

# Forced quasi-periodic oscillations in strongly dissipative systems of any finite dimension

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## Abstract

We consider a class of singular ordinary differential equations describing analytic systems of arbitrary finite dimension, subject to a quasi-periodic forcing term and in the presence of dissipation. We study the existence of response solutions, i.e. quasi-periodic solutions with the same frequency vector as the forcing term, in the case of large dissipation. We assume the system to be conservative in the absence of dissipation, so that the forcing term is — up to the sign — the gradient of a potential energy, and both the mass and damping matrices to be symmetric and positive definite. Further, we assume a non-degeneracy condition on the forcing term, essentially that the time-average of the potential energy has a strict local minimum. On the contrary, no condition is assumed on the forcing frequency; in particular we do not require any Diophantine condition. We prove that, under the assumptions above, a response solution always exists provided the dissipation is strong enough. This extends results previously available in literature in the one-dimensional case.

## 1 Introduction

Consider the singular non-autonomous ordinary differential equation on  $\mathbb{R}^m$

$$\varepsilon \ddot{x} + \Gamma \dot{x} + \varepsilon g(x) = \varepsilon f(\omega t), \quad (1.1)$$

where  $x = (x_1, \dots, x_m)$ , the dots denote derivatives with respect to time  $t$ ,  $\omega \in \mathbb{R}^d$  is the frequency vector of the forcing term,  $\varepsilon \in \mathbb{R}_+$  is a small parameter and  $\Gamma$  is a positive definite symmetric  $m \times m$  real matrix. The functions  $f : \mathbb{T}^d \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$  are assumed to be real analytic, with  $f$  quasi-periodic in  $t$ , so we can write its Fourier series expansion as

$$f(\psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i\nu \cdot \psi} f_\nu,$$

where  $\psi \in \mathbb{T}^d$  and  $f_\nu$  are the Fourier coefficients. Here and henceforth  $\cdot$  denotes the standard scalar product in  $\mathbb{R}^d$ , i.e.  $\nu \cdot \psi = \nu_1 \psi_1 + \dots + \nu_d \psi_d$ . Since the results we are going to discuss are local, we could assume  $g$  to be defined only in an open set  $\mathcal{A} \subset \mathbb{R}^m$ .

Systems described by equations of the form (1.1) naturally arise in classical mechanics and electronic engineering [32, 3, 1, 18, 4, 15, 6]. As a mechanical system, (1.1) describes

a collection of points of unit mass; as we show in Section 5, the analysis easily extends to the case of points with different masses. The points interact through a mechanical force  $-g$  and are subject to a forcing term  $f$ , while the term proportional to the velocity (drag) takes into account dissipation effects — described as dashpots or dampers — due, for instance, to air resistance, fluid viscosity or friction in the gears of a device. Thus,  $\Gamma$  is called the *damping matrix*. In (1.1) we assume the matrix  $\Gamma$  to be a positive definite symmetric matrix, as such a property is satisfied in many problems of physical relevance arising in fluid dynamics, electrodynamics and mechanics [1, 16]. While in principle one can also consider situations in which the matrix  $\Gamma$  is not symmetric [3], the results we shall present in this paper hold for symmetric damping matrices. The parameter  $\varepsilon$  plays the role of a *perturbation parameter* and measures the intensity of the dissipation (for this reason we take it to be positive): small  $\varepsilon$  means strong dissipation.

We also assume that the system is conservative in the absence of dissipation, so that there exists a real analytic function  $V: \mathbb{R}^m \rightarrow \mathbb{R}$  such that

$$g(x) = \frac{\partial V}{\partial x}(x), \tag{1.2}$$

i.e. the mechanical force  $-g$  is a gradient and  $V$  is the potential energy.

We are interested in studying the existence, for small values of  $\varepsilon$ , of *response solutions* to (1.1), i.e. quasi-periodic solutions with the same frequency vector  $\omega$  as the forcing [11]. If  $\varepsilon = 0$ , for any constant vector  $c \in \mathbb{R}^m$ ,  $x = c$  is a solution to (1.1). The problem we consider is whether it is possible to choose the constant vector  $c$  so that, for  $\varepsilon$  small enough, the equation (1.1) admits a response solution which tends to  $c$  as  $\varepsilon$  tends to zero.

The case  $d = 1$  has been already studied in the literature. Without any assumptions on the functions  $f$  and  $g$ , the answer in general is negative [19]. The analysis in [25] shows that, for any frequency vector  $\omega$ , if  $c$  is a simple zero of the function  $h(x) := g(x) - f_0$ , then for  $\varepsilon$  small enough there is a response solution which tends to  $c$  as  $\varepsilon$  tends to zero. This strengthens previous results, where non-resonance conditions were assumed on  $\omega$  [22, 9, 12, 13]. More or less strong non-resonance conditions assure some regularity in the dependence on  $\varepsilon$  of the solution; in particular, without assuming any condition on  $\omega$ , in general no more than a continuous dependence on  $\varepsilon$  can be obtained, while smoothness and analyticity – and even Borel-summability – in domains tangential to the imaginary axis follow if suitable Diophantine conditions are assumed on  $\omega$  [23, 9, 12]. On the other hand, on physical grounds, in the presence of dissipation one may expect a response solution to exist independently on the arithmetic properties of the frequency vector. However, from a mathematical point of view, quasi-periodicity typically involves small divisor problems, and to prove the existence of quasi-periodic solutions for any rotation vector may be a rather delicate problem, also in the context of singularly perturbed systems such as (1.1); see [25] for further references and comments.

Existence of quasi-periodic solutions in singular ordinary differential equations has been extensively studied in the literature. Usually one makes some non-degeneracy assumption on the unperturbed solution, which ensures that the response solution either smoothly appears by bifurcation, due to the loss of stability of an isolated solution, such as an equilibrium point or a periodic solution, or persists as a stable or hyperbolic central

manifold; see for instance [2, 7, 8, 30]. By contrast, in (1.1) any constant is a solution at  $\varepsilon = 0$ , so that the unperturbed bifurcating solution is not given a priori; in particular it does not satisfy any stability or hyperbolicity condition.

In this paper, we extend the result in [25] to the higher dimensional case  $m > 1$ . We may and shall only require that the components of  $\omega$  are rationally independent, i.e.  $\omega \cdot \nu \neq 0 \forall \nu \in \mathbb{Z}_*^d := \mathbb{Z}^d \setminus \{0\}$ . This is not restrictive, because one can always reduce to such a case, possibly changing the value of  $d$ ; indeed, if  $\omega \in \mathbb{R}^d$  is resonant with multiplicity  $r < d$  (i.e. if there is a rank  $r$  subgroup  $G$  of  $\mathbb{Z}^d$  such that  $\omega \cdot \nu = 0$  for all  $\nu \in G$  and  $\omega \cdot \nu \neq 0$  for all  $\nu \in \mathbb{Z}^d \setminus G$ ), then, in some coordinate system,  $\omega$  can be written as  $\omega = (\tilde{\omega}, 0)$ , such that  $\tilde{\omega} \in \mathbb{R}^{d_1}$ , with  $d_1 = d - r$ , is rationally independent and the function  $f$  may be viewed as a function of the form  $f(\omega t) = \tilde{f}(\tilde{\omega} t)$ .

Like in the one-dimensional case, some non-degeneracy condition involving the functions  $f$  and  $g$  must be required. Let us define the function

$$U(x) := V(x) - \langle f_0, x \rangle, \quad (1.3)$$

where  $V$  is the potential energy associated to  $g$  through (1.2) and  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product in  $\mathbb{R}^m$ .

**Hypothesis 1.** *The function  $U(x)$  in (1.3) has a strict local minimum point  $c$ .*

We shall prove the following result.

**Theorem 1.** *Consider the ordinary differential equation (1.1), with  $g$  as in (1.2), and assume Hypothesis 1. For any rationally independent frequency vector  $\omega \in \mathbb{R}^d$ , there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there is a quasi-periodic solution  $x_0(t) = c + X(\omega t, \varepsilon)$  to (1.1), such that  $X(\psi, \varepsilon)$  is analytic in  $\psi$  and goes to 0 as  $\varepsilon \rightarrow 0^+$ .*

Under the assumption that  $\omega$  satisfies the Bryuno condition (a weaker non-resonance condition, with respect to the standard Diophantine condition), the existence of response solutions has been proved in [28], by extending the results in [20] to higher dimensions. In this paper, we follow the approach introduced in [25] to remove the non-resonance condition on the frequency vector.

What we really need about the function  $U$  is that its Hessian matrix is strictly definite. Thus, the same result of existence holds if, instead of Hypothesis 1, we assume  $c$  to be a strict local maximum for  $c$ . Finally, we might allow  $\varepsilon$  to be negative as well, but we prefer to fix  $\varepsilon > 0$  because this corresponds to a physical system in the presence of dissipation.

