| + C|\varepsilon|^2] \), where the bound (5.13) has been used. This means that \( J^{(h+1)} \) is strictly contained in \( \tilde{J}^{(h+1)} \).

On the other hand it is obvious that there is a one-to-one correspondence between \( \{\nu_{h,m}^{(c)}\}_{[m] > 1, c = a,b} \) and the sequence \( \{\nu_{h+1,m}^{(c)}\}_{[m] > 1, c = a,b}, h + 1 \geq h \geq 0 \). Hence there is a set \( J^{(h+1)} \subset I^{(h)} \) such that, if \( \{\nu_{h,m}^{(c)}\}_{[m] > 1, c = a,b} \) spans \( I^{(h+1)} \), then \( \{\nu_{h+1,m}^{(c)}\}_{[m] > 1, c = a,b} \) spans the interval \( J^{(h)} \) and, for \( \varepsilon \) small enough, \( |\nu_{h}\| \leq C|\varepsilon| \) for \( \bar{h} + 1 \geq h \geq 0 \).
corresponding to

\( \beta \)

where (5.7) is even under the exchange

\( m \mid \)

an even function of \( m \).

\( \bar{\nu} \)

let us denote by \( \nu_{h,m}^{(c)} \) and \( \nu_{h,m}^{(q)} \), 0 \( \leq h \leq \bar{h} \), the sequences corresponding to \( \nu_{m}^{(c)} \) and \( \nu_{m}^{(q)} \) in \( I^{(h)} \). We have

\[
\nu_{h+1,m}^{(c)} - \nu_{h+1,m}^{(q)} = 2 \left( \nu_{h,m}^{(c)} - \nu_{h,m}^{(q)} \right) + \beta_{h,m}^{(c)} - \beta_{h,m}^{(q)},
\]

(A7.1)

where \( \beta_{h,m}^{(c)} \) and \( \beta_{h,m}^{(q)} \) are shorthands for the beta functions. Then, as \( |\nu_{h} - \nu_{k}^{q}| \leq |\nu_{h} - \nu_{h}^{q}| \) for all \( k \leq h \), we have

\[
|\nu_{h} - \nu_{h}^{q}| \leq \frac{1}{2}|\nu_{h+1} - \nu_{h+1}^{q}| + C|\varepsilon|^{2}|\nu_{h} - \nu_{h}^{q}|.
\]

(A7.2)

Hence if \( \varepsilon \) is small enough then one has

\[
|\nu - \nu^{q}| \leq (\sqrt{2})^{-(h+1)}|\nu_{h} - \nu_{h}^{q}|.
\]

(A7.3)

Since, by definition, if \( \nu \) spans \( I^{(h)} \), then \( \nu_{h} \) spans the interval \( J^{(h)} \), of size \( 2|\varepsilon| \), the size of \( I^{(h)} \) is bounded by \( 2|\varepsilon| (\sqrt{2})^{-(h+1)} \).

Finally note that one can choose \( \nu_{m}^{(c)} = \nu_{-m}^{(c)} \) and then \( \nu_{h,m}^{(c)} = \nu_{h,-m}^{(c)} \) for any \( m > 1 \) and any \( \bar{h} \geq h \geq 0 \); this follows from the fact that the function \( \beta_{h,m}^{(c)} \) in (5.7) is even under the exchange \( m \to -m \); it depends on \( m \) through \( \bar{\nu}_{m} \) (which is an even function of \( m \)), through the end-points \( v \in E(\theta) \) (which are odd under the exchange \( m \to -m \); but their number must be even) and finally through \( \nu_{k,m}^{(q-1)} \) which are assumed inductively to be even.

**Appendix A8. Proof of lemma 8**

(i) Assume \( \mu \) to be Diophantine with Diophantine constants \( C_{1}, \tau_{1} \), i.e.

\[
|\mu p + q| > C_{1}|p|^{-\tau_{1}} \quad \forall (p,q) \in \mathbb{Z}^{2} \setminus \{(0,0)\};
\]

(A8.1)

such \( \mu \) have full measure in \( \mathbb{R} \) if \( \tau_{1} > 1 \). From (A8.1), by recalling that \( \omega_{m} = m^{2} + \mu \), we have that, for all \( (n,m) \in \mathbb{Z}^{2} \setminus \{(0,0)\} \),

\[
|\omega_{n} - m\omega_{m}| = |\mu n^{2} + (n^{2} - m^{2})| > \frac{C_{1}}{|n|^{2\tau_{1}}},
\]

\[
|\omega_{n} - \omega_{m}| > \mu \left( 1 + (n^{2} - m^{2}) \right) > \frac{C_{1}}{|n|^{2\tau_{1}}},
\]

(A8.2)

which imply

\[
|\omega_{n} \pm m| > \frac{C_{2}}{|n|^{2\tau_{1}+1}},
\]

\[
|\omega_{n} \pm \omega_{m}| > \frac{C_{2}}{|n|^{2\tau_{1}+1}}.
\]

(A8.3)
from which the first two of (7.5) follow provided that one has $C_2 \geq C_0$ and $\tau_0 \geq 2\tau_1 + 1 > 3$.

(ii) Then we consider the last of (7.5). Suppose without loss of generality $n > 0$ and $m' \geq m > 0$. We discuss separately the cases $|\omega_1 n - (\omega_{m'} - \omega_m)|$ and $|\omega_1 n - (\omega_{m'} + \omega_m)|$.

