Conservation of resonant periodic solutions for the one-dimensional nonlinear Schrödinger equation

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Abstract. *We consider the one-dimensional nonlinear Schrödinger equation with Dirichlet boundary conditions in the fully resonant case (absence of the mass term). We investigate conservation of small amplitude periodic solutions for a large measure set of frequencies. In particular we show that there are infinitely many periodic solutions which continue the linear ones involving an arbitrary number of resonant modes, provided the corresponding frequencies are large enough, say greater than a certain threshold value depending on the number of resonant modes. If the frequencies of the latter are close enough to such a threshold, then they can not be too distant from each other. Hence we can interpret such solutions as perturbations of wave packets with large wave number.*

1. Introduction and set-up

We consider the *nonlinear Schrödinger equation* in $d = 1$ on the interval $[0, \pi]$, given by

$$\begin{cases} -iu_t + u_{xx} = \varphi(|u|^2)u, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{1.1}$$

where $\varphi(x)$ is any analytic function $\varphi(x) = \Phi x + O(x^2)$ with $\Phi \neq 0$. More generally we can take, instead of $\varphi(|u|^2)u$, any function $f(u, \overline{u})$ which is real analytic in its arguments, provided the two following conditions are fulfilled: the function f is odd in $(u, \overline{u})$ and the dominant order is still of the form $\Phi|u|^2u$. The first condition is likely only a technical one, while the second one is a non-degeneracy condition which is essential in the derivation of our results. In principle it could be removed, but then all the forthcoming analysis should be suitably changed.

We shall consider the problem of existence of resonant periodic solutions for (1.1), i.e. solutions arising from superpositions of several unperturbed harmonics, and we shall show how suitably adapting the techniques in Ref. [11] we can solve the problem.

The nonlinear Schrödinger equation has many physical applications, in nonlinear optics, plasma physics and fluid dynamics, and generally in any problem of evolution of quasi-monochromatic wave packets with moderate amplitude in strongly dispersive and weakly nonlinear media. The complex amplitude modulation of the packet is found to be described approximately by the cubic nonlinear Schrödinger equation. For a discussion of the applicative relevance of the equation we refer to Refs. [1], [5], [16], [17], [18], [19], [20], and papers quoted therein. The most studied equation of the form (1.1), with in principle any boundary conditions, is indeed the cubic one, and often equations containing high order terms are called generalised nonlinear Schrödinger equations. The particular relevance of the cubic equation is that it is completely solvable (i.e. integrable) on the line, as shown in Ref. [20], and on the interval with periodic boundary conditions, as shown in Ref. [14] by using the finite-gap approach first introduced in Ref. [15]. Other initial boundary values, including Dirichlet boundary condition on the semi-line and on the interval, have been recently considered in Refs. [8], [7] and [12], where the problem is shown to be reducible to the study of a system of ODE with algebraic right-hand side for the spectral data.

Existence of periodic (as well as quasi-periodic) solutions for (1.1) is well known; see for instance Refs. [13] and [3], and, very recently, Ref. [9], where more general nonlinearities are also considered, including the ones discussed after (1.1). The fact that no linear term as μu (mass term) appears in (1.1) introduces no further difficulties with respect to the case with mass. In this respect the situation is very different from the case of the nonlinear wave equation, where the completely resonant case ($\mu = 0$) can not be studied in the same way as the case with mass: there, when $\mu = 0$ only very special solutions of the linearized equation are found to be continuable in the presence of nonlinearities, for values of the periods which have those of the linearized equation (unperturbed periods) as Lebesgue density points [11] (the last result improves the previous ones where the unperturbed periods were found to be only accumulation points [2]). On the contrary in the case of the nonlinear Schrödinger equation, just because the cases $\mu = 0$ and $\mu \neq 0$ are faced in the same way, all the periodic solutions for $\mu = 0$ are obtained in the quoted papers as continuations of oscillations involving only one single mode.

Here we consider directly the case $\mu = 0$, and first we show how to recover the known results with a different technique, based on the Lindstedt series method introduced in Refs. [10] and [11]. Hence we discuss how to obtain other more complicated periodic solutions which arise from superposition of several (non-arbitrary) unperturbed modes. Such solutions look like perturbations of wave packets: the larger is the number N of involved harmonics the higher is the minimum allowed wave number of the corresponding wave packet. Moreover the width of the packet can be large with respect to the wave number only if the latter is large too, so that the smaller is the wave number the narrower is the wave packet itself. To give an idea of the phenomenology for instance, already for $N = 2$, we find that it is possible to continue for $\varepsilon \neq 0$ a packet involving for instance the two modes 7 and 8, but none with modes, say, 1 and 2. We note since now that the solutions that we find are special, as we impose that all amplitudes to be real. We do not think that such a constraint is necessary, but it rather simplifies the analysis of some non-degeneracy conditions of the unperturbed solutions.

If $\varphi = 0$, or $f = 0$, every solution of (1.1) can be written as

$$u(t,x) = \sum_{n=1}^{\infty} U_n e^{in^2 t} \sin nx = \sum_{n \in \mathbb{Z}_*} a_n e^{in^2 t} e^{inx}, \qquad a_{-n} = -a_n, \tag{1.2}$$

where we have set $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$. For $\varepsilon\Phi > 0$ we rescale $u \to \sqrt{\varepsilon/\Phi} u$ in (1.1), so obtaining

$$\begin{cases} iu_t + u_{xx} = \varepsilon |u|^2 u + O(\varepsilon^2), \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{1.3}$$

where $O(\varepsilon^2)$ denotes an analytic function of u, $\overline{u}$ and ε of order at least 2 in ε, and we define $\omega_\varepsilon = 1 + \varepsilon$.

We shall consider ε small and we shall show that for all $m_0 \in \mathbb{N}$ there exists a solution of (1.3), which is $2\pi/\omega_\varepsilon$-periodic in t and ε-close to the function

$$u_0(\omega_\varepsilon t, x) = a(\omega_\varepsilon t, x) - a(\omega_\varepsilon t, -x), \tag{1.4}$$

with

$$a(t,x) = A e^{im_0^2 t} e^{im_0 x}, \qquad A = \frac{m_0}{\sqrt{3}}, \tag{1.5}$$

provided ε is in an appropriate Cantor set (depending on m_0 and on the nonlinearity $f(u,\bar{u})$). We shall look for a solution of the form

$$u_\varepsilon(t,x) = \sum_{(n,m)\in\mathbb{Z}^2} e^{in\omega t+imx} u_{\varepsilon,n,m}, \qquad u_{\varepsilon,n,m}\in\mathbb{R}, \quad u_{\varepsilon,n,m} = -u_{\varepsilon,n,-m} \tag{1.6}$$

which is analytic both in x and t, and periodic in t. Then we shall use the norm

$$\|F(t,x)\|_r = \sum_{(n,m)\in\mathbb{Z}^2} F_{n,m} e^{r(|n|+|m|)} \tag{1.7}$$

for analytic functions.

Theorem 1. *Consider the equation*

$$\begin{cases} -iu_t + u_{xx} = f(u,\overline{u}), \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{1.8}$$

where $f(u,\overline{u})$ is any real analytic function, odd under the transformation $(u,\overline{u}) \to (-u,-\overline{u})$, such that $f(u,\overline{u}) = \Phi|u|^2u + O(|u|^5)$ with $\Phi\neq 0$. For all $m_0\in\mathbb{N}$, define $u_0(t,x) = a(t,x) - a(t,-x)$, with $a(t,x)$ as in (1.5). There are a positive constant ε_0 and a set $\mathcal{E}\in[0,\varepsilon_0]$, both depending on m_0, satisfying

$$\lim_{\varepsilon\to 0} \frac{\mathrm{meas}(\mathcal{E}\cap[0,\varepsilon])}{\varepsilon} = 1, \tag{1.9}$$

such that for all $\varepsilon\in\mathcal{E}$, by setting $\omega_\varepsilon = 1+\varepsilon$, there exists a $2\pi/\omega_\varepsilon$-periodic solution $u_\varepsilon(t,x)$ of (1.1), analytic in (t,x) and of the form (1.6), with

$$\left\| u_\varepsilon(t,x) - \sqrt{\varepsilon/\Phi}\, u_0(\omega_\varepsilon t,x) \right\|_\kappa \le C\,\varepsilon\sqrt{\varepsilon}, \tag{1.10}$$

with $\kappa = \kappa_0 \log 1/\varepsilon_0$, for some constants $C,\kappa_0>0$. If $f(u,\overline{u}) = \varphi(|u|^2)u$, with φ analytic, then one has $u_{\varepsilon,n,m}=0$ for all $n\neq m_0^2$ and $\mathcal{E}=[0,\varepsilon_0]$, that is no value of $\varepsilon\in[0,\varepsilon_0]$ has to be excluded.

As in Ref. [11] we start by considering the case $f(u,\overline{u}) = \varphi(|u|^2)u$, with $\varphi(x)=x$, which contains all the relevant features of the problem. In principle, with respect to the case (1.8), this introduces further symmetry properties which drastically simplify the problem, but if we ignore these simplifications we shall be able immediately to extend the analysis to more general nonlinearities, as in (1.8). This will be done in Section 4, where we shall also show how to deal with more general periodic solutions. The result we obtain at the end is the following one.