The results above can be extended to differential equations equations of the form

$$\varepsilon M \ddot{x} + \Gamma \dot{x} + \varepsilon h(x, \omega t) = 0, \quad (1.4)$$

where  $M$ , the mass matrix, is a diagonal or, more generally, a positive definite symmetric matrix, and  $h : \mathbb{R}^m \times \mathbb{T}^d \rightarrow \mathbb{R}^m$  is a real analytic function. For instance, if  $m = 3n$  and  $M$  is diagonal, with  $M_{ii} = M_1$  for  $i = 1, 2, 3$ ,  $M_{ii} = M_2$  for  $i = 4, 5, 6, \dots$ ,  $M_{ii} = M_n$  for  $i = 3n-2, 3n-1, 3n$ , then (1.4) describes a system of  $n$  points in  $\mathbb{R}^3$ , of masses  $M_1, M_2, \dots, M_n$ , respectively. We assume  $h$  to be the gradient of a time-dependent potential energy  $\tilde{V}$ , i.e.

$$h(x, \omega t) = \frac{\partial \tilde{V}}{\partial x}(x, \omega t). \quad (1.5)$$

Let us consider the Fourier expansion of the function  $\tilde{V}$ :

$$\tilde{V}(x, \psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i\nu \cdot \psi} \tilde{V}_\nu(x). \quad (1.6)$$

Then we assume the following condition on  $\tilde{V}$ .

**Hypothesis 2.** *The function  $\tilde{V}_0(x)$  in (1.6) has a strict local minimum point  $c$ .*

Then the following result generalises Theorem 1.

**Theorem 2.** *Consider the ordinary differential equation (1.4), with  $h$  as in (1.5), and assume Hypothesis 2. For any rationally independent frequency vector  $\omega \in \mathbb{R}^d$ , there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there is a quasi-periodic solution  $x_0(t) = c + X(\omega t, \varepsilon)$  to (1.4), such that  $X(\psi, \varepsilon)$  is analytic in  $\psi$  and goes to 0 as  $\varepsilon \rightarrow 0^+$ .*

The rest of the paper is organised as follows. We first discuss the proof of Theorem 1. In Section 2 we outline the overall strategy of the proof, by showing that the problem can be split into two different equations: the range equation and the bifurcation equation. In Section 3 we prove the existence of a solution to the range equation depending on a parameter  $\zeta$ . In Section 4 we fix the parameter  $\zeta$  as a function of  $\varepsilon$  in such a way that the bifurcation equation is solved as well. This completes the proof of Theorem 1. Finally, in Section 5 we discuss how to generalise the results and to prove Theorem 2.

On physical grounds, one may expect the response solution to be attracting. As a byproduct, local uniqueness would also follow. In the one-dimensional case, this can be easily proved by a rescaling of time [5, 25]. However, the argument does not extend to higher dimensions. So, uniqueness of the response solutions described in Theorems 1 and 2 is left as an open problem.

Another issue which deserves further investigation is how to deal with systems in which the mass matrix and the damping matrix are not symmetric. From a physical point of view, it is not restrictive to assume the mass matrix  $M$  to be symmetric since the kinetic energy is a quadratic form; more generally  $M$  may depend on the coordinates  $x$ , but the analysis below can be extended to cover such a case. As far as to the damping matrix, taking  $\Gamma$  to be positive means that we are assuming that the friction that a mass point experiences because of the presence of another mass point is the same as that it produces on that point. Usually, this is what happens in most dynamical structures [16]; however, there are situations in which the damping matrix are asymmetric [3]. From a technical point of view, the property of  $\Gamma$  being symmetric is used to control the inverse of a suitable matrix which appears along the analysis: renouncing such a property would introduce severe technical difficulties. This is not surprising, since such kind of problems are known to arise in the higher dimensional case – see for instance [17, 24, 14] and references therein. Similar considerations apply to the case in which the force is not conservative.

## 2 Plan of the proof of Theorem 1

We first prove Theorem 1: here we describe the strategy of the proof, with the details to be worked out in the forthcoming sections 3 and 4. Eventually, in Section 5, we discuss

how to adapt the argument so as to prove the more general Theorem 2.

Let us denote by  $\Sigma_\xi$  the strip around  $\mathbb{T}^d$  of width  $\xi$  and by  $B_\rho(c)$  the disk of center  $c = (c_1, \dots, c_m)$  and radius  $\rho$  in the  $m$ -dimensional complex plane,

$$\Sigma_\xi := \left\{ \psi \in \mathbb{C}^d : \operatorname{Re} \psi_i \in \mathbb{T}, \quad |\operatorname{Im} \psi_i| < \xi \right\}, \quad B_\rho(c) := \{x \in \mathbb{C}^m : |x - c| < \rho\}.$$

By the assumptions on  $f$  and  $g$ , for any  $c \in \mathbb{R}^m$  there exist  $\xi_0 > 0$  and  $\rho_0 > 0$  such that  $\psi \mapsto f(\psi)$  is analytic in  $\Sigma_{\xi_0}$  and  $x \mapsto g(x)$  is analytic in  $B_{\rho_0}(c)$ . Then for all  $\xi < \xi_0$  one has

$$f_i(\psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i\nu \cdot \psi} (f_\nu)_i, \quad |(f_\nu)_i| \leq \Phi e^{-\xi|\nu|}, \quad i = 1, \dots, m, \quad (2.1)$$

where  $\Phi$  is the maximum of  $|f(\psi)|$  for  $\psi \in \Sigma_\xi$ .

Define  $G_p(c)$  as the tensor of rank  $p + 1$  with components

$$[G_p(c)]_{i, i_1, \dots, i_p} = \frac{1}{p!} \frac{\partial^p}{\partial x_{i_1} \dots \partial x_{i_p}} g_i(x) \Big|_{x=c},$$

where  $i, i_1, \dots, i_p = 1, \dots, m$ , and write

$$g_i(x) = \sum_{p=0}^{\infty} [G_p(c)(x-c)^p]_i = \sum_{p=0}^{\infty} \sum_{i_1, \dots, i_p=1}^m [G_p(c)]_{i, i_1, \dots, i_p} \prod_{j=1}^p (x_{i_j} - c_{i_j}), \quad i = 1, \dots, m.$$

For all  $\rho < \rho_0$  one has

$$|[G_p(c)]_{i, i_1, \dots, i_p}| \leq \Delta \rho^{-p}, \quad (2.2)$$

with  $\Delta$  being the maximum of  $|g(x)|$  for  $x \in B_{\rho_0 - \rho}(c)$ .

Let us rewrite (1.1) as

$$\varepsilon \ddot{x} + \Gamma \dot{x} + \varepsilon g(c) + \varepsilon A(x-c) + \varepsilon G(x) = \varepsilon f(\omega t), \quad (2.3)$$

where  $A := G_1(c)$  is the  $m \times m$  matrix with entries

$$A_{i,j} = \frac{\partial g_i}{\partial x_j}(x) \Big|_{x=c} = \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \Big|_{x=c}, \quad i, j = 1, \dots, m,$$

and

$$G(x) := g(x) - g(c) - A(x-c) = \sum_{p=2}^{\infty} G_p(c)(x-c)^p,$$

By Hypothesis 1, the symmetric matrix  $A$  is positive definite; hence its eigenvalues are all positive.

We look for a quasi-periodic solution to (2.3), that is a solution of the form

$$x(t, \varepsilon) = c + \zeta + u(\omega t, \varepsilon, \zeta), \quad u(\psi, \varepsilon, \zeta) = \sum_{\nu \in \mathbb{Z}_*^d} e^{i\nu \cdot \psi} u_\nu, \quad (2.4)$$

where  $\zeta$  is a real parameter that has to be fixed eventually, and  $\psi \mapsto u(\psi, \varepsilon, \zeta)$  is a zero-average quasi-periodic function, with Fourier coefficients depending on both  $\varepsilon$  and  $\zeta$ .

It is more convenient to write (2.3) in Fourier space, so by choosing  $c$  as in Hypothesis 1, we obtain the following equations:

$$(-\varepsilon(\omega \cdot \nu)^2 \mathbb{1} + i\omega \cdot \nu \Gamma + \varepsilon A) u_\nu = -\varepsilon [G(c + \zeta + u)]_\nu + \varepsilon f_\nu, \quad \nu \neq 0, \quad (2.5a)$$

$$\varepsilon A \zeta = -\varepsilon [G(c + \zeta + u)]_0, \quad (2.5b)$$

where  $\mathbb{1}$  is the  $m \times m$  identity matrix and the notation  $[G(c + \zeta + u)]_\nu$  means that we first write  $u$  according to (2.4), then expand  $G(c + \zeta + u)$  in Fourier series in  $\psi$  and finally keep the Fourier coefficient with index  $\nu$ . We call (2.5a) the *range equation* and (2.5b) the *bifurcation equation* [10, 26, 31].