Let us consider first the case $|\omega_1 n - (\omega_{m'} - \omega_m)|$; if $m' = m + \ell$ we can write

$$\omega_{m + \ell} - \omega_m = \ell + c_{\ell,m}, \quad 0 > c_{\ell,m} > -K\mu_0 \frac{\ell}{m^\xi}, \quad (A8.4)$$

for $\xi = 2$ and a suitable positive $K$ (one can take $K = 2$), and we consider separately the case $|c_{\ell,m}| < C_2 |n|^{-2\tau_1 - 1}/2$ and $|c_{\ell,m}| \geq C_2 |n|^{-2\tau_1 - 1}/2$.

In the first case by the second of (A8.3) one has

$$|\omega_1 n - \omega_{m + \ell} + \omega_m| \geq |\omega_1 n - \ell| - |c_{\ell,m}| \geq \frac{1}{2} \frac{C_2}{|n|^{2\tau_1 + 1}} > \frac{C_0}{|n|^\tau_0}, \quad (A8.5)$$

where the last inequality requires $C_0 \leq C_2/2$.

In the second case we can assume $|\omega_1 n - \omega_{m + \ell} + \omega_m| < 1$, because otherwise (7.5) is trivially satisfied: hence we have to consider only values of $n, m, \ell$ such that $0 < \omega_{m + \ell} - \omega_m < \omega_1 n + 1$ (recall that we are assuming $\ell, m, \omega_1 n > 0$).

The condition $|c_{\ell,m}| \geq C_2 |n|^{-2\tau_1 - 1}/2$ and the last inequality in (A8.4) imply

$$|m| \leq M \equiv \left( \frac{2\mu_0 K|n|^{2\tau_1 + 1}}{C_2} \right)^{1/\xi}, \quad (A8.6)$$

whereas the inequality

$$\omega_{m + \ell} - \omega_m = \frac{\omega_m^{2 \ell + \ell} - \omega_m^{2 \ell}}{\omega_{m + \ell} + \omega_m} = \frac{\ell}{\sqrt{1 + \mu_0}} \frac{(\ell + m) \sqrt{1 + \mu} + m \sqrt{1 + \mu}}{\omega_{m + \ell} + \omega_m} \geq \frac{\ell}{\sqrt{1 + \mu_0}} \frac{\ell}{\sqrt{1 + \mu}} \quad (A8.7)$$

implies

$$\frac{\ell}{\sqrt{1 + \mu_0}} \leq \omega_{m + \ell} - \omega_m \leq \omega_1 n + 1, \quad (A8.8)$$

so that one must have

$$\ell < L(|\omega_1 n| + 1), \quad L = \sqrt{1 + \mu_0}. \quad (A8.9)$$

At fixed $n, m, m'$ we can define a map $t \rightarrow \mu(t)$ such that

$$f(\mu(t)) \equiv \omega_1 (\mu(t)) n - \omega_{m + \ell}(\mu(t)) + \omega_m(\mu(t)) = \ell \frac{C_0}{|n|^\tau_0}. \quad (A8.10)$$

Then one has, if $I$ is the set of $\mu$ not verifying the Diophantine condition $|\omega_1 n - \omega_{m + \ell} + \omega_m| \geq C_0 |n|^{-\tau_0}$,

$$\int_I d\mu = \sum_{n=1}^{\infty} \sum_{\ell=1}^{L(|\omega_1 n| + 1)} \sum_{|m| \leq M} \int_{-1}^{1} \left| \frac{d\mu(t)}{dt} \right|, \quad (A8.11)$$

33
where \( dp(t)/dt \) can be obtained by noting that deriving (A8.10) with respect to \( t \) gives

\[
\frac{df(\mu(t))}{dt} = \frac{df}{d\mu} \frac{d\mu}{dt} = \frac{C_0}{n!^{n}}
\]

(A8.12)

Moreover

\[
\frac{df}{d\mu} = \frac{1}{2 \omega_1 \omega_m + \mu \omega_m} \left( n \omega_m \omega_m + \omega_1 (\omega_m + \mu \omega_m) \right)
\]

(A8.13)

\[
\geq \frac{n \omega_m + \mu \omega_m}{2 \omega_1 \omega_m + \mu \omega_m} \geq \frac{1}{2} \frac{1}{1 + \mu} \geq \frac{1}{2} \frac{1}{1 + \mu_0} = \frac{1}{2L},
\]

so that we find

\[
\int \frac{d\mu}{L} \leq 4L \sum_{n=0}^{\infty} \sum_{\ell=1}^L \sum_{|m| \leq M} \frac{C_0}{n!^{n}}
\]

(A8.14)

We can bound (A8.14) with a constant times

\[
(1 + \mu_0)^{1/2} \sum_{n=1}^{\infty} n^{-(\tau_0 - 1 - (2\tau_1 + 2)/\xi)},
\]

(A8.15)

which is proportional to \( C_0 \) provided that one has

\[
\tau_0 > 2 + \frac{2\tau_1 + 2}{\xi} > 4.
\]

(A8.16)

(iii) Now we pass to the case \(|\omega_1 n - \omega_m - \omega_m| < 1\) only if \((\omega_1 + 1)n \geq 1 + \omega_1 n > \omega_m + \omega_m > \omega_m \geq |n|\), and for fixed \( m \) there is a unique \( |m'| \) such that \(|\omega_{m'} + \omega_m - \omega_m| \leq 1\) (so that the summation over \( m' \) gives only a factor 2). Moreover for \( n = 1 \) one has \( \omega_m + \omega_m' > 2 \omega_1 \), so that, if \( C_0 \leq 1 \leq \omega_1 \), the Diophantine condition \(|\omega_1 n - \omega_m' - \omega_m| \geq C_0|n|^{-\tau_0} \) is automatically satisfied for \( n = 1 \).