Theorem 2. *Consider the equation (1.1), where $\varphi(x) = \Phi x + O(x^2)$ is an analytic function, with $\Phi\neq 0$, or, more generally, the equation (1.8). For all $N\ge 2$ there are sets of N positive integers $\mathcal{M}_+$ and sets of real amplitudes $\{a_m\}_{m\in\mathcal{M}_+}$, such that the following holds. Define*

$$a(t,x) = \sum_{m\in\mathcal{M}_+} e^{im^2t+imx} a_m, \tag{1.11}$$

and set $u_0(t,x) = a(t,x) - a(t,-x)$. There are a positive constant ε_0 and a set $\mathcal{E}\in[0,\varepsilon_0]$, both depending on the set $\mathcal{M}_+$, satisfying

$$\lim_{\varepsilon\to 0} \frac{\mathrm{meas}(\mathcal{E}\cap[0,\varepsilon])}{\varepsilon} = 1, \tag{1.12}$$

such that for all $\varepsilon \in \mathcal{E}$, by setting $\omega_\varepsilon = 1 + \varepsilon$, there exist a $2\pi/\omega_\varepsilon$-periodic solution of (1.1) $u_\varepsilon(t,x)$ analytic in (t,x) and of the form (1.6), with

$$\left\| u_\varepsilon(t,x) - \sqrt{\varepsilon/\Phi}\, u_0(\omega_\varepsilon t, x) \right\|_\kappa \leq C\,\varepsilon\sqrt{\varepsilon}, \tag{1.13}$$

with $\kappa = \kappa_0 \log 1/\varepsilon_0$, for some constants $C, \kappa_0 > 0$.

In the proof of Theorem 2 a characterization of the sets $\mathcal{M}_+$ and of the amplitudes a_m will be provided. Hence the proof is constructive. What is found is that, by setting

$$M = \sum_{m \in \mathcal{M}_+} m^2, \qquad \|a\|^2 = \sum_{m \in \mathcal{M}_+} a_m^2, \qquad a_m \in \mathbb{R}, \tag{1.14}$$

one has to require $\|a\|^2 = M/(4N-1)$ and $a_m^2 = 4\|a\| - m^2$, which fixes the value of each amplitude, up to the sign (and up to an overall phase in the case (1.1), which, however, can be chosen to be zero under the request for the amplitudes to be real: we shall come back to this in a moment). In words, this means that the integers in $\mathcal{M}_+$ have to be large enough and close enough to each other, so that the solutions which can be continued appear as wave packets with large Fourier label (wave number). More precisely we shall find that for fixed N the harmonics have to be large enough – at best proportionally to N^2 –, while the width of the packet can not be smaller than $O(N)$. Then the wave packets with the smallest allowed wave numbers (which are the ones surviving for the largest values of ε) will have a width which is of order of the square root of the wave number.

Note that the level of difficulty of Theorem 1 in the case (1.1) is much lower than that of Theorem 2 (or even of Theorem 1 in the case (1.8)). The first is a result on the existence of a nonlinear ground state, and, with the Ansatz $u(t,x) = \exp(i\omega_\varepsilon t)\,\mathfrak{v}(x)$, it becomes a bifurcation problem for the function $\mathfrak{v}(x)$. Hence there is no small divisors problem, and other easier methods could be envisaged to prove that, no matter which linear eigenvalue is considered, there are nearby nonlinear solutions. The advantage of the method that we present is that it can be extended, with just a few slight adaptations, to prove existence of the periodic solutions of Theorem 1 in the case (1.8) and, mostly, of Theorem 2, which is on the contrary a substantially more difficult endeavour, as it mixes up different linear modes.

We can make a comparison between the nonlinear Schrödinger equation and the nonlinear wave equation, in the case of zero mass. For the wave equation only very special periodic solutions can be continued in the presence of nonlinearities, and this is due to the fact that the Q equation involves simultaneously all the harmonics. For the nonlinear Schrödinger equation one finds infinitely many periodic solutions, because there are infinitely many (but still not arbitrary) sets of harmonics which can be excited.

Note that for a fixed set of harmonics we construct explicitly only a few periodic solutions which continue the unperturbed ones (we have 2^N of them because of the arbitrariness of the amplitude signs). For simplicity we looked only for real values of the amplitudes a_m. Indeed in this way it turns out to be very easy to check some non-degeneracy conditions that we need in order to solve the equations to all orders. The drawback is that we find only special periodic solutions. It can be that, by relaxing the condition that the amplitudes be real, more general solutions can be found. We did not investigate further the problem, but we think that, at least in the case (1.1), one should be able to fix the unperturbed solution up to some arbitrary phases. This should mean that each amplitude a_m can be written as $\rho_m e^{i\theta_m}$, with $(\rho_m, \theta_m) \in \mathbb{R}_+ \times \mathbb{T}$, and likely there is some arbitrariness in choosing the angles θ_m (at best there can be N free parmeters, because of the symmetry $\theta_{-m} = \theta_m + \pi$). For $N = 1$ this simply yields that the unperturbed solution (1.5) is defined up to an arbitrary phase θ_0.

Finally we note that, as the case of non-zero mass μ can be reduced to that of zero-mass with the exponential substitution $u(t,x) = \exp(-i\mu t)\mathfrak{w}(t,x)$, Theorem 2 implies existence of families of quasi-periodic solutions with two-dimensional frequency vectors for the nonlinear Schrödinger equation with mass, or even further periodic solutions when the perturbed frequency ω_ε becomes commensurate with μ. Also such solutions are not known in literature.

From a technical point of view the discussion below heavily relies on [10] and [11]. We confine ourselves to explain how the renormalization group analysis developed in those papers applies to the nonlinear Schrödinger equation, by outlining the differences everywhere they appear and showing how they can be faced. Hence a full acquaintance with those paper is assumed to follow all the details of the technical parts. The discussion of Theorem 2 requires some new ideas, and involves problems which can be considered as typical of number theory and matrix algebra.

2. Lindstedt series expansion

2.1. Strategy of the proof. We proceed in three steps.

1. We perform a Lyapunov-Schmidt type decomposition. Namely we look for a solution of (1.3) of the form

$$\begin{aligned} u(t,x) &= \sum_{(n,m)\in\mathbb{Z}^2} e^{in\omega t+imx} u_{n,m} = v(\omega t,x) + w(\omega t,x), \\ v(t,x) &= \sum_{m\in\mathbb{Z}} e^{im^2 t+imx} v_m, \\ w(t,x) &= \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ n\neq m^2}} e^{int+imx} w_{n,m}, \end{aligned} \tag{2.1}$$

with $u_{n,m}\in\mathbb{R}$ and $\omega=\omega_\varepsilon = 1+\varepsilon$. In order to satisfy the Dirichlet boundary conditions, the solutions (if any), must verify

$$u_{n,m} = -u_{n,-m}, \tag{2.2}$$

for all $n,m\in\mathbb{Z}$. Another property that will be used in the following is that the subspace with $u_{n,m}$ real is invariant with respect to (1.1) and (1.8).

Inserting (2.1) into (1.3) gives two sets of equations, called the Q and P equations [6], which are given, respectively, by

$$\begin{array}{lll} \text{Q} & m^2 v_m = \left[f(v+w,\overline{v}+\overline{w})\right]_m , & \\ \text{P} & \left(\omega n - m^2\right) w_{n,m} = \varepsilon \left[f(v+w,\overline{v}+\overline{w})\right]_{n,m} , & n\neq m^2, \end{array} \tag{2.3}$$

where we denote by $[F]_{n,m}$ the Fourier component of the function $F(t,x)$ with labels (n,m), so that

$$F(t,x) = \sum_{(n,m)\in\mathbb{Z}^2} e^{in\omega t+mx}[F]_{n,m}, \tag{2.4}$$

and we shorthand $[F]_m$ the Fourier component of the function $F(t,x)$ with label $n=m^2$; hence $[F]_m = [F]_{m^2,m}$. We shall consider first the case $f(u,\overline{u}) = |u|^2 u$.

2. We study equation the Q equation in (2.3), in the limit $\varepsilon\to 0$ (and so $w_{n,m}\to 0$), and prove that it admits *non-degenerate* solutions $u_0(t,x)$ in appropriate *finite dimensional subspaces*; in particular in

the case of Theorem 1, for $\varepsilon = 0$ we get $v_m = 0$ for all $|m| \neq m_0$ and $v_{\pm m_0} = \pm A$ so that $u_0(t,x) = a(t,x) - a(t,-x)$.