To simplify the notation, we call

$$D(\varepsilon, s) := -\varepsilon s^2 \mathbb{1} + is\Gamma + \varepsilon A \quad (2.6)$$

and write (2.5a) as

$$D(\varepsilon, \omega \cdot \nu) u_\nu = -\varepsilon [G(c + \zeta + u)]_\nu + \varepsilon f_\nu, \quad \nu \neq 0.$$

In order to study the equations (2.5), we start by ignoring the bifurcation equation (2.5b) and considering only the range equation (2.5a). We treat  $\zeta$  as a free parameter, close enough to 0, and look for a solution depending on such a parameter, by using perturbation theory. To this aim, we modify (2.5a) introducing the auxiliary parameter  $\mu$ ,

$$D(\varepsilon, \omega \cdot \nu) u_\nu = -\mu [\varepsilon G(c + \mu\zeta + u)]_\nu + \mu\varepsilon f_\nu, \quad \nu \neq 0. \quad (2.7)$$

Of course the original equation (2.5a) is recovered only for  $\mu = 1$ , so eventually we have to check that the solution to (2.7) is well-defined for  $\mu$  as large as 1.

Then, in Section 3, we show that a quasi-periodic solution to (2.7) exists in the form of a power series in  $\mu$ ,

$$u(\omega t, \varepsilon, \zeta, \mu) = \sum_{k=1}^{\infty} \sum_{\nu \in \mathbb{Z}_*^d} \mu^k e^{i\nu \cdot \psi} u_\nu^{(k)}, \quad (2.8)$$

and we prove that the radius of convergence of (2.8) is strictly greater than 1.

Once a solution  $u(\omega t, \varepsilon, \zeta) := u(\omega t, \varepsilon, \zeta, 1)$  to the range equation has been proved to exist, we pass to the bifurcation equation (2.5b). The latter is studied in Section 4 as an implicit function problem to determine the constant  $\zeta$  in terms of  $\varepsilon$ . The solution provides a value  $\zeta = \zeta(\varepsilon)$ , depending continuously on  $\varepsilon$ , such that  $x(t, \varepsilon) = c + \zeta(\varepsilon) + u(\omega t, \varepsilon, \zeta(\varepsilon), 1)$  is a solution to (2.3) and hence to (1.1).

This completes the proof of Theorem 1.

### 3 The range equation

In this and in the following section we provide the details of the proof of Theorem 1. Here we show that, for  $\varepsilon$  and  $\zeta$  small enough, there exists a solution of the form (2.4) to the

range equation (2.5a). Moreover, such a solution  $u(\omega t, \varepsilon, \zeta)$  is analytic in  $\psi = \omega t$  and  $\zeta$ , and tends to  $c$  as  $\varepsilon$  tends to 0.

### 3.1 Recursive equations

The coefficients  $u_\nu^{(k)}$  are obtained recursively by inserting the expansion (2.8) into (2.7). If we set  $u_0^{(1)} = \zeta$  and  $u_0^{(k)} = 0 \ \forall k \geq 2$ , to make the notations uniform, then, one obtains, formally,

$$D(\varepsilon, \omega \cdot \nu) u_\nu^{(1)} = \varepsilon f_\nu \quad (3.1a)$$

$$D(\varepsilon, \omega \cdot \nu) u_\nu^{(k)} = -\varepsilon \sum_{p=2}^{\infty} G_p(c) \sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k-1}} \sum_{\substack{\nu_1, \dots, \nu_p \in \mathbb{Z}^d \\ \nu_1 + \dots + \nu_p = \nu}} u_{\nu_1}^{(k_1)} \dots u_{\nu_p}^{(k_p)}, \quad k \geq 2, \quad (3.1b)$$

where  $\nu \neq 0$ . We assume that the sums over the empty set are meant as zero. It is easily checked that  $u_\nu^{(2)} = 0 \ \forall \nu \in \mathbb{Z}_*^d$ .

### 3.2 Tree representation for the coefficients

By iterating (3.1b), one obtains a diagrammatic representation of the coefficients  $u_\nu^{(k)}$  in terms of trees; we refer to [21, 25] and references therein for further details.

A *rooted tree*  $\theta$  is a graph with no cycle, such that all the lines are oriented toward a unique point, called the *root*, which has only one incident line, called the root line. All the points in  $\theta$  except the root are called *nodes*. Given a rooted tree  $\theta$  we denote by  $N(\theta)$  the set of its nodes and by  $L(\theta)$  the set of its lines.

The orientation of the lines in  $\theta$  induces a partial ordering relation  $\preceq$  between the nodes. If  $v$  and  $w$  are two nodes of the tree, we write  $w \prec v$  every time  $v$  is along the path of lines which connects  $w$  to the root; we shall write  $w \prec \ell$  if  $w \preceq v$ , where  $v$  is the unique node that the line  $\ell$  exits. For any node  $v$  let  $p_v$  denote the number of lines entering  $v$ . One has  $p_v \geq 0$ ; if  $p_v = 0$  we say that  $v$  is an *end node*, while if  $p_v \geq 1$  we say that  $v$  is an *internal node*. Denote by  $E(\theta)$  the set of end nodes and by  $V(\theta)$  the set of internal nodes in  $\theta$ ; one has  $N(\theta) = E(\theta) \amalg V(\theta)$ .

For any discrete set  $A$ , we denote by  $|A|$  its cardinality; we define the *order* of  $\theta$  as  $k(\theta) := |N(\theta)|$ .

We associate with each end node  $v \in E(\theta)$  a *mode* label  $\nu_v \in \mathbb{Z}^d$ . According to the values of the mode labels, we split  $E(\theta)$  into two disjoint sets by writing  $E(\theta) = E_0(\theta) \amalg E_1(\theta)$ , with  $E_0(\theta) = \{v \in E(\theta) : \nu_v = 0\}$  and  $E_1(\theta) = \{v \in E(\theta) : \nu_v \neq 0\}$ . With each line  $\ell \in L(\theta)$  we associate a *momentum*  $\nu_\ell \in \mathbb{Z}^d$  with the constraint

$$\nu_\ell = \sum_{\substack{w \in E(\theta) \\ w \prec \ell}} \nu_w. \quad (3.2)$$

Finally we impose the further constraints that

- $p_v \geq 2 \forall v \in V(\theta)$ ,
- $\nu_\ell \neq 0$  for any line  $\ell$  exiting a node in  $V(\theta)$ .

We call *equivalent* two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. Since, in the following, we shall consider only inequivalent labelled rooted trees, for simplicity's sake they will be called simply 'trees'.

Next, with each node  $v \in N(\theta)$  we associate a *node factor*

$$F_v := \begin{cases} -\varepsilon G_{p_v}, & v \in V(\theta), \\ \varepsilon f_{\nu_v}, & v \in E_1(\theta), \\ \zeta, & v \in E_0(\theta), \end{cases} \quad (3.3)$$

and with each line  $\ell \in L(\theta)$  we associate a *propagator*

$$\mathcal{G}_\ell := \begin{cases} D^{-1}(\varepsilon, \omega \cdot \nu_\ell), & \nu_\ell \neq 0, \\ \mathbf{1}, & \nu_\ell = 0, \end{cases} \quad (3.4)$$

with  $D(\varepsilon, s)$  defined according to (2.6). Note that the node factor (3.3) is a tensor of rank  $p_v + 1$ , while the propagator (3.4) is a matrix. Finally we define the value of the tree  $\theta$  as

$$\mathcal{V}(\theta) := \left( \prod_{v \in N(\theta)} F_v \right) \left( \prod_{\ell \in L(\theta)} \mathcal{G}_\ell \right), \quad (3.5)$$

where

- the indices of both node factors and propagators are such that, if  $[F_v]_{i_0, i_1, \dots, i_{p_v}}$  is the component of  $F_v$ , then the second index of the propagator of the line  $\ell_v$  is  $i_0$  and the first indices of the propagators of the  $p_v$  lines entering  $v$  are  $i_1, \dots, i_{p_v}$ ;
- all the indices have been contracted, so that (3.5) gives a vector, whose component index  $i$  equals the first index of the propagator of the root line.

If we denote by  $\mathcal{T}_{k, \nu}$  the set of non-equivalent trees of order  $k$  and momentum  $\nu$  associated with the root line, and define the coefficients  $u_\nu^{(k)}$  in (2.8) as

$$u_\nu^{(k)} := \sum_{\theta \in \mathcal{T}_{k, \nu}} \mathcal{V}(\theta), \quad \nu \in \mathbb{Z}_*^d, \quad (3.6)$$

then it is straightforward to see that the equations (3.1) are solved for all  $k \in \mathbb{N}$ .