At fixed \( n, m, m' \), with \( n \geq 2 \), by defining a map \( t \rightarrow \mu(t) \) through

\[
f(\mu(t)) \equiv \omega_1 (\mu(t)) n - \omega_m + \epsilon(\mu(t)) - \omega_m (\mu(t)) = t \frac{C_0}{|\mu|^{\tau_0}}, \quad t \in [-1, 1],
\]

(A8.17)

and using that for \( n \geq 2 \) one has

\[
n \omega_m \omega_m' \geq n \frac{\omega_m + \omega_m'}{2} \omega_m \geq \omega_m (\omega_m' + \omega_m),
\]

(A8.18)

we have

\[
\frac{df}{d\mu} = \frac{1}{2} \frac{1}{\omega_1 \omega_m + \mu \omega_m} \left( n \omega_m \omega_m + \omega_1 (\omega_m + \mu \omega_m) \right)
\]

(A8.19)

\[
\geq \frac{1}{4} \omega_1 \omega_m \frac{\omega_m + \omega_m'}{\omega_m'} \geq \frac{1}{4} \omega_1 \omega_m \left( \omega_m + \omega_1 \right)
\]

\[
\geq \frac{1}{16} \frac{m^2}{\omega_1} \geq \frac{1}{16} \frac{1}{1 + \mu^{3/2}} = \frac{1}{16L^3}.
\]
so that we have to exclude further a set $I'$ of values of $\mu$ with measure bounded by
\[
\int_{I'} d\mu \leq 32L^3 \sum_{n=1}^{\infty} \frac{C_0}{|n|^\tau_0},
\]
which is bounded by a constant times
\[
(1 + \mu_0)^2 C_0 \sum_{n=1}^{\infty} n^{-\tau_0 + 1},
\]
which is proportional to $C_0$ if $\tau_0$ satisfies (A8.16).

**Appendix A9. Proof of lemma 9**

*Proof. (i) We shall prove inductively on $p$ the bounds (7.7). From (7.4) we have
\[
|\tilde{\omega}^{(p)}_m(\omega, \varepsilon) - \tilde{\omega}^{(p-1)}_m(\omega, \varepsilon)| \leq C|\nu_m(\tilde{\omega}^{(p-1)}(\omega, \varepsilon), \varepsilon) - \nu_m(\tilde{\omega}^{(p-2)}(\omega, \varepsilon), \varepsilon)|,
\]
as we can bound $|\tilde{\omega}^{(p)}_m(\omega, \varepsilon) + \tilde{\omega}^{(p-1)}_m(\omega, \varepsilon)| \geq 1$ for $\varepsilon \in \mathcal{E}(p)$.

We set, for $|m| > 1$
\[
\Delta \nu_{h,m} \equiv \nu_{h,m}(\tilde{\omega}^{(p-1)}(\omega, \varepsilon), \varepsilon) - \nu_{h,m}(\tilde{\omega}^{(p-2)}(\omega, \varepsilon), \varepsilon) = \lim_{q \to \infty} \Delta \nu_{h,m}^{(q)},
\]
where we have used the notations (6.7) to define
\[
\Delta \nu_{h,m}^{(q)} = \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-1)}(\omega, \varepsilon), \varepsilon) - \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-2)}(\omega, \varepsilon), \varepsilon).
\]

We want to prove inductively on $q$ the bound
\[
|\Delta \nu_{h,m}^{(q)}| \leq C\varepsilon^2 \|\tilde{\omega}^{(p-1)}(\omega, \varepsilon) - \tilde{\omega}^{(p-2)}(\omega, \varepsilon)\|_\infty,
\]
for some constant $C$, uniformly in $q, h$ and $m$.

(ii) For $q = 0$ the bound (A9.4) is trivially satisfied. Then assume that (A9.4) hold for all $q' < q$.

For simplicity we set $\tilde{\omega} = \tilde{\omega}^{(p-1)}(\omega, \varepsilon)$ and $\tilde{\omega}' = \tilde{\omega}^{(p-2)}(\omega, \varepsilon)$. We can write, from (6.7) and (6.6), for $|m| > 1$
\[
\Delta \nu_{h,m}^{(q)} = -\sum_{k=h}^{h-1} 2^{-k-2} \left( \beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)) - \beta_{k,m}^{(q)}(\tilde{\omega}', \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}', \varepsilon)) \right),
\]
where we recall that $\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon))$ depend only on $\nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)$ with $k' \leq k - 1$, and we can set
\[
\beta_{k,m}^{(q)}(\tilde{\omega}, \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)) = \beta_{k,m}^{(a)(q)}(\tilde{\omega}, \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)) - \beta_{k,m}^{(b)(q)}(\tilde{\omega}, \varepsilon, \nu_{k'}^{(q-1)}(\tilde{\omega}, \varepsilon)),
\]

35
to the analysis performed in Section 5, is that we cannot apply directly lemma 5.