3. We solve iteratively the P equation for $w(t,x)$ using renormalization techniques; similarly, using the non-degeneracy of $u_0(t,x)$, we solve the Q equation for

$$v(t,x) - u_0(x,t) \equiv V(x,t) = \sum_{m\in\mathbb{Z}} e^{im^2t+imx} V_m. \tag{2.5}$$

2.2. The Q equation. The Q equation in (2.3), in the limit $\varepsilon \to 0$ (and so $w_{n,m} \to 0$) is

$$m^2 v_m(\varepsilon = 0) = [|v(\varepsilon = 0)|^2 v(\varepsilon = 0)]_m, \tag{2.6}$$

therefore a non-trivial infinite dimensional equation for the coefficients $v_m(\varepsilon = 0)$. Following the scheme proposed in the previous section we set: $v_m(\varepsilon = 0) = 0$ for all $|m| \neq m_0$ and $v_{\pm m_0}(\varepsilon = 0) = \pm A$, with A defined as in (1.5), so that $u_0(t,x) = a(t,x) - a(t,-x)$, which is clearly a solution of (2.6).

Let us now consider the Q equation at $\varepsilon \neq 0$. With the notations of (2.4), and recalling that we are considering for the moment being $f(u,\overline{u}) = \varphi(|u|^2)u$, with $\varphi(x) = x$, we have

$$\begin{aligned} \big[|v+w|^2(v+w)\big]_m &= [|v|^2 v]_m + [|w|^2 w]_m + [2|v|^2 w + \overline{w} v^2]_m + [2|w|^2 v + \overline{v} w^2]_m \\ &\equiv [|v|^2 v]_m + [G_2(v,w)]_m, \end{aligned} \tag{2.7}$$

where $G_2(v,w)$ is at least linear in w.

As said in the previous section we write $v = a + b + V$, with $b(\omega t, x) = -a(\omega t, -x)$, i.e.

$$b(t,x) = B e^{im_0^2 t - im_0 x}, \qquad B = -A, \tag{2.8}$$

so that we obtain

$$\begin{aligned} [|v|^2 v]_m = {} & |A|^2 A \delta_{m,m_0} + |B|^2 B \delta_{m,-m_0} + 2|A|^2 B \delta_{m,-m_0} + 2|B|^2 A \delta_{m,m_0} \\ & + 2|A|^2 V_m + 2|B|^2 V_m + 2\overline{A} B V_{-m} \delta_{m,-m_0} + 2\overline{B} A V_{-m} \delta_{m,m_0} \\ & + \overline{V}_m A^2 \delta_{m,m_0} + 2\overline{V}_{-m} AB \delta_{m,\pm m_0} + \overline{V}_m B^2 \delta_{m,-m_0} + [G_1(v)]_m, \end{aligned} \tag{2.9}$$

where $G_1(v)$ is at least quadratic in V.

Then, by setting $G(v,w) = G_1(v) + G_2(v,w)$, the Q equation in (2.3) can be rewritten for $m = m_0$ as

$$m_0^2 V_{m_0} = 2|A|^2 V_{m_0} + 2|B|^2 V_{m_0} + \overline{V}_{m_0} A^2 + 2AB\overline{V}_{-m_0} + 2\overline{B} A V_{-m_0} + [G(v,w)]_{m_0}, \tag{2.10}$$

and for positive $m \neq m_0$ as

$$m^2 V_m = 2|A|^2 V_m + 2|B|^2 V_m + [G(v,w)]_m, \tag{2.11}$$

while the equation for negative values of m can be obtained by using the symmetry properties (2.2), which imply $V_{-m} = -V_m$.

By defining $\alpha = |A|^2 = \overline{A} A$ and using the identities

$$\alpha = \overline{A} A = \overline{B} B = -\overline{A} B = -\overline{B} A, \qquad \beta = AA = BB = -AB = -BA, \tag{2.12}$$

which follow trivially from the definitions of A and B in (1.5) and (2.8) respectively, we can rewrite (2.10) as

$$m_0^2 V_{m_0} = 4\alpha V_{m_0} + \beta \overline{V}_{m_0} - 2\beta \overline{V}_{-m_0} - 2\alpha V_{-m_0} + [G(v,w)]_{m_0}, \tag{2.13}$$

where we have used that $\alpha = |A|^2 = m_0^2/3$. By using once more the identities (2.2) and imposing that the coefficients V_m be real, so that $\alpha = \beta$ in (2.12), we can write (2.13) and the equation (2.11), respectively, as

$$\begin{cases} m_0^2 V_{m_0} = 9\alpha V_{m_0} + [G(v,w)]_{m_0}, \\ m^2 V_m = 4\alpha V_m + [G(v,w)]_m, \end{cases} \tag{2.14}$$

so that we find

$$\begin{cases} V_{m_0} = -\dfrac{1}{6\alpha}[G(v,w)]_{m_0}, \\ V_m = \dfrac{1}{m^2 - 4\alpha}[G(v,w)]_m, \end{cases} \tag{2.15}$$

respectively for positive m_0 and $m \neq m_0$.

2.3. The P equation. The P equation in (2.3) involves small divisors; as in Refs. [10] and [11] we handle them by appropriately renormalizing the frequencies.

Given a sequence $\{\nu_m(\varepsilon)\}_{|m|\geq 1}$, such that $\nu_m = \nu_{-m}$, we define the renormalized frequencies as

$$\tilde{\omega}_m^2 \equiv \omega_m^2 - \nu_m, \qquad \omega_m = |m|, \tag{2.16}$$

and the quantities ν_m will be called the counterterms.

By the above definition and the parity properties (2.2) the P equation in (2.3) can be rewritten as

$$\begin{aligned} \left(\omega n - \tilde{\omega}_m^2\right) w_{n,m} &= \nu_m w_{n,m} + \varepsilon [f(v+w, \overline{v} + \overline{w})]_{n,m} \\ &= \nu_m^{(a)} w_{n,m} + \nu_m^{(b)} w_{n,-m} + \varepsilon [f(v+w, \overline{v} + \overline{w})]_{n,m}, \end{aligned} \tag{2.17}$$

where

$$\nu_m^{(a)} - \nu_m^{(b)} = \nu_m. \tag{2.18}$$

Finally we write

$$w_{n,m} = g(n,m)\left(\mu \nu_m^{(a)} w_{n,m} + \mu \nu_m^{(b)} w_{n,-m} + \mu\varepsilon [|v+w|^2 (v+w)]_{n,m}\right), \tag{2.19}$$

where

$$g(n,m) = \frac{1}{\omega n - \tilde{\omega}_m^2}, \qquad n \neq m^2, \tag{2.20}$$

and we look for a solution $u_{n,m}$ in the form of a power series expansion in μ,

$$u_{n,m} = \sum_{k=0}^{\infty} \mu^k u_{n,m}^{(k)}, \tag{2.21}$$

with $u_{n,m}^{(k)}$ depending on ε and on the parameters $\nu_m^{(c)}$, with $c = a, b$ and $|m| \geq 1$.

2.4. Recursive equations. So we obtain recursive definitions of the coefficients $u_{n,m}^{(k)}$. The coefficients $w_{n,m}^{(k)}$ verify for $k \geq 1$ the equations

$$w_{n,m}^{(k)} = g(n,m)\left(\nu_m^{(a)} w_{n,m}^{(k-1)} + \nu_m^{(b)} w_{n,-m}^{(k-1)} + [|v+w|^2 (v+w)]_{n,m}^{(k-1)}\right), \tag{2.22}$$

where

$$[|v+w|^2(v+w)]^{(k)}_{n,m} = \sum_{k_1+k_2+k_3=k} \sum_{\substack{-n_1+n_2+n_3=n \\ -m_1+m_2+m_3=m}} \overline{u}^{(k_1)}_{n_1,m_1} u^{(k_2)}_{n_2,m_2} u^{(k_3)}_{n_3,m_3}, \tag{2.23}$$

with

$$\begin{aligned} u^{(0)}_{n,m} &= \begin{cases} A, & \text{if } n=m^2 \text{ and } m=m_0, \\ -A, & \text{if } n=m^2 \text{ and } m=-m_0, \\ 0, & \text{otherwise} \end{cases} \\ u^{(k)}_{n,m} &= \begin{cases} V^{(k)}_m, & \text{if } n=m^2\ , \\ w^{(k)}_{n,m}, & \text{if } n\neq m^2, \qquad k\geq 1\ , \end{cases} \end{aligned} \tag{2.24}$$

while the coefficients $V^{(k)}_m$ verify for $k \geq 1$ the equations

$$V^{(k)}_m = g(m^2,m) \sum_{k_1+k_2+k_3=k} \sum^{*}_{\substack{-n_1+n_2+n_3=m \\ -m_1+m_2+m_3=m}} \overline{u}^{(k_1)}_{n_1,m_1} u^{(k_2)}_{n_2,m_2} u^{(k_3)}_{n_3,m_3}, \tag{2.25}$$

where

$$g(m^2,m) = \begin{cases} -\dfrac{1}{2m_0^2}, & \text{if } |m|=m_0\ , \\ \dfrac{3}{3m^2-4m_0^2}, & \text{if } |m|\neq m_0\ , \end{cases} \tag{2.26}$$

and the $*$ means that there appear only contributions either with at least one coefficient with $n \neq m^2$ or with at least two labels $k_i \geq 1$.