### 3.3 Bounds on the propagators

We can estimate the norm of the propagators as follows. By construction  $\Gamma$  is symmetric and positive definite, so we write  $\Gamma := K^2$ , with  $K$  a symmetric positive definite matrix. Let us denote by  $\kappa_1 \leq \dots \leq \kappa_m$  the eigenvalues of  $K$ . Hence, we can write  $D = D(\varepsilon, s)$  as

$$D = K (is\mathbf{1} + K^{-1}(-\varepsilon s^2 \mathbf{1} + \varepsilon A) K^{-1}) K,$$



so that

$$D^{-1} = K^{-1} (is\mathbb{1} + K^{-1}(-\varepsilon s^2\mathbb{1} + \varepsilon A)K^{-1})^{-1} K^{-1}.$$

By defining

$$S := \varepsilon K^{-1}AK^{-1} - \varepsilon s^2K^{-2} \quad \text{and} \quad R := S + is\mathbb{1},$$

we obtain  $D^{-1} = K^{-1}R^{-1}K^{-1}$ . Let  $b_1 \leq \dots \leq b_m$  be the eigenvalues of  $K^{-1}AK^{-1}$ . Since both  $A$  and  $K$  are positive definite,  $b_1, \dots, b_m$  are positive. By Rellich's theorem [29], it follows that the eigenvalues of  $S$  are  $b_k\varepsilon + O(\varepsilon s^2)$ ,  $k = 1, \dots, m$ , and hence the eigenvalues of  $R$  are  $\beta_k + is$ ,  $k = 1, \dots, m$ , with  $\beta_k = b_k\varepsilon + O(\varepsilon s^2)$ . Set  $\beta := \min\{|\beta_1|, \dots, |\beta_m|\}$ .

Let us denote by  $|\cdot|$  the Euclidean norm on  $\mathbb{R}^m$  and by  $\|R\| := \sup_{|x|=1} |Rx|$  the induced norm; one has

$$\|R^{-1}\| \leq \frac{1}{\sqrt{\beta^2 + |s|^2}}.$$

For any  $s \in \mathbb{R}$  one has  $|\beta_k - b_k\varepsilon| \leq W|\varepsilon s^2|$  for a suitable constant  $W$ . Thus, if  $|s| \leq \alpha$ , where  $\alpha := \sqrt{b_1/2W}$ , one obtains

$$\|R^{-1}\| \leq \min \left\{ \frac{2}{|b_1\varepsilon|}, \frac{1}{|s|} \right\},$$

otherwise one has  $\|R^{-1}\| \leq 1/\alpha$ . By collecting together the bounds above, we arrive at

$$\|D^{-1}\| \leq \frac{1}{\kappa_1^2} \max \left\{ \frac{1}{\alpha}, \min \left\{ \frac{1}{|s|}, \frac{2}{|b_1\varepsilon|} \right\} \right\}. \quad (3.7)$$

Since we are interested in results for  $\varepsilon$  small enough, in the following we assume  $\varepsilon < \varepsilon_1$ , where  $\varepsilon_1 := \alpha/b_1$ , so that we can replace (3.7) with

$$\|D^{-1}\| \leq \frac{1}{\kappa_1^2} \min \left\{ \frac{2}{|b_1\varepsilon|}, \max \left\{ \frac{1}{\alpha}, \frac{1}{|s|} \right\} \right\}. \quad (3.8)$$

### 3.4 Bounds on the coefficients

For future convenience, set

$$C_0 := m^2\rho^{-1} \max \left\{ \Phi, 1, \frac{2\Delta}{\kappa_1^2|b_1|} \right\}, \quad (3.9)$$

and define

$$s_N := \min\{|\omega \cdot \nu| : 0 < |\nu| \leq N\}, \quad r_N := \min\{s_N, \alpha\}. \quad (3.10)$$

By reasoning as in the proof of Lemma 2.3 in [13], for any tree  $\theta$  one obtains the bound

$$|E(\theta)| \geq \frac{1}{2} (k(\theta) + 1). \quad (3.11)$$

The following result is crucial for the proof of Theorem 1. In order to deal with the small divisors problem without strong non-resonance conditions on  $\omega$ , it is fundamental

to bound in a different way the propagator associated to a line exiting an internal node or an end node. Moreover, here (which is not the case of [13] for instance), one needs to estimate in a different way, following the ideas in [25], also the propagators  $D(\varepsilon, \omega \cdot \nu_v)$  with  $v \in E_1(\theta)$  and studying separately the different regimes of  $\nu$ . The key point is that, due to the presence of the dissipation, the eigenvalues of the matrix  $R$  have both real and imaginary parts; hence one is able to prove an estimate like (3.8).

**Lemma 3.1.** *Let  $\xi$  be as in (2.1). For any fixed  $B \in (0, C_0)$  there exist two positive constants  $\bar{\varepsilon}$  and  $\bar{\zeta}$  such that for any  $k \geq 1$  and any  $\nu \in \mathbb{Z}_*^d$  one has*

$$\left| u_\nu^{(k)} \right| \leq B_0 B_1^k e^{-\xi|\nu|/2},$$

where  $B_0$  and  $B_1$  are two positive constants proportional to  $B$ , provided  $|\varepsilon| < \bar{\varepsilon}$  and  $|\zeta| < \bar{\zeta}$ .

*Proof.* One bounds (3.5) as

$$\begin{aligned} |\mathcal{V}_i(\theta)| &\leq m^{2k-1} \left( \prod_{v \in V(\theta)} |\varepsilon| \Delta \rho^{-p_v} \right) \left( \prod_{v \in E_1(\theta)} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| |\varepsilon f_{\nu_v}| \right) \times \\ &\quad \times \left( \prod_{v \in E_0(\theta)} |\zeta| \right) \left( \prod_{v \in V(\theta)} \frac{2}{\kappa_1^2 |b_1 \varepsilon|} \right) \\ &\leq m^{2k-1} |\zeta|^{|E_0(\theta)|} \Delta^{|V(\theta)|} \rho^{-(|N(\theta)|-1)} \Phi^{|E_1(\theta)|} \times \\ &\quad \times \left( \frac{2}{\kappa_1^2 |b_1|} \right)^{|V(\theta)|} \left( \prod_{v \in E_1(\theta)} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| |\varepsilon| e^{-\xi|\nu_v|} \right), \end{aligned}$$

where we have bounded  $f_{\nu_v}$  according to (2.1) and  $G_{p_v}$  according to (2.2), and we have used the bound  $\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq 2/\kappa_1^2 |b_1 \varepsilon|$  for the propagators of the lines exiting the nodes  $v \in V(\theta)$ . For each end node  $v \in E_1(\theta)$  we extract a factor  $e^{-3\xi|\nu_v|/4}$ , so that, if we define  $C_0$  as in (3.9), we obtain

$$|\mathcal{V}(\theta)| \leq \rho C_0^k |\zeta|^{|E_0(\theta)|} \left( \prod_{v \in E_1(\theta)} e^{-3\xi|\nu_v|/4} \right) \left( \prod_{v \in E_1(\theta)} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| |\varepsilon| e^{-\xi|\nu_v|/4} \right). \quad (3.12)$$

By the definition (3.10), for any given  $N \in \mathbb{N}$  we have  $|\omega \cdot \nu| \geq s_N$  for all  $\nu \in \mathbb{Z}_*^d$  such that  $|\nu| \leq N$ . Set

$$\delta = \delta(N) := e^{-\xi N/4}.$$

Let  $B$  be such that  $0 < B < C_0$ . We first fix  $N$  such that  $2\delta C_0^2 \leq B^2 \kappa_1^2 |b_1|$ , then we fix  $\bar{\varepsilon}$  and  $\bar{\zeta}$  by requiring that  $C_0^2 \bar{\varepsilon} / \kappa_1^2 r_N \leq B^2$  and  $C_0^2 \bar{\zeta} < B^2$ .

By (3.8), in (3.12), for all  $v \in E_1(\theta)$ , we can bound  $\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq 1/\kappa_1^2 r_N$  if  $|\nu_v| \leq N$  and  $\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq 2/\kappa_1^2 |b_1 \varepsilon|$  if  $|\nu_v| > N$ . Thus, for all  $v \in E_1(\theta)$ , one has

$$|\varepsilon| e^{-\xi|\nu_v|/4} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq \frac{1}{\kappa_1^2} \max \left\{ \frac{2\delta}{|b_1|}, \frac{|\varepsilon|}{r_N} \right\}, \quad (3.13)$$

so that in (3.12), when  $2\delta/|b_1| \geq |\varepsilon|/r_N$ , we can bound

$$C_0^k |\zeta|^{|E_0(\theta)|} \left( \prod_{v \in E_1(\theta)} |\varepsilon| e^{-\xi|\nu_v|/4} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \right) \leq C_0^k \bar{\zeta}^{|E_0(\theta)|} \left( \frac{2\delta}{|b_1| \kappa_1^2} \right)^{|E_1(\theta)|} \leq \frac{B}{C_0} B^k,$$

while, when  $2\delta/|b_1| < |\varepsilon|/r_N$ , we can bound

$$C_0^k |\zeta|^{|E_0(\theta)|} \left( \prod_{v \in E(\theta)} |\varepsilon| e^{-\xi|\nu_v|/4} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \right) \leq C_0^k \bar{\zeta}^{|E_0(\theta)|} \left( \frac{|\varepsilon|}{\kappa_1^2 r_N} \right)^{|E_1(\theta)|} \leq \frac{B}{C_0} B^k,$$

where we have used twice (3.11) with  $k(\theta) = k$ .