Note that the above formula holds for any value of \( \tilde{\omega} \), support properties of the functions \( \chi \) inductively for \( L \) according to the settings in (1.17). Then we can split the differences in (A9.5) into

\[
\beta^{(q)}_{k,m}(\tilde{\omega}, \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) - \beta^{(q)}_{k,m}(\tilde{\omega}', \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) \\
= \left( \beta^{(q)}_{k,m}(\tilde{\omega}, \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) - \beta^{(q)}_{k,m}(\tilde{\omega}, \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) \right) \\
+ \left( \beta^{(q)}_{k,m}(\tilde{\omega}', \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) - \beta^{(q)}_{k,m}(\tilde{\omega}', \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) \right),
\]

and we bound separately the two terms.

(iii) The second term can be expressed as sum of trees \( \theta \) which differ from the previously considered ones as, among the nodes \( v \) with only one entering line, \( v \)there are some with \( \nu_{k}(v)(\tilde{\omega}', \epsilon) \), some with \( \nu_{k}(v)(\tilde{\omega}', \epsilon) \) and one with \( \nu_{k}(v)(\tilde{\omega}', \epsilon) - \nu_{k}(v)(\tilde{\omega}', \epsilon) \). Then, by the inductive hypothesis, we can bound

\[
\beta^{(q)}_{k,m}(\tilde{\omega}, \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) - \beta^{(q)}_{k,m}(\tilde{\omega}, \epsilon, \{\nu_{k}^q(\tilde{\omega}', \epsilon)\}) \\
\leq D_1 \varepsilon^2 \sup_{h' \geq 0} \sup_{|\eta| \geq 1} \Delta_{v}(\tilde{\omega}') \leq D_1 C \varepsilon^4 \|\tilde{\omega}' - \tilde{\omega}\|_{\infty},
\]

where we have used that for all \( q' \) the trees contributing to \( \beta^{(q)}_{k,m} \) contain at least two end-points, so that the first nonvanishing contribution to the left hand side of (A9.8) is to second order.

(iv) We are left with the first term in (A9.7). We write \( \nu_{k}^{(c,q)}(v) \) as in (5.8), with \( k \) replacing \( h' \); for each \( T \in \mathcal{T}_k^{(c)} \) we can order the lines in \( L(T) \) and construct a set of \( k_T \) subsets \( L_1(T), \ldots, L_{k_T}(T) \) of \( L(T) \), with \( |L_j(T)| = j \), in the following way. Set \( L_1(T) = \emptyset \), \( L_2(T) = \ell_1 \), if \( \ell_1 \) is a line connected to the outcoming line of \( T \), and, inductively for \( k_T \geq 3 \) and \( 2 \leq j \leq k_T - 1 \), \( L_{j+1}(T) = L_j(T) \cup \ell_j \), where the line \( \ell_j \in L(T) \setminus L_j(T) \) connected to \( L_j(T) \); of course \( L_{k_T}(T) = L(T) \). Then

\[
\nu_{T}^{k+1}(\pm \tilde{\omega}, m) = \varepsilon^k \left( \prod_{v \in V(T)} \eta_v \right) \\
\sum_{j=1}^{k_T} \left( \prod_{T \in L_j(T)} g_{\ell_j}^{(h_j)}(\tilde{\omega}') \right) \left( g_{\ell_j}^{(h_j)}(\tilde{\omega}') - g_{\ell_j}^{(h_j)}(\tilde{\omega}) \right) \left( \prod_{T \in L_j(T)} g_{\ell_j}^{(h_j)}(\tilde{\omega}) \right),
\]

where we have use the shortened notation \( g_{\ell_j}^{(h_j)}(\tilde{\omega}) = g^{(h_j)}(\tilde{\omega}) \), and, by construction, the sets \( L_j(T) \) are connected (while the sets \( L(T) \setminus L_j(T) \) in general are not).

In (A9.9) one has, by proceeding as in (A2.3)

\[
\left| g_{\ell_j}^{(h_j)}(\tilde{\omega}') - g_{\ell_j}^{(h_j)}(\tilde{\omega}) \right| \leq C \varepsilon \|\tilde{\omega}' - \tilde{\omega}\|_{\infty},
\]

Note that the above formula holds for any value of \( \tilde{\omega}, \tilde{\omega}' \), thanks to the compact support properties of the functions \( \chi \) in the definition of \( g_{\ell_j}^{(h_j)} \).

The main difference in bounding the other factors appearing in (A9.9), with respect to the analysis performed in Section 5, is that we cannot apply directly lemma 5,
because in (A9.9) there are some propagators with $\tilde{\omega}$ and some other propagators with $\tilde{\omega}'$. However this is not a problem as we can divide the tree in the union of (properly modified, see lemma 4) subtrees, each one having all propagators with the same kind of frequency (that is either $\tilde{\omega}$ or $\tilde{\omega}'$), and we can bound separately each of such trees.