It is easy to realize that to any order k one has $u^{(k)}_{n,m} = 0$ whenever $n \neq m_0^2$, and the same remains still true if we replace $f(u,\overline{u}) = \Phi|u|^2u$ with any function of the form $f(u,\overline{u}) = \varphi(|u|^2)u$ (one can check this obvious property, for instance, by induction on k). For the time being, however, we ignore such a property, and we proceed as if every value of n was possible. The reason to do this is that in such a way the results that we can find for the case $f(u,\overline{u}) = \varphi(|u|)^2u$ can be extended immediately to the more general case of equation (1.8).

To prove Theorem 1 we can proceed in two steps as in Ref. [11]. The first step consists in looking for the solution of the recursive equations by considering $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1}$ as a given set of parameters satisfying the Diophantine conditions (called respectively the first and the second Mel′nikov conditions)

$$\begin{aligned} &\left|\omega n \pm \tilde{\omega}^2_m\right| \geq C_0|n|^{-\tau} \qquad \forall n\in\mathbb{Z}_* \text{ and } \forall m\in\mathbb{Z}_* \text{ such that } n\neq m^2, \\ &\left|\omega n \pm \left(\tilde{\omega}^2_m \pm \tilde{\omega}^2_{m'}\right)\right| \geq C_0|n|^{-\tau} \qquad \forall n\in\mathbb{Z}_* \text{ and } \forall m,m'\in\mathbb{Z}_* \text{ such that } |n|\neq|m^2\pm(m')^2|, \end{aligned} \tag{2.27}$$

with positive constants C_0, τ. We can assume without loss of generality $C_0 \leq 1/2$.

We shall show in Section 3 how to adapt the discussion in Ref. [11] in order to obtain the following result.

Proposition 1. *Consider a sequence $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1}$ verifying (2.27), with $\omega = \omega_\varepsilon = 1+\varepsilon$ and such that $|\tilde{\omega}^2_m - m^2| \leq C_1\varepsilon$ for some constant C_1. For all $\mu_0 > 0$ there exists $\varepsilon_0 > 0$ such that for $|\mu| \leq \mu_0$ and $0 < \varepsilon < \varepsilon_0$ there is a sequence $\nu(\tilde{\omega},\varepsilon;\mu) = \{\nu_m(\tilde{\omega},\varepsilon;\mu)\}_{|m|\geq 1}$, where each $\nu_m(\tilde{\omega},\varepsilon;\mu)$ is analytic in μ, such* *that there are coefficients $u^{(k)}_{n,m}$ which solve the recursive equations (2.22) and (2.25), with $\nu_m = \nu_m(\tilde{\omega}, \varepsilon)$, and define a function $u(t, x; \tilde{\omega}, \varepsilon; \mu)$ which is analytic in μ, analytic in (t, x) and $2\pi/\omega_\varepsilon$-periodic in t.*

Then in Proposition 1 one can fix $\mu_0 = 1$, so that one can choose $\mu = 1$ and set $u(t, x; \tilde{\omega}, \varepsilon) = u(t, x; \tilde{\omega}, \varepsilon; 1)$ and $\nu_m(\tilde{\omega}, \varepsilon) = \nu_m(\tilde{\omega}, \varepsilon; 1)$.

The second step, also to be proved in Section 3, consists in inverting (2.1), with $\nu_m = \nu_m(\tilde{\omega}, \varepsilon)$ and $\tilde{\omega}$ verifying (2.27). This requires some preliminary conditions on ε, given by the Diophantine conditions

$$|\omega n \pm m| \geq 2\, C_0 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z}_* \text{ and } \forall m \in \mathbb{Z}_* \text{ such that } n \neq m, \tag{2.28}$$

with a positive constant $\tau_0 > 1$. Then we can solve iteratively (2.1), by imposing further non-resonance conditions besides (2.28). At each iterative step one has to exclude some further values of ε, and at the end the left values fill a Cantor set $\mathcal{E}$ with large relative measure in $[0, \varepsilon_0]$ and $\tilde{\omega}$ verify (2.27). Of course in the case (1.1), which yields $n = m_0^2$, no further condition has to be imposed as one has $|m^2 - m_0^2| \geq 1 > |\varepsilon m_0^2|$ for fixed m_0 and ε small enough.
The result of this second step can be summarized as follows.

Proposition 2. *In the case (1.8) here are $\delta > 0$ and a set $\mathcal{E} \subset [0, \varepsilon_0]$ with complement of relative Lebesgue measure of order ε_0^δ such that for all $\varepsilon \in \mathcal{E}$ there exists $\tilde{\omega} = \tilde{\omega}(\varepsilon)$ which solves (2.1) and satisfy the Diophantine conditions (2.27) with $|\tilde{\omega}_m^2 - m^2| \leq C_1 \varepsilon$ for some positive constant C_1. In the case (1.1) the same result holds for all $\varepsilon \in [0, \varepsilon_0]$.*

The proof follows the same strategy as in Ref. [11]. The slight changes will be discussed in Section 3.

3. Renormalization and proof of Theorem 1

We refer to Section 3 in Ref. [11] for the basic definitions of trees (cf. in particular Definition 2). With respect to that paper the diagrammatic rules are changed as follows.

(1) We call *nodes* the vertices such that there is at least one line entering them. We call *end-points* the vertices which have no entering line. We denote with $L(\theta)$, $V(\theta)$ and $E(\theta)$ the set of lines, nodes and end-points, respectively. For any vertex $\mathbb{v}$ (node or end-point) there is one and only one line ℓ exiting it, so that we can set $\ell = \ell$.

(2) There can be two types of lines, w-lines and v-lines, so we associate with each line $\ell \in L(\theta)$ a *badge* label $\gamma_\ell \in \{v, w\}$ and a *momentum* $(n_\ell, m_\ell) \in \mathbb{Z}^2$, to be defined in item (6) below. One has $\gamma_\ell = v$ if $n_\ell = m_\ell^2$, and $\gamma_\ell = w$ otherwise. One can not have $(n_\ell, m_\ell) = (0, 0)$. All the lines coming out from the end-points are v-lines.

(3) With each line ℓ coming out from a node we associate a *propagator* $g_\ell = g(n_\ell, m_\ell)$, with $g(n, m)$ defined in (2.20) and (2.26) if the line comes out from a node, while one has $g_\ell = 1$ if the line ℓ comes out from an end-point.

(4) If we denote by s the number of lines entering the node $\mathbb{v}$ one can have either $s = 1$ or $s = 3$. In the latter case we call L the set of lines entering $\mathbb{v}$: we associate with each line $\ell \in L$ a label $s(\ell) \in \{\pm 1\}$ with the constraint $\sum_{\ell \in L} s(\ell) = 1$. Also the nodes $\mathbb{v}$ can be of w-type and v-type: we say that a node is of v-type if the line ℓ coming out from it has label $\gamma_\ell = v$; analogously the nodes of w-type are defined. We can write $V(\theta) = V_v(\theta) \cup V_w(\theta)$, with obvious meaning of the symbols; we also call $V_w^s(\theta)$, $s = 1, 3$, the

set of nodes in $V_w(\theta)$ with s entering lines, and analogously we define $V_v^s(\theta)$, $s=1,3$. One has $s=3$ for all $\mathbb{v}\in V_v$ (so that $V_v^1=\emptyset$ unlike Ref. [11]). If $\mathbb{v}\in V_v^3(\theta)$ and two entering lines come out of end points then the remaining line entering $\mathbb{v}$ has to be a w-line. If $\mathbb{v}\in V_w^1(\theta)$ then the line entering $\mathbb{v}$ has to be a w-line.

(5) With each end-point $\mathbb{v}$ we associate a mode label (n,m), with $m=\pm m_0$ and $n=m_0^2$, and an *end-point factor*

$$V=\begin{cases} A, & m=m_0 \ , \\ -A, & m=-m_0 \ . \end{cases} \tag{3.1}$$

while with each node $\mathbb{v}$ we associate a *node factor*

$$\eta=\begin{cases} 1/3, & \mathbb{v}\in V_v(\theta) \ , \\ \varepsilon, & \mathbb{v}\in V_w^3(\theta) \ , \\ \nu_{m_\ell}^{(c)}, & \mathbb{v}\in V_w^1(\theta) \ , \end{cases} \tag{3.2}$$

where $V_v=V_v^3$ (cf. item (4)), and $c=a$ if $m_\ell=m_\ell$, where ℓ is the line entering $\mathbb{v}$, while $c=b$ if $m_\ell=-m_\ell$, with the same notations.

(6) The momentum (n_ℓ,m_ℓ) of a line $\ell=\ell$ coming out from a node $\mathbb{v}$ is given by

$$n_\ell=\sum_{\substack{\in E(\theta)\\ \preceq}}(-1)^{S(,\ell)}n, \qquad m_\ell=\sum_{\substack{\in E(\theta)\\ \preceq}}(-1)^{S(,\ell)}m+\sum_{\substack{\in V_w^1(\theta):c=b\\ \preceq}}(-2m_\ell), \tag{3.3}$$

where $S(\mathbb{w},\ell)$ is the number of lines ℓ with $s(\ell)=-1$ between $\mathbb{w}$ and ℓ.

Note that the rules given above look simpler with respect to Ref. [11]. This is due essentially to the fact that the Q equation is much simpler in the present case.