Summarising, for all  $k \in \mathbb{N}$  and all  $\nu \in \mathbb{Z}_*^d$  we have obtained

$$|\mathcal{V}(\theta)| \leq B_0 B^k \left( \prod_{v \in E(\theta)} e^{-3\xi|\nu_v|/4} \right), \quad B_0 := \rho \frac{B}{C_0}.$$

The latter bound can be used to provide an estimate for the coefficients  $u_\nu^{(k)}$ . According to (3.6), we have to sum over all trees in  $\mathcal{T}_{k,\nu}$ : the sum over the Fourier labels  $\{\nu_v\}_{v \in E_1(\theta)}$  is performed by using the factors  $e^{-3\xi|\nu_v|/4}$  associated with the end nodes in  $E_1(\theta)$ , and gives a bound  $C_1^{|E_1(\theta)|} e^{-\xi|\nu|/2}$ , for some positive constant  $C_1$ . The sum over the other labels produces a factor  $C_2^{|N(\theta)|}$ , with  $C_2$  another suitable positive constant. Set  $B_1 := BC_1C_2$ ; then the assertion follows.  $\blacksquare$

### 3.5 Analyticity in $\psi$

The bounds on the coefficients  $u_\nu^{(k)}$  given in Lemma 3.1 assure that the series (2.8), with  $\mu = 1$ , converges to a well-defined function  $u(\omega, \varepsilon, \zeta)$ , which depends analytically on  $\psi = \omega t$ . This is implied by the following result.

**Lemma 3.2.** *For any  $\omega \in \mathbb{R}^d$  there exist two positive constants  $\bar{\varepsilon}$  and  $\bar{\zeta}$  such that the series (2.8) converges to a function  $u(\psi, \varepsilon, \zeta) := u(\psi, \varepsilon, \zeta, 1)$ , with the following properties:*

1.  $u(\psi, \varepsilon, \zeta)$  is analytic in  $\psi$  in a strip  $\Sigma_{\xi'}$ , with  $\xi' < \xi/2$ ,
2.  $u(\omega t, \varepsilon, \zeta)$  solves (2.5a),

for  $\mu = 1$ ,  $|\varepsilon| < \bar{\varepsilon}$  and  $|\zeta| < \bar{\zeta}$ .

*Proof.* In Lemma 3.1 we can fix  $B$  so that  $B_1 < 1$ . This implies the convergence of the series (2.8) provided  $\mu$  is taken to satisfy  $B_1\mu < 1$ , which allows  $\mu = 1$ . The function (2.8) solves (2.7) order by order by construction, and, since the series converges uniformly, then it is also a solution *tout court* to (2.7) with  $\mu = 1$  and hence of (2.5a).

The bound on the Fourier coefficients given by Lemma 3.1 assures that the solution is analytic in  $\psi \in \Sigma_{\xi'}$  for any  $\xi' < \xi/2$ .  $\blacksquare$

### 3.6 Continuity in $\varepsilon$

The bounds of Lemma 3.1 are not enough to prove that the solution  $u(\omega t, \varepsilon, \zeta)$  vanishes in the limit  $\varepsilon \rightarrow 0$ . Indeed, the constants  $B$  and  $B_0$  are independent of  $\varepsilon$ . However, a slight adaptation of the proof of Lemma 3.1 leads to the following result.

**Lemma 3.3.** *Let  $\bar{\varepsilon}$  and  $\bar{\zeta}$  be as in Lemma 3.2. For any  $|\zeta| < \bar{\zeta}$ , the function  $u(\psi, \varepsilon, \zeta)$  is continuous in  $\varepsilon \in [0, \bar{\varepsilon})$  and it tends to 0 as  $\varepsilon \rightarrow 0$ .*

*Proof.* Let  $\zeta$  be such that  $|\zeta| < \bar{\zeta}$ . Continuity in  $\varepsilon$  is obvious for  $\varepsilon > 0$ . To study the continuity at  $\varepsilon = 0$ , set

$$F(\varepsilon, \zeta) = \|u(\cdot, \varepsilon, \zeta)\|_\infty := \sup\{u(\psi, \varepsilon, \zeta) : \psi \in \Sigma_{\xi'}\},$$

with  $\xi'$  as in Lemma 3.2. Since  $F(0, \zeta) = 0$ , we have only to prove that  $F(\varepsilon, \zeta) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , that is that for all  $\eta > 0$  there exists  $\varepsilon_1 > 0$  such that  $0 < \varepsilon < \varepsilon_1$  implies  $|F(\varepsilon, \zeta)| < \eta$ .

Let  $\bar{\varepsilon}$  be as in Lemma 3.2. By writing  $u(\omega t, \varepsilon, \zeta)$  as in (2.8) with  $\mu = 1$ , we obtain

$$F(\varepsilon, \zeta) \leq \sum_{k=1}^{\infty} \sum_{\nu \in \mathbb{Z}^d} |u_\nu^{(k)}| e^{\xi'|\nu|},$$

where  $\xi' < \xi/2$ . By reasoning as in the proof of Lemma 3.1, we can bound

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}^d} |u_\nu^{(k)}| e^{\xi'|\nu|} &\leq \sum_{\nu \in \mathbb{Z}^d} \sum_{\theta \in \mathcal{T}_{k,\nu}} |\mathcal{V}(\theta)| e^{\xi'|\nu|} \\ &\leq \rho C_0^k \sum_{\nu \in \mathbb{Z}^d} e^{\frac{\xi}{2}|\nu|} \sum_{\theta \in \mathcal{T}_{k,\nu}} |\zeta|^{|E_0(\theta)|} \times \\ &\quad \times \left( \prod_{v \in E_1(\theta)} e^{-\frac{3}{4}\xi|\nu_v|} \right) \left( \prod_{v \in E_1(\theta)} \|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| |\varepsilon| e^{-\frac{\xi}{4}|\nu_v|} \right) \\ &\leq \rho C_0^k \sum_{\nu \in \mathbb{Z}^d} e^{\frac{\xi}{2}|\nu|} \sum_{\theta \in \mathcal{T}_{k,\nu}} |\zeta|^{|E_0(\theta)|} \left( \frac{1}{\kappa_1^2} \max \left\{ \frac{2\delta}{|b_1|}, \frac{|\varepsilon|}{r_N} \right\} \right)^{|E_1(\theta)|-1} \times \\ &\quad \times \left( \prod_{v \in E_1(\theta) \setminus \{\bar{v}\}} e^{-\frac{3}{4}\xi|\nu_v|} \right) \sum_{\nu_{\bar{v}} \in \mathbb{Z}_*^d} |\varepsilon| e^{-\frac{\xi}{4}|\nu_{\bar{v}}|} \|D^{-1}(\varepsilon, \omega \cdot \nu_{\bar{v}})\| \\ &\leq \rho C_0 B^{-1} B_1^k \sum_{\nu \in \mathbb{Z}_*^d} |\varepsilon| e^{-\frac{\xi}{4}|\nu|} \|D^{-1}(\varepsilon, \omega \cdot \nu)\|, \end{aligned}$$

with  $B_1$  as in Lemma 3.1. We have used that any tree  $\theta$  contains at least one end node  $\bar{v}$  with  $\nu_{\bar{v}} \neq 0$ , since  $\nu \neq 0$ . In particular, this implies  $|E_1(\theta)| \geq 1$ . Therefore we may use the bound (3.13) for all the end nodes  $v \in E_1(\theta)$  except for one. As a consequence, we obtain, for any  $N \in \mathbb{N}$ ,

$$F(\varepsilon, \zeta) \leq \frac{\rho C_0 B^{-1}}{1 - B_1} \sum_{\substack{\nu \in \mathbb{Z}_*^d \\ |\nu| \leq N}} |\varepsilon| e^{-\xi|\nu|/4} \|D^{-1}(\varepsilon, \omega \cdot \nu)\| + D_0 e^{-\xi N/8}, \quad (3.14)$$

where we have set

$$D_0 := \frac{\rho C_0 B^{-1}}{1 - B_1} \sum_{\nu \in \mathbb{Z}^d} e^{-\xi|\nu|/8}.$$

Fix  $\eta > 0$  and take  $N_1$  such that  $D_0 e^{-\xi N_1/8} < \eta/2$ . If we define  $r_N$  as in (3.10) and choose  $\varepsilon_1$  small enough, we bound

$$\frac{\rho C_0 B^{-1}}{1 - B_1} \sum_{\substack{\nu \in \mathbb{Z}^d \\ |\nu| \leq N_1}} |\varepsilon| e^{-\xi|\nu|/4} \|D^{-1}(\varepsilon, \omega \cdot \nu)\| \leq \frac{|\varepsilon| D_0}{r_{N_1}} < \frac{\eta}{2},$$

provided one has  $0 < \varepsilon < \varepsilon_1$ . This implies  $|F(\varepsilon, \zeta)| < \eta$ . Hence the thesis follows.  $\blacksquare$

## 4 The bifurcation equation

In this section we show that, given the solution  $u(\omega t, \varepsilon, \zeta)$  to the range equation (2.5a), it is possible to fix  $\zeta$  as a function of  $\varepsilon$  in such a way that the bifurcation equation (2.5b) is solved as well.