Consider a self-energy graph $T$. For fixed $j$ (see (A9.9)) call $\theta_0$ the tree obtained in the following way. Call $L_0$ the set of lines containing the line $\ell_2^j$ coming out from $T$, all the lines in $L_j(T)$ and all the $s \geq 0$ lines in $L_0'$, where $L_0'$ is the set of lines in $L(T) \setminus L_j(T)$ connected to $L_j(T)$. Call $V_0$ the set of nodes connected by lines in $L_j(T)$. We add $s$ end-points attaching them at the beginning of the $s$ lines in $L_0'$; we call $E''$ the set of such end-points and $E'$ the set of end-points connected to lines in $L_j(T)$, and we set $\tilde{E} = E' \cup E''$. Then we call $\theta_0$ the tree such that $L(\theta_0) = L_0$, $V(\theta_0) = V_0$ and $\tilde{E}(\theta_0) = \tilde{E}$; note that, by construction, such a tree has also end-points with mode labels different from $(\pm 1, \pm 1)$, and $|E_0(\theta_0)| = 1$ and $|E_1(\theta_0)| \geq 0$ (we use the notations introduced before lemma 4).

Consider also the subtrees which have as root lines the $s$ lines in $L_0$, respectively. One of such trees, say $\theta'_1$, contains the node $v$ such that the entering line of $T$ enters $v$; call $\theta_1$ the tree obtained from $\theta'_1$ by replacing the line $\ell_2^j$ with a line emerging from an end-point $v_0$ carrying a mode label $(\overline{m}, m)$. Then we denote with $\theta_1, \ldots, \theta_s$ the $s$ subtrees obtained in this way. For all $i = 1, \ldots, s$ one has $\tilde{E}(\theta_i) = E(\theta_i)$, while for $s = 1$ one has $\tilde{E}(\theta_i) = E(\theta_i) + E_0(\theta_i)$.

Finally associate to each end-point $v_i$ in $E_1(\theta)$ a mode label $(n_i, m_i)$ if $(n_i, m_i)$ is the momentum flowing through the root line of the subtree $\theta_i$. Moreover if $v_1$ is the end-point in $E_0(\theta_0)$ the corresponding mode labels are $(\overline{m} + \overline{m}, \overline{m} + m)$, with $\overline{m} = \sum_{v \in E(\theta)} n_v = n_1$ and $\overline{m} = \sum_{v \in E(\theta)} m_v = m_1$.

One among the trees $\theta_1, \ldots, \theta_s$ has as root line the line $\ell_j$; suppose for simplicity that $\theta_j$ is such a tree (if not the discussion proceeds in a similar way). For an example and an explicit construction see Figures A9.1 and A9.2.

(v) We can distinguish two cases. If there are no self-energy graphs $T'$ internal to $T$ with outcome line in $L_0$ and entering line outside $L_0$, then one can write

$$V^{k+1}_T(x_m, m) = \text{Val}'(\theta_0) \left( B_{(\theta_1)}(\tilde{\omega}') - g_{(\theta_1)}(\tilde{\omega}) \right) \text{Val}'(\theta_1) \left( \prod_{i=2}^s \text{Val}(\theta_i) \right), \quad (A9.11)$$

where $\text{Val}'(\theta)$ is defined after (A4.10).

Note that $\text{Val}(\theta_0)$, $\text{Val}'(\theta_1)$ and $\text{Val}(\theta_i)$, for $i = 2, \ldots, s$, admit the bounds of lemma 2 and lemma 4. Therefore one has

$$\prod_{i=2}^s |\text{Val}(\theta_i)| \leq D_0 \prod_{i=1}^s D^{K(\theta_i)}|\mathcal{E}|^{K(\theta_i)}, \quad (A9.12)$$

$$|\text{Val}'(\theta_1)| \leq D_0 D^{K(\theta_1)}|\mathcal{E}|^{K(\theta_1)},$$

where, for all $i = 1, \ldots, s$, the mode labels $n_i$ and $m_i$ are given by the sum of the labels $n_i$ and $m_i$, respectively, of the end-points of $\theta_i$. The first bound in (A9.12) follows from lemma 2 (as for $i = 2, \ldots, s$ the trees $\theta_i$ have $\tilde{E}(\theta_i) = E(\theta_i)$), while the
Call $\ell^{(1)}, \ldots, \ell^{(5)}$ the five lines connected to $\ell^T_T$, ordered up to down. Consider the summand $j = 4$ in (A9.9), and suppose that one has $\ell_j = \ell_4 = \ell^{(1)}$. The set $L_0$ is formed by the line $\ell^T_T$ coming out from $T$ and by the lines $\ell^{(2)}, \ell^{(3)}$ and $\ell^{(4)}$ in $L_4(T)$, while $L_0'$ is formed by the lines $\ell^{(1)}$ and $\ell^{(5)}$. The trees $\theta'_1$ and $\theta'_2$ have $\ell_1$ and $\ell_5$ as root lines, respectively. In particular $\theta_1$ contains the node $v$ such that the line $\ell^T_T$ enters $v$, and the root line of $\theta_1$ is the line $\ell^T_4$.

The second one follows from lemma 4 and the fact that one has $\tilde{E}(\theta_1) = E(\theta_1) + E_0(\theta_1)$ and $(\overline{n}, \overline{m}) = (0, 0)$ for the node $v_0 \in E_0(\theta_1)$, so that $\tilde{K}(\theta_1) = K(\theta_1)$.

Analogously the bound of $\text{Val}'(\theta_0)$ can be obtained from lemma 4, which gives

$$|\text{Val}'(\theta_0)| \leq D_0 D \tilde{K}(\theta_0) |\varepsilon| K(\theta_0), \quad (A9.13)$$

where $\tilde{K}(\theta_0) = K(\theta_0) + |n_1| + \ldots + |n_s| \leq K(\theta_0) + K(\theta_1) + \ldots + K(\theta_s)$.