Introducing a multiscale decomposition as in Section 4 of Ref. [11] we can define for the lines ℓ with $\gamma_\ell=w$ the propagator on scale $h\ge -1$ as

$$g_\ell^{(h)}=\chi_h(|\omega n_\ell-\tilde{\omega}_m^2|)\,g_\ell=\frac{\chi_h(|\omega n_\ell-\tilde{\omega}_m^2|)}{\omega n_\ell-\tilde{\omega}_m^2}, \tag{3.4}$$

where $\chi_h(x)$ is a C^∞ function non-vanishing for $2^{-h-1}C_0<|x|<2^{-h+1}C_0$ if $h\ge 0$ and for $|x|>C_0$ if $h=-1$.

This leads to new diagrammatic rules, which differ with respect to the previous ones because item (6) has to be replaced by the following one.

(6′) Each line carries, besides the momentum $(n_\ell,m_\ell)\in\mathbb{Z}^2$, also a scale label $h_\ell\ge -1$ for $\gamma_\ell=w$ and a scale label $h_\ell=-1$ for $\gamma_\ell=v$. The corresponding propagator $g_\ell^{(h_\ell)}$ is given by (3.4) with $h=h_\ell$ for $\gamma_\ell=w$, while is the same as before for $\gamma_\ell=v$.

Then for each tree θ one can define the tree value as

$$\mathrm{Val}(\theta)=\left(\prod_{\ell\in L(\theta)}g_\ell^{(h_\ell)}\right)\left(\prod_{\in V(\theta)}\eta\right)\left(\prod_{\in E(\theta)}V\right), \tag{3.5}$$

so that one has

$$u^{(k)}_{n,m} = \sum_{\theta \in \Theta^{(k)}_{n,m}} \mathrm{Val}(\theta), \tag{3.6}$$

where $\Theta^{(k)}_{n,m}$ is the set of tress θ of order k, that is with $|V_w(\theta)| = k$, and with momentum (n, m) associated with the root line (we omit the proof, as it proceeds exactly as for Lemma 2 in Ref. [11]). Note that one has $|V_v(\theta)| \le 2|V_w(\theta)| = 2k$ and $|E(\theta)| \le 2(|V_w(\theta)| + |V_v(\theta)|) + 1 \le 6k + 1$ (cf. Lemma 3 in Ref. [11]).

Clusters and self-energy graphs are defined as in Ref. [11] (cf. Definitions 6 and 7). In particular we call ℓ^1_T and ℓ^2_T the lines exiting and entering (respectively) the self-energy graph T. Given a self-energy graph T with momentum (n, m) associated to the line ℓ^2_T the corresponding self-energy value is given by

$$\mathcal{V}^h_T(\omega n, m) = \Big(\prod_{\ell \in T} g^{(h_\ell)}_\ell \Big) \Big(\prod_{\in V(T)} \eta \Big) \Big(\prod_{\mathbb{v} \in E(T)} V \Big), \tag{3.7}$$

where $h = h^{(e)}_T$ is the minimum between the scales of the two external lines of T (they can differ at most by a unit and $h^{(e)}_T \ge 0$), and, given a self-energy graph, one has

$$\begin{aligned} n(T) &\equiv \sum_{\in E(T)} (-1)^{S(,\ell^1_T)} n = 0, \\ m(T) &\equiv \sum_{\in E(T)} (-1)^{S(,\ell^1_T)} m + \sum_{\substack{\in V^1_w(T) \\ c=b}} (-2m_\ell) \in \{0, 2m\}, \end{aligned} \tag{3.8}$$

by definition of self-energy graph. One says that T is a self-energy graph of type $c = a$ when $m(T) = 0$ and a resonance of type $c = b$ when $m(T) = 2m$.

The following results hold.

Lemma 1. *Assume that there is a constant C_1 such that $|\tilde{\omega}^2_m - m^2| < C_1 \varepsilon$ for all $m \ge 1$. If $|\omega n_\ell - \tilde{\omega}^2_m| < 1/2$ and ε is small enough then $\min\{n_\ell, m^2_\ell\} > 1/4\varepsilon$.*

Proof. One has $\omega n - \tilde{\omega}^2_m = \varepsilon n + (n - m^2) + \nu_m$, so that $|\omega n - \tilde{\omega}^2_m| > 1/2$ for $n \ne m^2$ and $0 < n < 1/3\varepsilon$. Moreover if $|\omega n - \tilde{\omega}^2_m| < 1/2$ then one has $n > 0$ and $m^2 > \omega n - |\nu_m| - 1/2 > 1/4\varepsilon$. ■

Hence if $n_\ell < 1/4\varepsilon$ we can bound $|g(n_\ell, m_\ell)| \le 2$ while if $n_\ell \ge 1/4\varepsilon$ in general we can bound $|g(n_\ell, m_\ell)| \le 2^{h+1} C_0^{-1}$. To any line ℓ with $n_\ell < 1/4\varepsilon$ we can assign a scale label $h_\ell = -1$.

Lemma 2. *Assume that there is a constant C_1 such that $|\tilde{\omega}^2_m - m^2| < C_1 \varepsilon$. Define h_0 such that $2^{h_0} < 16 C_0 / \sqrt{\varepsilon} < 2^{h_0+1}$. Then for $h \ge h_0$ one has*

$$N_h(\theta) \le 4k 2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M^\nu_h(\theta), \tag{3.9}$$

where $N_h(\theta)$ is the number of lines in $L(\theta)$ on scale h, $C_h(\theta)$ is the number of clusters in θ on scale h, $S_h(\theta)$ the number of self-energy graphs in θ with $h^{(e)}_T = h$ and $M^\nu_h(\theta)$ is the number of ν-vertices (i.e. nodes $\mathbb{v}$ of w-type with $s = 1$) in θ; for more details cf. Definition 8 and Lemma 5 of Ref. [11].

Proof. The proof as for Lemma 5 of Ref. [11]. Again the only case which deserves attention is when one has a cluster T with two external lines ℓ and ℓ_1 both on scales $\ge h$, so that, with the same notations as in Ref. [11], one has

$$2^{-h+2} C_0 \ge \big| \omega(n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}^2_{m_\ell} + \eta_{\ell_1} \tilde{\omega}^2_{m_{\ell_1}} \big|. \tag{3.10}$$

Then $|n_\ell - n_{\ell_1}| = |m_\ell^2 \pm m_{\ell_1}^2|$ would require $|n_\ell - n_{\ell_1}| \ge |m_\ell| + |m_{\ell_1}| > 1/\sqrt{\varepsilon}$, while (3.10) would become $2^{-h+2}C_0 > |\varepsilon(n_\ell - n_{\ell_1})| - 2C_1\varepsilon$. Combining the two inequalities one would obtain $C_0 2^{-h+3} > \sqrt{\varepsilon}$, which contradicts the condition $h \ge h_0$. Then one proceeds as in Ref. [11]. ■

Lemma 3. *Assume that there is a constant C_1 such that $|\tilde{\omega}_m^2 - m^2| < C_1\varepsilon$. Then one has*

$$\prod_{h=0}^{h_0-1} \prod_{\substack{\ell\in L(\theta) \\ h_\ell = h}} |g_\ell^{(h_\ell)}| \le C_2^k \varepsilon^{-k/2}, \tag{3.11}$$

for some positive constant C_2.

Proof. If $h < h_0$ one has $|g_\ell^{(h_\ell)}| \le C_0 2^{-h_0+1} < \sqrt{\varepsilon}/4$, and the number of lines ℓ with scales $0 \le h_\ell < h_0$ can be bounded by the total number of lines ℓ with label $\gamma_\ell = w$, which is less than k. ■

The renormalized expansion is defined as in Section 5 Ref. [11], with the only difference that now the action of the localization operator $\mathcal{L}$ is such that

$$\mathcal{L}\mathcal{V}_T^h(\omega n, m) = \mathcal{V}_T^h(\tilde{\omega}_m^2, m), \tag{3.12}$$

so that, in the definition of the set $E_0(\theta)$ (see item (7′) in Section 5 of Ref. [11]), we set $\overline{\omega}_m = \tilde{\omega}_m^2/\omega$. Up to these notational changes no other difference appears with respect to the discussion carried out in Ref. [11], hence we introduce the set $\Theta_{n,m}^{(k)\mathcal{R}}$ of renormalized trees by adding the following further rules to the previous ones.

(7) With the nodes $\mathtt{v}$ of w-type with $s = 1$ (ν-vertices) and with $h \ge 0$ the minimal scale among the lines entering or exiting $\mathtt{v}$, we associate a factor $2^{-h}\nu_{h,m}^{(c)}$, $c = a, b$, where (n, m) and $(n, \pm m)$, with $|m| \ge 1$, are the momenta of the lines, and a corresponds to the sign $+$ and b to the sign $-$ in $\pm m$.

(8) The set $\{h_\ell\}$ of the scales associated to the lines $\ell \in L(\theta)$ must satisfy the following constraint (which we call *compatibility*): fixed (n_ℓ, m_ℓ) for any $\ell \in L(\theta)$ and replaced $\mathcal{R}$ with $\mathbb{1}$ at each self-energy graph, one must have $\chi_{h_\ell}(|\omega n_\ell| - \tilde{\omega}_{m_\ell}) \ne 0$.