**Lemma 4.1.** *Let  $\bar{\varepsilon}$  and  $\bar{\zeta}$  be as in Lemma 3.1 and let  $u = u(\omega t, \varepsilon, \zeta)$  be the function in Lemma 3.2. There exist neighbourhoods  $U \subset (-\bar{\varepsilon}, \bar{\varepsilon})$  and  $V \subset B_{\bar{\zeta}}(0)$  and a function  $\zeta : U \rightarrow V$  with the following properties:*

1. *the function  $\varepsilon \mapsto \zeta(\varepsilon)$  is continuous in  $U$ ,*
2. *for all  $\varepsilon \in U$  the equation (2.5b) is satisfied for  $\zeta = \zeta(\varepsilon)$ ,*
3.  *$\zeta(\varepsilon)$  is the only solution to (2.5b) in  $V$ .*

*Proof.* Write (2.5b) as

$$\mathcal{H}(\zeta, \varepsilon) := A\zeta + [G(c + \zeta + u)]_0 = 0.$$

By construction, the function  $u(\psi, \varepsilon, \zeta)$  depends analytically on  $\zeta$  in a neighbourhood of the origin. Therefore,  $\mathcal{H}(\zeta, \varepsilon)$  is analytic in  $\zeta$  and, by Lemma 3.3, is continuous in  $\varepsilon$ . One has  $\mathcal{H}(0, 0) = 0$  and

$$\frac{\partial \mathcal{H}_i}{\partial \zeta_j}(0, 0) = \frac{\partial}{\partial \zeta_j} (A\zeta + G(c + \zeta + u))_i(0, 0) = A_{i,j}, \quad i, j = 1, \dots, m.$$

Since  $\det A \neq 0$  by Hypothesis 1, we can apply the implicit function theorem, in the version of Loomis and Sternberg [27], so as to conclude that there exist neighbourhoods  $U \subset (-\bar{\varepsilon}, \bar{\varepsilon})$  and  $V \subset B_{\bar{\zeta}}(0)$  such that for all  $\varepsilon \in U$  one can find a unique value  $\zeta(\varepsilon) \in V$ , depending continuously on  $\varepsilon$ , such that  $\mathcal{H}(\zeta(\varepsilon), \varepsilon) = 0$ .  $\blacksquare$

**Lemma 4.2.** *Let the functions  $u(\psi, \varepsilon, \zeta)$  and  $\zeta(\varepsilon)$  be as defined in Lemma 3.2 and in Lemma 4.1, respectively. There exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  the function  $x(t, \varepsilon) = c + \zeta(\varepsilon) + u(\omega t, \varepsilon, \zeta(\varepsilon))$  solves (2.3). Moreover  $x(t, \varepsilon) \rightarrow c$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* The result follows immediately from Lemma 3.2 and Lemma 4.1.  $\blacksquare$

## 5 Extension of the results to more general nonlinearities

In this section, we show how to extend the results of the previous section, so as to prove Theorem 2.

### 5.1 Recursive equations

Let the function  $h(x, \psi)$  in (1.4) be analytic on the domain  $B_{\rho_0}(c) \times \Sigma_{\xi_0}$ , with  $B_{\rho_0}(c)$  and  $\Sigma_{\xi_0}$  as in Section 2. We define  $H_p(c, \psi)$ , the tensor of rank  $p + 1$  with components

$$[H_p(c, \psi)]_{i, i_1, \dots, i_p} = \frac{1}{p!} \frac{\partial^p h_i}{\partial x_{i_1} \dots \partial x_{i_p}}(x, \psi) \Big|_{x=c}, \quad i = 1, \dots, m,$$

where  $H_0(\psi, c) = h(\psi, c)$  and  $i_1, \dots, i_p = 1, \dots, m$  for  $p \geq 1$ , and write

$$h_i(x, \psi) = \sum_{p=0}^{\infty} [H_p(c, \psi)(x-c)^p]_i = \sum_{p=0}^{\infty} \sum_{i_1, \dots, i_p=1}^m [H_p(c, \psi)]_{i, i_1, \dots, i_p} \prod_{j=1}^p (x_{i_j} - c_{i_j}).$$

Then we consider the Fourier expansions

$$h(x, \psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i\nu \cdot \psi} h_{\nu}(x), \quad H_p(c, \psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i\nu \cdot \psi} H_{p, \nu}(c),$$

so that, for any positive  $\rho < \rho_0$  and  $\xi < \xi_0$ ,

$$|[H_{p, \nu}(c)]_{i, i_1, \dots, i_p}| \leq \Delta \rho^{-p} e^{-\xi |\nu|}, \quad (5.1)$$

for a suitable positive constant  $\Delta$ . We rewrite (1.4) as

$$\begin{aligned} \varepsilon M \ddot{x} + \Gamma \dot{x} + \varepsilon A(x-c) + \varepsilon \sum_{\nu \in \mathbb{Z}_*^d} e^{i\nu \cdot \omega t} h_{\nu}(c) \\ + \varepsilon \sum_{\nu \in \mathbb{Z}_*^d} H_{1, \nu}(c) e^{i\nu \cdot \omega t} (x-c) + \varepsilon \sum_{p=2}^{\infty} \sum_{\nu \in \mathbb{Z}^d} H_{p, \nu}(c) e^{i\nu \cdot \omega t} (x-c)^p = 0, \end{aligned}$$

where we have used that  $h_0(c) = 0$  and the  $m \times m$  matrix  $A := H_{1,0}(c)$ , with entries

$$A_{i,j} = \frac{\partial h_{i,0}(x)}{\partial x_j} \Big|_{x=c} = \frac{\partial^2 \tilde{V}_0(x)}{\partial x_i \partial x_j} \Big|_{x=c}, \quad i, j = 1, \dots, m,$$

is positive definite by Hypothesis 2.

We look for a solution of the form (2.4). If we pass to Fourier space, setting

$$\alpha_1 = \alpha_1(c, \psi) := \sum_{\nu \in \mathbb{Z}_*^d} H_{1, \nu}(c) e^{i\nu \cdot \psi}, \quad \alpha_p = \alpha_p(c, \psi) := \sum_{\nu \in \mathbb{Z}^d} H_{p, \nu}(c) e^{i\nu \cdot \psi},$$

and defining  $D(\varepsilon, s)$  as (compare with (2.6))

$$D(\varepsilon, s) := -\varepsilon s^2 M + is\Gamma + \varepsilon A, \quad (5.2)$$

we obtain the equations

$$D(\varepsilon, \omega \cdot \nu) u_\nu = -\varepsilon h_\nu(c) - \varepsilon [\alpha_1(x-c)]_\nu - \varepsilon \sum_{p=2}^{\infty} [\alpha_p(x-c)^p]_\nu, \quad \nu \neq 0, \quad (5.3a)$$

$$\varepsilon A \zeta = -[\alpha_1(x-c)]_0 - \varepsilon \sum_{p=2}^{\infty} [\alpha_p(x-c)^p]_0. \quad (5.3b)$$

By following the same strategy as in Section 3, we shall study first (5.3a) and look for a solution depending on the parameter  $\zeta$ . Thereafter we shall fix  $\zeta$  by requiring that (5.3b) is solved as well.