We can take into account the bound (A9.10) simply by noting that the factor $2^{-2h_{\ell_j}}$ can be associated to the tree $\theta_1$, so that in (A9.11) we can replace $\text{Val}'(\theta_1)$ with a quantity which can be bounded by replacing the propagators of $\text{Val}'(\theta_1)$ by their squares (of course we are using that the scales are all positive).

At the end we obtain the bound

$$\left| \text{Val}'(\theta_0) \left( g^{(h_{\ell_j})}_{\ell_{\ell_j}}(\tilde{\omega}') - g^{(h_{\ell_j})}_{\ell_{\ell_j}}(\tilde{\omega}) \right) \text{Val}'(\theta_1) \left( \prod_{i=2}^s \text{Val}(\theta_i) \right) \right| \leq C^{K(T)} D^{2K(T)} |\varepsilon|^{K(T)} \|\tilde{\omega}' - \tilde{\omega}\|_{\infty}$$

$$\leq C^{K(T)} D^{2K(T)} |\varepsilon|^{K(T)/2} |\varepsilon|^{2(k-1)/2} \|\tilde{\omega}' - \tilde{\omega}\|_{\infty}, \quad (A9.14)$$
Figure A9.2. The trees $\theta_0$, $\theta_1$ and $\theta_2$ obtained from $T$ according to the prescription given in the text: the tree $\theta_1$ is obtained from $\theta'_1$ by replacing the line $\ell^2_T$ with a line emerging from a end-point $v_0$.

where $K(T)$ is defined in the statement of lemma 5, and the last inequality is obtained as explained at the end of the proof of lemma 6.

(vi) Suppose now that there are self-energy graphs with outcoming line in $L_0$ and entering line outside $L_0$; in such a case we split the renormalized part of the value of the maximal $T'$ among such self-energy graphs (note that there cannot be two such maximal self-energy graphs) as $RV_{T'} = V_{T'} - L V_{T'}$, so obtaining two contributions. Moreover if in the contribution in which $V_{T'}$ was selected there is still other self-energy graphs contained in $T'$ with outcoming line in $L_0$ and entering line outside $L_0$ again we split the maximal $T''$ among them as $RV_{T''} = V_{T''} - L V_{T''}$, and so on. At the end we obtain a sum of contributions such that no renormalized self-energy graph can have the outcoming line in $L_0$ and the incoming line outside $L_0$. If we are left with a localized self-energy graph $LV_{T'}$, with external lines one in $L_0$ and the other outside $L_0$, then it can be factorized away and the remaining part can be bounded as in item (v) above. In order to bound $LV_{T'}$ we note that it has exactly the form (A9.9), i.e. it contains a product of propagators $g^{(h_0)}(\tilde{\omega}')$, a product of propagators $g^{(h_1)}(\tilde{\omega})$ and a difference of propagators $g^{(h_{i_1})}(\tilde{\omega}') - g^{(h_{i_1})}(\tilde{\omega})$, so that it can bounded exactly as above.

At the end, for a fixed choice of the scales, the total number of terms so obtained is bounded by $\tilde{C}^k$, for some constant $\tilde{C}$, so that a bound $\tilde{C}^k$ times the right hand side of (A9.14) follows again. In the course of the above procedure we are loosing the factor $2^{-h_{i_1}^T + h_T}$, see (A3.1), for all the self energy graphs $T', T'', ...$ with an external line in $L_0$ and another outside $L_0$; but by definition such clusters are a sequence of clusters contained one in the other, hence we have at most an extra factor $2^h$ for each cluster.
with scale $h$, hence the final bound is still given by (A3.2) with log 4 replacing log 2.

(vii) A similar, but simpler, analysis holds in the case $m = 1$; in that case $\nu_1$ is given by (2.7).

(viii) Now we can draw the conclusion from the construction above. We can bound $\|\tilde{\phi}_{(p)}(\omega, \varepsilon) - \tilde{\phi}_{(p-1)}(\omega, \varepsilon)\|_\infty$ with a constant times $|\varepsilon|^2$ times the same expression with $p$ replaced with $p - 1$, i.e. $\|\tilde{\phi}_{(p-1)}(\omega, \varepsilon) - \tilde{\phi}_{(p-2)}(\omega, \varepsilon)\|_\infty$, so that, by the inductive hypothesis, the bound (7.7) follows.

Appendix A10. Proof of lemma 10

As $\nu_m = \nu_{-m}$ we can consider only $m \geq 1$. For $m \geq 2$ one has

$$\nu_m = \nu_m^{(a)} - \nu_m^{(b)}, \quad \nu_m^{(a)} = O(\varepsilon^2), \quad \nu_m^{(b)} = O(\varepsilon^{\min(2m, M-1)}),$$  \hspace{1cm} (A10.1)

and, if $2m > M$, a trivial computation gives

$$\nu_m^{(a)} = (-1)^{2(M-1)/2} M \left( \frac{M-1}{(M-1)/2} \right)^2 \left( \frac{1}{4t} \right)^{M-1} \varepsilon^{M-1} + O(\varepsilon^M),$$  \hspace{1cm} (A10.2)

where the factor $M$ is due to the possible choices of the line entering the self-energy graph contributing to $\nu_m$, while the square of the binomial factor is due to the possible choices of the labels $(n_1, m_1), \ldots, (n_{M-1}, m_{M-1})$ of the two nodes internal to the self-energy graph such that $n_1 + \ldots + n_{M-1} = 0$ and $m_1 + \ldots + m_{M-1} = 0$.