In terms of the renormalized trees we can write

$$u_{n,m} = u_{n,m}^{(0)} + \sum_{k=1}^{\infty} \mu^k \sum_{\theta\in\Theta_{n,m}^{(k)\mathcal{R}}} \mathrm{Val}(\theta), \tag{3.13}$$

where, for $|m| \ge 1$ and $h \ge 0$, $\nu_{h,m}^{(c)}$ is given by

$$2^{-h}\mu\nu_{h,m}^{(c)} = \mu\nu_m^{(c)} + \frac{1}{2}\sum_{\sigma=\pm}\sum_{T\in\mathcal{T}_{<h}^{(c)}} \mu^{k_T}\mathcal{V}_T^h(\sigma\tilde{\omega}_m, m), \tag{3.14}$$

with $c = a, b$, and $\mathcal{T}_{<h}^{(c)}$ denoting the set of self-energy graphs T of type c with $h_T < h$. The tree value $\mathrm{Val}(\theta)$ is defined recursively as in equations (5.5) to (5.7) in Ref. [11]. Call $L_0(\theta)$, $V_0(\theta)$ and $E_0(\theta)$ the sets of lines, nodes and end-points, respectively, in θ which are not contained in any self-energy graph, and $S_0(\theta)$ the set of *maximal self-energy graphs*, *i.e.* self-energy graphs which are not contained in any self-energy graphs. We can write $\mathrm{Val}(\theta)$ in (3.13) as

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L_0(\theta)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{\mathbb{V} \in V_0(\theta)} \eta \Big) \Big(\prod_{\mathbb{V} \in E_0(\theta)} V \Big) \Big(\prod_{T \in S_0(\theta)} \mathcal{R}\mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T}) \Big), \tag{3.15}$$

where ℓ_T denotes the line entering T, and we have set

$$\mathcal{R}\mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T}) = \mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T}) - \mathcal{L}\mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T}), \tag{3.16}$$

with $\mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T})$ given by

$$\mathcal{V}_T^{h_T^{(e)}}(\omega n_{\ell_T}, m_{\ell_T}) = \Big(\prod_{\ell \in L_0(T)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{\mathbb{V} \in V_0(T)} \eta \Big) \Big(\prod_{\mathbb{V} \in E_0(T)} V \Big) \Big(\prod_{T' \in S_0(T)} \mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\omega n_{\ell_{T'}}, m_{\ell_{T'}}) \Big), \tag{3.17}$$

if $L_0(T)$, $V_0(T)$, $E_0(T)$ and $S_0(T)$ are the sets of lines, nodes, end-points and maximal self-energy graphs in T which are not contained in any self-energy graph internal to T.

Now we proceed exactly as in Ref. [11].

First (by Lemma 7 of Ref. [11]) we have that the expansion (3.13) is well defined, for $\nu_{h,m} = O(\varepsilon)$; namely we have that that the coefficients (3.13) are bounded by $|u_{n,m}| \le D_0 \varepsilon^{|m|/4m_0} \varepsilon^{|n|/4m_0^2}$.

The presence of the factor $\varepsilon^{|m|/4m_0} \varepsilon^{|n|/4m_0^2}$, instead of the factor $\varepsilon\, e^{-\kappa(|n|+|m|)/4}$ in equation (5.8) of Ref. [11], is due to the fact that the end-points carry a mode label $(\pm m_0, m_0^2)$, so that in order to have a momentum (n,m) flowing through the root line one needs at least k_* nodes, with $k_* > \min\{|m|/m_0, |n|/m_0^2\}$, and for $\theta \in \Theta^{(k))\mathcal{R}}_{n,m}$ the tree value $\mathrm{Val}(\theta)$ is in general proportional to $\varepsilon^{k/2}$ by (3.11) (the discussion is very similar to that performed in Ref. [10], which we refer to for further details).

Then (by Lemma 8 and Lemma 9 of Ref. [11]) we have that under the same conditions also the r.h.s. of (3.14) is well defined. Moreover (by Lemma 10 of Ref. [11]) we find that it is indeed possible to choose $\nu_m^{(c)}$ such that $\nu_{h,m} = O(\varepsilon)$ for any h. We omit the other details which can be easily worked out by looking at the quoted reference.

This completes the proof of Proposition 1.

We now pass to Proposition 2: we look for the perturbed frequencies $\tilde\omega_m(\varepsilon)$ which solve (2.16) and satisfy the Diophantine conditions (2.27) for $\varepsilon \in \mathcal{E}$, where $\mathcal{E}$ is a Cantor set of relative large measure. We proceed as in Section 6 of Ref. [11], with some minor differences (which are in fact simplifications) that we outline below.

The condition (2.28) on ε can be imposed exactly as in Ref. [11] (Lemma 14), and requires $\varepsilon \in \mathcal{E}^{(0)}$, with $\mathcal{E}^{(0)}$ a suitable subset of $[0, \varepsilon_0]$, with ε_0 as in Proposition 1, of large relative measure provided one sets $\tau_0 > 1$.

We define a sequence $\tilde\omega = \{\tilde\omega_m^{(p)}\}_{m=1}^\infty$ by setting

$$\begin{aligned} &(\omega_m^{(0)})^2 = \omega_m^2, \\ &(\tilde\omega_m^{(p)})^2 = \omega_m^2 - \nu_m(\tilde\omega_m^{(p-1)}, \varepsilon), \qquad p \ge 1, \end{aligned} \tag{3.18}$$

with $\nu_m(\tilde\omega_m^{(p)}, \varepsilon)$ well defined on a Cantor set $\mathcal{E}^{(p)}$ where, as in equation (6.11) of Ref. [11], the Mel′nikov conditions (2.27) are satisfied with $\tilde\omega_m = \tilde\omega_m^{(p)}$.

By reasoning as for Lemma 16 of Ref. [11] the following result is immediately obtained.

Lemma 4. *For all $p \geq 0$ there exists a positive constant C_3 such that $|\nu_m(\tilde{\omega}^{(p)}, \varepsilon)| < C_3\varepsilon$.*

Hence for all fixed p the hypotheses of Proposition 1 are satisfied and $\nu_m(\tilde{\omega}_m^{(p+1)}, \varepsilon)$ is well defined on some smaller Cantor set $\mathcal{E}^{(p+1)} \subset \mathcal{E}^{(p)}$.

By reasoning as for Lemma 15 of Ref. [11] (Section 6) we have that the difference between the frequencies at two subsequent steps decreases exponentially:

$$\max_{m\in\mathbb{Z}} |\nu_m(\tilde{\omega}^{(p)}, \varepsilon) - \nu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \leq C\varepsilon^p, \tag{3.19}$$

so that the sequence $\{\tilde{\omega}^{(p)}\}_{p=0}^{\infty}$ has a limit $\tilde{\omega}^{(\infty)} \equiv \tilde{\omega}^{(\infty)}(\varepsilon)$ which satisfies the equations (2.18) and is well defined on a Cantor set $\mathcal{E} = \mathcal{E}^{(\infty)}$.

To conclude the proof of Proposition 2 we still need to impose the Mel′nikov conditions in (2.27), with $\tilde{\omega}_m = \tilde{\omega}_m^{(p)}(\varepsilon)$ and to verify that the set of ε which satisfy such conditions for all p is of large relative measure and has the origin as a density point. To do this, we evaluate the measure of the complementary set to $\mathcal{E}^{(p)}$ in $[0, \varepsilon_0]$ defined by

$$f_1(\varepsilon(t)) \equiv (1 + \varepsilon(t))n - (\tilde{\omega}_m^{(p)}(\varepsilon(t)))^2 = t\frac{C_0}{|n|^\tau}, \qquad t \in [-1, 1], \qquad \varepsilon(t) \in (0, \varepsilon_0), \tag{3.20}$$

when dealing with the first Mel′nikov conditions (cf. equation (6.30) of Ref. [11]), and through

$$f_2(\varepsilon(t)) \equiv (1+\varepsilon(t))|n_\ell| - |(\tilde{\omega}_m^{(p)}(\varepsilon(t)))^2 \pm (\tilde{\omega}_{m'}^{(p)}(\varepsilon(t)))^2| = t\frac{C_0}{|n|^\tau}, \qquad t \in [-1, 1], \qquad \varepsilon(t) \in (0, \varepsilon_0), \tag{3.21}$$

when dealing with the second Mel′nikov conditions. The functions f_1 and f_2 depend also on n, m and n, m, m', respectively: we are not making explicit such a dependence in order not to overwhelm the notations.

The measure of the complementary set is then bounded by

$$\sum_{n,m\in\mathbb{Z}}^{*} \frac{C_0}{|n|^\tau} \max_{\varepsilon\in(0,\varepsilon_0)} |\partial f_1/\partial\varepsilon|^{-1} + \sum_{n,m,m'\in\mathbb{Z}}^{*} \frac{C_0}{|n|^\tau} \max_{\varepsilon\in(0,\varepsilon_0)} |\partial f_2/\partial\varepsilon|^{-1}, \tag{3.22}$$

where $*$ means that in the sums over n, m, m' one only has to consider those values n, m, m' such that both $f_1(\varepsilon)$ and $f_2(\varepsilon)$ can be small, say smaller than $1/4$. As in Ref. [11] one has to use that $|\partial f_j/\partial\varepsilon| \geq |n|/2$ for $j = 1, 2$.