Instead of (5.3a) we consider the equation

$$D(\varepsilon, \omega \cdot \nu) u_\nu = -\mu \varepsilon h_\nu(c) - \mu \varepsilon [\alpha_1(x-c)]_\nu - \mu \varepsilon \sum_{p=2}^{\infty} [\alpha_p(x-c)^p]_\nu, \quad \nu \neq 0, \quad (5.4)$$

and look for a quasi-periodic solution to (5.4) in the form of a power series in  $\mu$ ,

$$x(t, \varepsilon, \mu) = c + \zeta + u(\omega t, \varepsilon, \zeta, \mu), \quad u(\omega t, \varepsilon, \zeta, \mu) = \sum_{k=1}^{\infty} \sum_{\nu \in \mathbb{Z}_*^d} \mu^k e^{i\nu \cdot \psi} u_\nu^{(k)}. \quad (5.5)$$

We find the recursive equations

$$D(\varepsilon, \omega \cdot \nu) u_\nu^{(1)} = -\varepsilon h_\nu(c) \quad (5.6a)$$

$$D(\varepsilon, \omega \cdot \nu) u_\nu^{(k)} = -\varepsilon \sum_{\nu_0 \in \mathbb{Z}_*^d} H_{1, \nu_0}(c) u_{\nu - \nu_0}^{(k-1)} - \varepsilon \sum_{p=2}^{\infty} \sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k-1}} \sum_{\substack{\nu_0, \nu_1, \dots, \nu_p \in \mathbb{Z}^d \\ \nu_0 + \nu_1 + \dots + \nu_p = \nu}} H_{p, \nu_0}(c) u_{\nu_1}^{(k_1)} \dots u_{\nu_p}^{(k_p)}, \quad k \geq 2, \quad (5.6b)$$

where  $\nu \neq 0$  and we have set once more  $u_0^{(1)} = \zeta$  and  $u_0^{(k)} = 0 \forall k \geq 2$ .

## 5.2 Trees and chains

As in Section 3, we can represent the coefficients  $u_\nu^{(k)}$  with tree diagrams; there are just few differences with the previous case. Define the sets  $N(\theta)$ ,  $E(\theta)$ ,  $E_0(\theta)$ ,  $E_1(\theta)$ ,  $V(\theta)$  and  $L(\theta)$  as previously and call  $k(\theta) := |N(\theta)|$  the order of  $\theta$ . Now we split  $V(\theta) = V_1(\theta) \amalg V_2(\theta)$ , with  $V_1(\theta) = \{v \in V(\theta) : p_v = 1\}$  and  $V_2(\theta) = \{v \in V(\theta) : p_v \geq 2\}$ . By contrast with Section 3, now in general  $V_1(\theta) \neq \emptyset$ .

We associate with each node  $v \in N(\theta)$  a *mode* label  $\nu_v \in \mathbb{Z}^d$  and with each line  $\ell \in L(\theta)$  a *momentum*  $\nu_\ell \in \mathbb{Z}^d$  defined by

$$\nu_\ell = \sum_{\substack{w \in N(\theta) \\ w \prec \ell}} \nu_w.$$

Finally we impose the constraints that

- $\nu_v \neq 0 \forall v \in V_1(\theta)$ ,
- $\nu_\ell \neq 0$  for any line  $\ell$  exiting a node in  $V(\theta)$ .

We associate with each node  $v \in N(\theta)$  a *node factor*

$$F_v := \begin{cases} -\varepsilon H_{p_v, \nu_v}(c), & v \in V(\theta), \\ -\varepsilon h_{\nu_v}(c), & v \in E_1(\theta), \\ \zeta, & v \in E_0(\theta), \end{cases} \quad (5.7)$$

and with each line  $\ell \in L(\theta)$  a *propagator*

$$\mathcal{G}_\ell := \begin{cases} D^{-1}(\varepsilon, \omega \cdot \nu_\ell), & \nu_\ell \neq 0, \\ \mathbf{1}, & \nu_\ell = 0. \end{cases} \quad (5.8)$$

We define the value of the tree  $\theta$  as in (3.5) and write  $u_\nu^{(k)}$  as in (3.6), where  $\mathcal{T}_{k, \nu}$  denotes the set of non-equivalent trees of order  $k$  and momentum  $\nu$  associated with the root line, constructed according to the new rules given above. Then, by construction, the coefficients  $u_\nu^{(k)}$  solve formally (5.6).

The main difference with respect to the trees considered in Section 3 is that, now, trees may contain ‘chains’. In a tree  $\theta$  we define a *chain*  $\mathcal{C}$  as a subset of  $\theta$  formed by a connected set of nodes  $v \in V(\theta)$  with  $p_v = 1$  and by the lines exiting them. Therefore, if  $V(\mathcal{C})$  and  $L(\mathcal{C})$  denote the set of nodes and the set of lines of  $\mathcal{C}$  and  $V(\mathcal{C}) = \{v_1, v_2, \dots, v_p\}$ , with  $v_1 \succ v_2 \succ \dots \succ v_p$ , then  $L(\mathcal{C}) = \{\ell_1, \ell_2, \dots, \ell_p\}$ , where  $\ell_i$  is the line exiting  $v_i$ , for  $i = 1, \dots, p$ , and  $\ell_i$  enters the node  $v_{i-1}$  for  $i = 2, \dots, p$ . We call  $p = |V(\mathcal{C})| = |L(\mathcal{C})|$  the *length* of the chain  $\mathcal{C}$ . Finally we define the *value* of the chain  $\mathcal{C}$  as the matrix

$$\mathcal{V}(\mathcal{C}) := \left( \prod_{v \in V(\mathcal{C})} F_v \right) \left( \prod_{\ell \in L(\mathcal{C})} \mathcal{G}_\ell \right),$$

with component indices  $i$  and  $j$  equal to the first index of the propagator of the line  $\ell_1$  and to the first index of the propagator of the line entering  $v_p$ , respectively, and denote by  $\mathfrak{C}(\theta)$  the set of all maximal chains contained in  $\theta$ .



### 5.3 Bounds

We can see that the bounds on the propagators are the same as in Section 3.3. Indeed let  $K$  and  $\kappa_1 \leq \dots \leq \kappa_m$  be as in the aforementioned section, so that the inverse of  $D = D(\varepsilon, s)$  can be written as

$$D^{-1} = K^{-1} (is\mathbb{1} + K^{-1}(-\varepsilon s^2 M + \varepsilon A)K^{-1})^{-1} K^{-1}.$$

By defining

$$S := \varepsilon K^{-1} A K^{-1} - \varepsilon s^2 K^{-1} M K^{-1} \quad \text{and} \quad R := S + is\mathbb{1},$$

we obtain  $D^{-1} = K^{-1} R^{-1} K^{-1}$ . Again, let  $b_1 \leq \dots \leq b_m$  be the eigenvalues of  $K^{-1} A K^{-1}$ : the eigenvalues of  $S$  are  $b_k \varepsilon + O(\varepsilon s^2)$ ,  $k = 1, \dots, m$ , and hence the eigenvalues of  $R$  are  $b_k \varepsilon + is + O(\varepsilon s^2)$ ,  $k = 1, \dots, m$ . By repeating the same argument as in Section 3.3, we find, for  $\varepsilon$  small enough,

$$\|D^{-1}\| \leq \frac{1}{\kappa_1^2} \min \left\{ \frac{2}{|b_1 \varepsilon|}, \max \left\{ \frac{1}{\alpha}, \frac{1}{|s|} \right\} \right\}, \quad (5.9)$$

that is the bound that we had in (3.8).

We define  $s_N$  and  $r_N$  as in (3.10), and fix  $N$  and  $\delta$  as in Section 3.4. With respect to (3.9), we redefine the constant  $C_0$  as

$$C_0 := m^2 \rho^{-1} \max \left\{ \frac{2\Delta}{\kappa_1^2 |b_1|}, 1 \right\}. \quad (5.10)$$

with  $\rho$  and  $\Delta$  as in (5.1), and set

$$\beta := \max \left\{ \delta, \frac{2|\varepsilon b_1|}{r_N} \right\}. \quad (5.11)$$

The following lemma provides us a bound for the values of the chains.

**Lemma 5.1.** *For any  $p \geq 1$  and for any chain  $\mathcal{C}$  of length  $p$ , one has*

$$\|\mathcal{Y}(\mathcal{C})\| \leq C_0^p \beta^{(p-1)/2} \prod_{v \in V(\mathcal{C})} e^{-3\xi|\nu_v|/4}.$$

*Proof.* We proceed by induction on  $p$ . If  $p = 1$  then  $\mathcal{C}$  contains only one node  $v$ , so that

$$\|\mathcal{Y}(\mathcal{C})\| \leq m|\varepsilon|\Delta\rho^{-1}e^{-\xi|\nu_v|}\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq C_0 e^{-3\xi|\nu_v|/4},$$

where we have bounded  $\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \leq 2/\kappa_1^2 |b_1 \varepsilon|$ , according to (5.9).