For $m = 1$ one has

$$\nu_1 = (-1)^{2(M-1)/2} \left( \frac{M}{(M-1)/2} \right)^2 \left( \frac{1}{4t} \right)^{M-1} \varepsilon^{M-1} + O(\varepsilon^M),$$  \hspace{1cm} (A10.3)

where the square of the combinatorial factor is due to the possible choices of the labels $(n_1, m_1), \ldots, (n_M, m_M)$ of the $M$ nodes internal to the tree such that $n_1 + \ldots + n_M = 1$ and $m_1 + \ldots + m_M = 1$.

Therefore, for $2m > M$, the lower bound on $|n\alpha_M - \beta_M|$, $n \in \mathbb{Z}$, is an easy check from the explicit expressions in (A10.2) and (A10.3), which yield

$$|n\alpha_M - \beta_M| \geq \frac{M}{4M-1} \left( \frac{M-1}{(M-1)/2} \right)^2 \left| \frac{M}{(M+1)/2} - n \right| \varepsilon^{M-1},$$  \hspace{1cm} (A10.4)

where the term in the last parantheses is greater than 1/4 (and it equalt to 1/4 for $M = 3$ and $n = 1$).

If $2m < M$ becomes a little more involved, as one has to take into account also the terms $\nu_m^{(b)}$, which has a dominant contribution given by the same expression as for $\nu_m^{(a)}$ times the factor

$$(-1)^{-m} \left( \frac{M - 2}{2} + 1 \right) \ldots \left( \frac{M - 2}{2} + m \right) \left( \frac{M - 2}{2} - 1 \right) \ldots \left( \frac{M - 2}{2} - m \right).$$  \hspace{1cm} (A10.5)

40
but it is easy to realize that the same conclusions still hold (with 1/4 replaced with 1/9, which is the value of the difference \(|n\alpha_M - \beta_M|\) computed at \(M = 5\) and \(n = 1\) for \(m = 1\)).

**Appendix A11. Proof of lemma 11**

In order to prove the bounds (7.15) note that

\[
\left| g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - \partial_{\omega} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})(\tilde{\omega} - \tilde{\omega}') \right| \leq C n_j 2^{-3h_{\ell_j}} \|\tilde{\omega}' - \tilde{\omega}\|_{\infty}^2, \tag{A11.1}
\]

from the compact support properties of the propagator.

Let us consider the quantity \(\nu(\tilde{\omega}', \epsilon) - \nu(\tilde{\omega}, \epsilon) - \partial_{\omega} \nu(\tilde{\omega}, \epsilon)(\omega - \omega')\), where \(\partial_{\omega} \nu(\tilde{\omega}, \epsilon)\) denotes the derivative in the sense of Whitney, and note that it can be expressed as a sum over trees each one containing a line with propagator \(g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - \partial_{\omega} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})(\tilde{\omega} - \tilde{\omega}')\), by proceeding as in the proof of lemma 9. Then we find

\[
\|\nu(\tilde{\omega}', \epsilon) - \nu(\tilde{\omega}, \epsilon) - \partial_{\omega} \nu(\tilde{\omega}, \epsilon)(\omega - \omega')\|_{\infty} \leq C_1 \epsilon^2 (M-1) \|\tilde{\omega}' - \tilde{\omega}\|_{\infty}^2, \tag{A11.2}
\]

and the first bound in (7.15).

In order to prove the first bound in (7.15) we proceed by induction by assuming that it holds for \(\hat{\omega}^{(p-1)}\); then from (7.4) we have

\[
2\hat{\omega}^{(p)} \partial_{\omega} \hat{\omega}^{(p)}(\omega, \epsilon) = -\partial_{\omega} \nu_m(\hat{\omega}^{(p-1)}(\omega, \epsilon), \epsilon)
\]

\[
= -\partial_{\omega} \nu_m(\hat{\omega}^{(p-1)}(\omega, \epsilon), \epsilon) \partial_{\omega} \hat{\omega}^{(p-1)}(\omega, \epsilon) - \partial_{\omega} \nu_m(\hat{\omega}^{(p-1)}(\omega, \epsilon), \eta) \bigg|_{\eta = \epsilon},
\]

hence from the inductive hypothesis and the proved bound in (7.15), we obtain

\[
\|\hat{\omega}^{(p)}(\omega, \epsilon') - \hat{\omega}^{(p)}(\omega, \epsilon) - \partial_{\omega} \hat{\omega}^{(p)}(\omega, \epsilon)(\epsilon - \epsilon')\|_{\infty} \leq C |\epsilon|^{M-3} |\epsilon' - \epsilon|^2, \tag{A11.4}
\]

so that also the second bound in (7.15) follows.

**Appendix A12. Proof of lemma 12**

First of all we check that, if we call \(\varepsilon_j^{(p)}(n)\) the centers of the intervals \(\mathcal{T}_j^{(p)}(n)\), with \(j = 1, 2, 3\), then one has

\[
|\varepsilon_j^{(p+1)}(n) - \varepsilon_j^{(p)}(n)| \leq D \varepsilon_0^p, \tag{A12.1}
\]

for a suitable constant \(D\).