In the case of the first Mel′nikov conditions one has to consider only the values of n such that $n \geq N_0 = O(\varepsilon_0^{-1})$, as $|\nu_m| < C_1\varepsilon$, and for each n the set $\mathcal{M}_0(n)$ of m's such that $f_1(\varepsilon) < 1/4$ contains at most $2 + \varepsilon_0\sqrt{n}$ values. Therefore the measure of the set of excluded values of ε turns out to be bounded proportionally to

$$\sum_{n=N_0}^{\infty} \frac{C_0}{n^{\tau+1}} \left(2 + \varepsilon_0\sqrt{n}\right) \leq \text{const.}\, \varepsilon_0^{1+\delta_1}, \tag{3.23}$$

with $\delta_1 > 0$ provided one takes $\tau > 1$.

In the case of the second Mel′nikov conditions one has to use that if $|n|$ is close to $|m^2 - (m')^2|$ then $|n|$ is of order $||m| - |m'||\,(|m| + |m'|)$, with $|m| - |m'| \neq 0$, so that $|m| + |m'| \leq |n|$. This means that for each n the number of pairs (m, m') one has to sum over is at most proportional to $2 + \varepsilon_0|n|^2$. The same happens (trivially) when $|n|$ is close to $m^2 + (m')^2$. In both cases one has to sum only on the values of n such that $|n| \geq N_0 = O(\varepsilon_0^{-1})$, so that one has to exclude a set of values of ε whose measure is bounded proportionally to

$$\sum_{n=N_0}^{\infty} \frac{C_0}{n^{\tau+1}} \left(2 + \varepsilon_0 n^2\right) \leq \text{const.}\, \varepsilon_0^{1+\delta_2}, \tag{3.24}$$

with $\delta_2 > 0$ provided one takes $\tau > 2$.

The argument above is for fixed p. By taking into account that the centers of the intervals of excluded values of ε get closer and closer at each iterative step p (cf. Lemma 17 in Ref. [11]), we find that we have to apply the construction above only for a finite number of steps $p_0(n)$ (growing proportionally to $\log|n|$), at the price of enlarging the sizes of the first $p_0(n)$ intervals. The conclusion is that if we set $\tau > 2$ we have that $\mathcal{E}^{(\infty)}$ has large relative measure. Again we refer to Ref. [11] for further details.

4. Extension of the results and proof of Theorem 2

The extension of the results of the previous sections to the case in which $\varphi(x)$ is any analytic function with $\varphi'(0) \neq 0$, can be easily dealt with by reasoning as in Section 8 of Ref. [11]. Essentially the diagrammatic rules change as one has to take into account also the contributions of order higher than three arising from the nonlinearity, which means that now s can be any odd positive integer and for each node $\mathtt{v}$ with $s > 1$ the node factors η depend on the function φ.

If we use that $w_{n,m}^{(k)} = 0$ for $n \neq m_0^2$ then we see that in fact we have no small divisor problem: the quantity $|\omega n \pm \tilde{\omega}_m^2| = |\omega m_0^2 \pm \tilde{\omega}_m^2|$ is bounded from below for all $m \in \mathbb{Z} \setminus \{\pm m_0\}$, provided ε is small enough. Hence in the case (1.1) the discussion can be substantially simplified.

The advantage of the proof given in Section 3 is that it still applies to any $f(u, \overline{u})$ as in (1.8). In such a case there is no longer a symmetry property which imposes $n = m_0^2$, so that n can really assume any value. Therefore we need the Diophantine conditions (2.27) and we have to exclude some values of ε. This leads naturally to the set $\mathcal{E}$ defined in the statement of Theorem 1. As for the diagrammatic rules now also the constraints on $\sum_{\ell \in L} s(\ell)$, besides the node factors, depend on f.

No extra difficulty arises with respect to the analysis of Section 3, so we pass directly to discuss the case of more general periodic solutions to be continued.

For $\varepsilon = 0$ we call $v_0 = v(\varepsilon = 0) = a + b$ the solution of the Q equation in (2.3), by writing

$$a(t, x) = \sum_{m=1}^{\infty} a_m e^{im^2 t + imx}, \tag{4.1}$$

with coefficients $a_m \in \mathbb{R}$ to be determined, and setting $b(t, x) = -a(t, -x)$. Then the Q equation becomes

$$m^2 v_{0,m} = \sum_{\substack{-m_1+m_2+m_3=m \\ -m_1^2+m_2^2+m_3^2=m^2}} \overline{v}_{0,m_1} v_{0,m_2} v_{0,m_3} = 2 v_{0,m} \sum_{m' \neq m} \overline{v}_{0,m'} v_{0,m'} + \overline{v}_{0,m} v_{0,m} v_{0,m}, \tag{4.2}$$

so that we obtain

$$v_{0,m} \left(m^2 - 2\|v_0\|^2 + |v_{0,m}|^2\right) = 0, \tag{4.3}$$

where we have defined

$$\mathcal{M} = \{m \in \mathbb{Z} : v_{0,m} \neq 0\}, \qquad \mathcal{M}_+ = \{m \in \mathcal{M} : m > 0\}, \tag{4.4}$$

and set

$$\|v_0\|^2 \equiv \sum_{m\in\mathbb{Z}} v_{0,m}^2 = \sum_{m\in\mathcal{M}} v_{0,m}^2. \tag{4.5}$$

Hence (4.3) can be satisfied either if $v_{0,m} = 0$ or, when $v_{0,m} \neq 0$, if

$$\|v_0\|^2 = \frac{2M}{4N-1}, \tag{4.6}$$

where we have set

$$2N = |\mathcal{M}| = \#\{m \in \mathcal{M}\}, \qquad 2M = \sum_{m\in\mathcal{M}} m^2. \tag{4.7}$$

By inserting (4.6) into (4.3), setting

$$a_m = v_{0,m}, \quad m > 0, \qquad \|a\|^2 = \sum_{m\in\mathcal{M}_+} a_m^2 = \frac{1}{2}\|v_0\|^2, \tag{4.8}$$

and writing $\mathcal{M}_+ = \{m_1, m_2, \ldots, m_N\}$, with $m_k < m_{k+1}$, $k = 1, \ldots, N-1$, we obtain

$$a_{m_k}^2 = 4\|a\|^2 - m_k^2 = \frac{4}{4N-1}\left(m_1^2 + m_2^2 + \ldots + m_N^2\right) - m_k^2, \qquad k = 1, \ldots, N, \tag{4.9}$$

which makes sense as long as

$$\max_{m\in\mathcal{M}_+} m^2 \le \frac{4}{4N-1} \sum_{m\in\mathcal{M}_+} m^2. \tag{4.10}$$

The following result is easily proved.

Lemma 5. *For all $N \ge 2$ there are solutions of (4.9) such that $4\|a\|^2$ is not an integer.*

Proof. To obtain a solution one can take $m_k = m_N - (N-k)$ for $k = 1, \ldots, N$, and choose $m_N \ge 4N(N-1)$. Choose $m_N = (4N+j)(N-1)$, with $j \in \{0, 1\}$: then $4(m_1^2 + \ldots + m_N^2)$ can not be a multiple of $4N-1$ for both $j = 0$ and $j = 1$. ■

Here we are confined ourselves only to an existence result. Of course more general solutions can be envisaged, with more spacing between the involved wave numbers m_k. The result above can indeed be strengthened as follows.

Lemma 6. *For all $N \ge 2$ and for all increasing lists of positive integers $I := \{i_1, \ldots, i_{N-1}\}$ there exists $m_N(I)$ (m_N for short) such that (4.10) has a solution in the set $\mathcal{M}_+ = \{m_N - i_{N-1}, \ldots, m_N - i_1, m_N\}$ with $4\|a\|^2 \neq m^2$ for all $m \notin \mathcal{M}$.*

Proof. Fix the set of integers $I = \{i_1, \ldots, i_{N-1}\}$, and consider the expression $M - (N-1/4)j^2$ for $j \in \mathbb{N}$. For $j = m_N$ it becomes a polynomial of degree two in m_N, with positive leading coefficient $1/4$ and positive discriminant. Hence there is an integer K_1 such that for all $m_N > K_1$ one has $f_1(m_N) \equiv M - (N-1/4)m_N^2 > 0$, hence (4.10) is satisfied.

The inequality $M - (N-1/4)j^2 > 0$ is trivially satisfied for $j \le m_N$, so that it is enough to look an integer $m_N > K_1$ such that one has $f_2(m_N) \equiv M - (N-1/4)(m_N+1)^2 < 0$. Again f_2 is a polynomial of degree two in m_N, with positive leading coefficient $1/4$ and positive discriminant, so that there exist two integers $K_2 < K_3$ such that $f_2(m_N) < 0$ for $K_2 < m_N < K_3$. Moreover $K_3 - K_2 \geq 4(N-1)$, so that there is m_N satisfying (4.10) such that $4\|a\| \neq j^2$ for all $j \in \mathbb{N}$. ∎

The condition $4\|a\|^2 \neq m^2$ for all $m \in \mathcal{M}$, required in Lemma 6 and implied by Lemma 5, implies that the solution $v_0(t,x)$ is non-degenerate, namely the linearized operator acting on V is invertible, so that V_m turns out to be defined iteratively to all orders (compare (2.15) in the case of Theorem 1 with Lemma 8 below). The request for the amplitides a_m to be real was motivated just with the aim of making straightforward the check of the non-degeneracy condition.