Let us denote by  $v_1 \succ v_2 \succ v_3 \succ \dots \succ v_p$  the nodes in  $\mathcal{C}$  and by  $\ell_1, \ell_2, \ell_3, \dots, \ell_p$  the lines exiting such nodes. If  $p \geq 2$ , the nodes  $\{v_2, v_3, \dots, v_p\}$  and the lines  $\{\ell_2, \ell_3, \dots, \ell_p\}$  form a chain  $\mathcal{C}'$  (contained in  $\mathcal{C}$ ) of length  $p - 1$ ; moreover, if  $p \geq 3$ , the nodes  $\{v_3, \dots, v_p\}$  and the lines  $\{\ell_3, \dots, \ell_p\}$  form a chain  $\mathcal{C}''$  (contained in  $\mathcal{C}'$ ) of length  $p - 2$ .

Assume the bound to hold up to  $p - 1$ . If  $|\omega \cdot \nu_{\ell_1}| \geq s_N/2$ , one has

$$\begin{aligned} \|\mathcal{V}(\mathcal{C})\| &\leq m|\varepsilon|\Delta\rho^{-1}e^{-\xi|\nu_{v_1}|}\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \|\mathcal{V}(\mathcal{C}')\| \\ &\leq C_0\beta \left(C_0^{p-1}\beta^{(p-2)/2}\right) \prod_{v \in V(\mathcal{C})} e^{-3\xi|\nu_v|/4}, \end{aligned}$$

which yields the bound for  $p$ . If  $|\omega \cdot \nu_{\ell_1}| < s_N/2$  and  $p \geq 3$  we distinguish between two cases. If  $|\omega \cdot \nu_{\ell_2}| \geq s_N/2$ , we can bound

$$\begin{aligned} \|\mathcal{V}(\mathcal{C})\| &\leq m^2|\varepsilon|\Delta\rho^{-1}e^{-\xi|\nu_{v_1}|}\|D^{-1}(\varepsilon, \omega \cdot \nu_{\ell_1})\| |\varepsilon|\Delta\rho^{-1}e^{-\xi|\nu_{v_2}|}\|D^{-1}(\varepsilon, \omega \cdot \nu_{\ell_2})\| \|\mathcal{V}(\mathcal{C}'')\| \\ &\leq C_0^2\beta \left(C_0^{p-2}\beta^{(p-3)/2}\right) \prod_{v \in V(\mathcal{C})} e^{-3\xi|\nu_v|/4}, \end{aligned}$$

so that the bound follows once more.

If  $|\omega \cdot \nu_{\ell_2}| < s_N/2$ , then

$$\begin{aligned} |\omega \cdot \nu_{v_1}| &= |\omega \cdot \nu_{v_1} + \omega \cdot \nu_{\ell_2} - \omega \cdot \nu_{\ell_2}| \leq |\omega \cdot \nu_{v_1} + \omega \cdot \nu_{\ell_2}| + |\omega \cdot \nu_{\ell_2}| \\ &= |\omega \cdot \nu_{\ell_1}| + |\omega \cdot \nu_{\ell_2}| < s_N, \end{aligned}$$

so that, since  $\nu_{v_1} \neq 0$  and hence  $\omega \cdot \nu_{v_1} \neq 0$ , we conclude that  $|\nu_{v_1}| > N$ , which allows us to bound  $e^{-\xi|\nu_{v_1}|} \leq \delta e^{-3\xi|\nu_{v_1}|/4}$ . Therefore we obtain

$$\|\mathcal{V}(\mathcal{C})\| \leq C_0^2\delta \|\mathcal{V}(\mathcal{C}'')\| \leq C_0^2\delta \left(C_0^{p-2}\beta^{(p-3)/2}\right) \prod_{v \in V(\mathcal{C})} e^{-3\xi|\nu_v|/4},$$

which gives the bound for  $p$  in this case too.

Finally if  $p = 2$  we can reason as in the case  $p \geq 3$ , the only difference being that the matrix  $\mathcal{V}(\mathcal{C}'')$  has to be replaced with  $\mathbf{1}$  – since there is no further chain  $\mathcal{C}''$  for  $p = 2$ . Therefore we obtain  $\|\mathcal{V}(\mathcal{C})\| \leq C_0^2\beta$ , which is the desired bound.  $\blacksquare$

**Lemma 5.2.** *For any tree  $\theta$  one has  $4|E(\theta)| + 2|V_1(\theta)| - 2|\mathfrak{C}(\theta)| \geq k(\theta) + 2$ .*

*Proof.* See [25, page 12].  $\blacksquare$

A result analogous to Lemma 3.1 still holds. This can be proved as follows. For any tree  $\theta$  we have

$$\begin{aligned} |\mathcal{V}_i(\theta)| &\leq m^{2k-1-2|V_1(\theta)|} \left( \prod_{v \in V_2(\theta)} \frac{2|\varepsilon|\Delta\rho^{-p_v}e^{-\xi|\nu_v|}}{\kappa_1^2|\varepsilon b_1|} \right) \times \\ &\quad \times \left( \prod_{v \in E_1(\theta)} |\varepsilon|\Delta e^{-\xi|\nu_v|}\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \right) \left( \prod_{v \in E_0(\theta)} |\zeta| \right) \left( \prod_{\mathcal{C} \in \mathfrak{C}(\theta)} \|\mathcal{V}(\mathcal{C})\| \right) \\ &\leq m^{-1}\rho C_0^k |\zeta|^{|E_0(\theta)|} \left( \prod_{v \in E_1(\theta)} |\varepsilon| e^{-\xi|\nu_v|/4}\|D^{-1}(\varepsilon, \omega \cdot \nu_v)\| \right) \times \\ &\quad \times \left( \prod_{\mathcal{C} \in \mathfrak{C}(\theta)} \beta^{(|V(\mathcal{C})|-1)/2} \right) \left( \prod_{v \in V(\theta)} e^{-3\xi|\nu_v|/4} \right), \end{aligned}$$

for  $i = 1, \dots, m$ , so that

$$|\mathcal{V}(\theta)| \leq \rho C_0^k \bar{\zeta}^{|E_0(\theta)|} \left( \frac{1}{\kappa_1^2} \max \left\{ \frac{2\delta}{|b_1|}, \frac{\bar{\varepsilon}}{r_N} \right\} \right)^{|E_1(\theta)|} \beta^{(|V_1(\theta)| - |\mathcal{C}(\theta)|)/2} \left( \prod_{v \in V(\theta)} e^{-3\xi|\nu_v|/4} \right).$$

Therefore, for any constant  $B \in (0, C_0)$ , if we fix first  $N$  and hence  $\bar{\varepsilon}$  and  $\bar{\zeta}$  so that for any  $|\varepsilon| < \bar{\varepsilon}$  and any  $|\zeta| < \bar{\zeta}$  one has

$$C_0^4 \max \left\{ \bar{\zeta}, \frac{2\delta}{\kappa_1^2 |b_1|}, \frac{\bar{\varepsilon}}{\kappa_1^2 r_N}, \beta \right\} < B^4, \quad (5.12)$$

we obtain

$$|\mathcal{V}(\theta)| \leq \rho C_0^k \left( \frac{B}{C_0} \right)^{4|E(\theta)| + 2|V_1(\theta)| - 2|\mathcal{C}(\theta)|} \left( \prod_{v \in V(\theta)} e^{-3\xi|\nu_v|/4} \right).$$

By using Lemma 5.2 we arrive at

$$|\mathcal{V}(\theta)| \leq \rho C_0^k \left( \frac{B}{C_0} \right)^{k+2} \left( \prod_{v \in V(\theta)} e^{-3\xi|\nu_v|/4} \right) \leq B_0 B^k \left( \prod_{v \in V(\theta)} e^{-3\xi|\nu_v|/4} \right), \quad B_0 := \rho \frac{B^2}{C_0^2}.$$

From here on, we can reason as in Section 3 and we find that the coefficients  $u_\nu^{(k)}$  can be bounded as

$$\left| u_\nu^{(k)} \right| \leq B_0 B_2^k e^{-\xi|\nu|/2},$$

for a suitable constant  $B_2$  proportional to the constant  $B$  in (5.12).

## 5.4 Conclusion of the proof

For  $\bar{\varepsilon}$  and  $\bar{\zeta}$  small enough,  $B_2$  can be made arbitrarily small; in particular one obtains  $B_2 \leq \tilde{B}$ , with  $\tilde{B} < 1$ , so that the series (5.5) converges for  $\mu = 1$ . As a consequence, the function  $x(t, \varepsilon, \zeta)$  solves the range equation (5.3a) for any  $\zeta$  small enough.

By reasoning as in Section 3.6 and 4, respectively, one proves the continuity in  $\varepsilon$  of the function  $x(t, \varepsilon, \zeta)$  and fixes the parameter  $\zeta$  when dealing with the bifurcation equation (5.3b). This concludes the proof of Theorem 2.

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