The center \(\varepsilon_1^{(p)}(n)\) is defined by the condition

\[
\omega_1 n - \mu_1(\hat{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n)) n - \omega_m + \mu_m(\hat{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n)) = 0, \tag{A12.2}
\]

where Whitney extensions are considered outside \(\mathcal{E}^{(p-1)}\); then, by subtracting (A12.2) from the equivalent expression for \(p + 1\), we have

\[
\left(\mu_1(\hat{\omega}^{(p)}(\varepsilon_1^{(p+1)}(n)), \varepsilon_1^{(p+1)}(n)) - \mu_1(\hat{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n))\right) n
\]

\[
- \left(\mu_m(\hat{\omega}^{(p)}(\varepsilon_1^{(p+1)}(n)), \varepsilon_1^{(p+1)}(n)) - \mu_m(\hat{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)), \varepsilon_1^{(p)}(n))\right) = 0. \tag{A12.3}
\]
In (7.28) we can write, by setting for simplicity \( \varepsilon = \varepsilon^{(p)}_1(n), \varepsilon' = \varepsilon^{(p+1)}_1(n), \bar{\omega} = \bar{\omega}^{(p)} \) and \( \bar{\omega} = \bar{\omega}^{(p-1)} \),

\[
\begin{align*}
(\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon)) n \\
= (\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon'), \varepsilon')) n + (\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon')) n
\end{align*}
\]  
\[\text{(A12.4)}\]

where we can write, from lemma 11,

\[
\begin{align*}
|\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon)) n| &\leq C|\varepsilon|^{2(M-1)}|n| \|\omega' - \omega\|_\infty, \\
|\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon')) n| &\leq C|\varepsilon|^{2(M-1)}|n| |\varepsilon' - \varepsilon|,
\end{align*}
\]  
\[\text{(A12.5)}\]

and by lemma 10 one has

\[
(\mu_1(\bar{\omega}(\varepsilon), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon)) n = (n \alpha_M (M - 1) \varepsilon^{m-2} (1 + O(\varepsilon^2))) (\varepsilon' - \varepsilon). \quad \text{(A12.6)}
\]

Moreover

\[
\begin{align*}
\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon) &= \mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon'), \varepsilon') \\
&+ \mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon') + (\mu_1(\bar{\omega}(\varepsilon), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon)),
\end{align*}
\]  
\[\text{(A12.7)}\]

and, from lemma 11,

\[
\begin{align*}
|\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon'), \varepsilon')| &\leq C|\varepsilon|^{2(M-1)} \|\omega' - \omega\|_\infty, \\
|\mu_1(\bar{\omega}(\varepsilon'), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon')| &\leq C|\varepsilon|^{2(M-1)}|n| |\varepsilon' - \varepsilon|,
\end{align*}
\]  
\[\text{(A12.8)}\]

and by lemma 10 one has

\[
\mu_1(\bar{\omega}(\varepsilon), \varepsilon') - \mu_1(\bar{\omega}(\varepsilon), \varepsilon) = (2\beta_M \varepsilon + O(\varepsilon^2)) (\varepsilon - \varepsilon'). \quad \text{(A12.9)}
\]

Then we get, by lemma 9,

\[
|\varepsilon^{(p+1)}_1(n) - \varepsilon^{(p)}_1(n)| \leq C\varepsilon^{p+M}_0,
\]  
\[\text{(A12.10)}\]

for a suitable positive constant \( C \). This proves the bound (A12.1).

Define \( p_0 \equiv p_0(n, j) \) such that

\[
|\varepsilon^{(p+1)}_j(n) - \varepsilon^{(p)}_j(n)| \leq C_0 |n|^{-r}. \quad \text{(A12.11)}
\]

By (A12.10) one can choose

\[
p_0 = p_0(n, j) \leq \text{const. log } |n|. \quad \text{(A12.12)}
\]

For all \( p \leq p_0 \) define \( J^{(p)}_j(n) \) as the set of values \( \varepsilon \) such that (7.11), (7.12) and (7.13) are satisfied with \( C_0 \) replaced with \( 2C_0 \). By the definition (A12.11) all the intervals
\(\mathcal{I}_j^{(p)}(n)\) fall inside the union of the intervals \(\mathcal{J}_j^{(0)}(n), \ldots, \mathcal{J}_j^{(p_0)}(n)\) as soon as \(p > p_0\). This means that, by calling

\[
\mathcal{I} = \bigcup_{p=0}^{\infty} \bigcup_{n \in \mathbb{Z}^d} \bigcup_{j=1,2,3} \mathcal{I}_j^{(p)}(n),
\]

so that \(\mathcal{E} = \mathcal{E}^{(0)} \setminus \mathcal{I}\), we can bound \(\text{meas}(\mathcal{I})\) with

\[
\text{meas}(\mathcal{I}) \leq \sum_{|n| \geq N_0} \sum_{j=1,2} \text{meas}(\mathcal{I}_j^{(p)}(n)) \leq \text{const.} \sum_{|n| \geq N_0} \sum_{j=1,2} \sum_{p=0}^{p_0(n,j)} \frac{2C_0}{|n|^\tau'} \leq \sum_{n=N_0} n^{-\tau'} \log n \leq b \varepsilon_0^{1+\xi},
\]

with \(\tau'\) verifying (7.33) and (7.46), and a constant \(\xi\) smaller than \(\xi'\) in order to take into account the logarithmic corrections due to (A12.12). Then the bound (7.48) is proved.

References