To have solutions of (4.9) requires the integers in $\mathcal{M}_+$ to be large enough, and not too distant from each other. Indeed, at best, the distance between the harmonics is $O(N)$, while the harmonics themselves are greater than some threshold value, which is $O(N^2)$ – cf. the proof of Lemma 5. Hence the solutions whose existence is stated in Lemma 5 have the form of wave packets centered around some harmonic (wave number) large enough, with a width proportional to the square root of the wave number. In general, if the harmonics are very large with respect to the threshold value, then also the width of the packet can be large. Not however that in order to be sure that the quantity $4\|a\|^2$ is not a squared integer in the proof of Lemma 6 we required m_N (hence the wave number of the corresponding packet) to be not too large. Moreover we are more interested in wave packets with wave number not too large, as the larger is the latter the smaller is the corresponding value ε_0 appearing in the statement of Theorem 2, as we shall see.

Hence we have proved the following result.

Lemma 7. *For any N there are sets $\mathcal{M}$ and functions $v_0(t,x) = a(\omega t, x) - a(\omega t, -x)$, with*

$$a(t,x) = \sum_{m \in \mathcal{M}_+} e^{im^2\omega t + imx} a_m, \tag{4.11}$$

which solve the Q equation with $\varepsilon = 0$.

Moreover, by using once more the parity properties $V_{-m} = -V_m$, one obtains, generalising (2.14) to the case $N > 1$ and with the same meaning as there for the function $G(v,w)$, for $m \in \mathcal{M}_+$

$$\sum_{m' \in \mathcal{M}_+} \mathcal{A}_{m,m'} V_{m'} = \left[G(v,w)\right]_m , \tag{4.12}$$

where $\mathcal{A}$ is an $N \times N$ matrix with entries

$$\mathcal{A}_{m,m'} = \begin{cases} m^2 - 4\|a\|^2 - 5a_m^2, & m = m' \ , \\ -8a_m a_{m'}, & m \neq m' \ , \end{cases} \tag{4.13}$$

It is also easy to check that, if the amplitudes a_m are chosen to be real, then also the amplitudes V_m and $w_{n,m}$ can be found to be real (as remarked in Section 2 the condition $u_{n,m} \in \mathbb{R}$ is consistent with equation (1.8)).

Then the following result holds.

Lemma 8. *For $\mathcal{M}$ chosen according to Lemma 6 (or Lemma 5), one has for $m \in \mathcal{M}_+$*

$$V_m = \sum_{m' \in \mathcal{M}_+} \mathcal{D}_{m,m'} [G(v,w)]_{m'}, \tag{4.14}$$

with $\mathcal{D}$ a $N \times N$ non-singular matrix. For positive $m \notin \mathcal{M}_+$, an analogous, simpler expression is found of the form (4.14) with

$$V_m = (m^2 - 4\|a\|^2)^{-1} \left[G(v,w)\right]_m \tag{4.15}$$

where the coefficients $(m^2 - 4\|a\|^2)$ are not zero by Lemma 6 (or Lemma 5). The amplitudes V_m for negative m are easily obtained by noting that $V_m = -V_{-m}$.

Proof. To obtain (4.14) it is sufficient to prove that the matrix $\mathcal{A}$ with entries (4.13) is not singular. By using (4.3) we can write the diagonal entries of $\mathcal{A}$ as $\mathcal{A}_{m,m} = -6a_m^2$. Then one realizes immediately that one has

$$\det \mathcal{A} = (-1)^N \det D_N(6,8) \prod_{m=1}^{N} a_m^2, \tag{4.16}$$

where $D_N(p,q)$ is the $N \times N$ matrix with diagonal entries p and all off-diagonal entries q. One can easily prove that $\det D_N(p,q) = (p-q)^{N-1}(p + (N-1)q)$. As in our case $p = 6$ and $q = 8$ (so that $p \neq q$ and $p < (N-1)q$ for all $N \geq 2$) the assertion follows. Finally equation (4.15) is a direct generalisation of (2.9). ∎

This allows us to extend the analysis of the previous section to the case in which the function v_0 is of the form considered here. At the end Theorem 2 is obtained, with the set $\mathcal{M}$ chosen according to Lemma 6.

Following Section 2 we insert the series expansion (2.21) in the P equation in (2.19) and in the new Q equation, as given by (4.14) and (4.15). The iterative P equation (2.22) is unchanged, while, by equation (4.15) and by Lemma 8, the iterative Q equation (2.25) should be substituted by

$$V_m^{(k)} = g(m^2, m) \sum_{k_1+k_2+k_3=k} \sideset{}{^*}\sum_{\substack{-n_1+n_2+n_3=m^2 \\ -m_1+m_2+m_3=m}} \overline{u}_{n_1,m_1}^{(k_1)} u_{n_2,m_2}^{(k_2)} u_{n_3,m_3}^{(k_3)}, \tag{4.17}$$

with $g(m^2, m) = (m^2 - 4\|a\|^2)^{-1}$, for $m \notin \mathcal{M}$, and

$$V_m^{(k)} = \sum_{m' \in \mathcal{M}_+} \mathcal{D}_{m,m'} \sum_{k_1+k_2+k_3=k} \sideset{}{^*}\sum_{\substack{-n_1+n_2+n_3=(m')^2 \\ -m_1+m_2+m_3=m'}} \overline{u}_{n_1,m_1}^{(k_1)} u_{n_2,m_2}^{(k_2)} u_{n_3,m_3}^{(k_3)} \tag{4.18}$$

for $m \in \mathcal{M}_+$, while $V_m^{(k)} = -V_{-m}^{(k)}$ for $m \in \mathcal{M} \setminus \mathcal{M}_+$.

Let us consider first the case $f(u, \overline{u}) = |u|^2 u$. The tree expansion is as in Section 3, with the following differences. In item (5) now to each end-point a mode label (n, m), with $m \in \mathcal{M}$ and $n = m^2$, and an end-point factor $V = \sigma a_m$, with $\sigma = \operatorname{sgn} m$ are associated. Moreover each line ℓ carries two momentum labels (n_ℓ, m_ℓ) and (n'_ℓ, m'_ℓ), with m_ℓ and $m_{\ell'}$ both in $\mathcal{M}$ or in its complement. If $m_\ell, m'_\ell \in \mathcal{M}$ they are both either in $\mathcal{M}_+$ or in $\mathcal{M} \setminus \mathcal{M}_+$, and the propagator g_ℓ is not diagonal any more, as it is given by $g_\ell = \mathcal{D}_{m,m'}$ (see (4.15)), while if $m_\ell, m_{\ell'} \notin \mathcal{M}$ then $m_\ell = m'_\ell$ and $g_\ell = g(m^2, m)$, with $g(m^2, m)$ given as after (4.17); like in the previous case one has $n_\ell = m_\ell^2$ and $n'_\ell = (m'_\ell)^2$ for such lines. Also for the lines of type w we set $n_\ell = n'_\ell$ and $m_\ell = m'_\ell$, but one has $n_\ell \neq m_\ell^2$ in such a case.

The momentum (n'_ℓ, m'_ℓ) is recursively defined as

$$n'_\ell = \sum_{\ell \in L} (-1)^{s(\ell)} n_\ell, \qquad m'_\ell = \sum_{\ell \in L} (-1)^{s(\ell)} m_\ell, \tag{4.19}$$

for $\mathbb{v} \in V_w^3(\theta) \cup V_v^3(\theta)$, and

$$n'_\ell = n_\ell, \qquad m_\ell = \delta_{c,a} m_\ell - \delta_{c,b} m_\ell, \tag{4.20}$$

for $\mathbb{v} \in V_w^1$, if ℓ denotes the line entering $\mathbb{v}$.

The self-energy graphs T with momentum (n, m) associated to the line ℓ_T^2 are characterized by the relations $n(T) = n_{\ell_T^1} - n_{\ell_T^2} = 0$ and $m(T) = m_{\ell_T^1} - m_{\ell_T^2} \in \{0, 2m\}$. No other differences arise with respect to Section 3, so that the analysis can be carried out in the same way.

For more general f one reasons as at the beginning of this Section. We do not describe in detail the obvious changes of notations.

Of course the value of ε_0 depends on the set $\mathcal{M}$, and in particular it goes to zero when $N \to \infty$ (as M diverges in such a case) and, for fixed N, when $M \to \infty$.

The conclusion is that infinitely many unperturbed solutions which are trigonometric polynomial with an arbitrary number of harmonics can be continued in presence of nonlinearities. The case of polynomials of degree 1 (Theorem 1) is the one usually considered in literature, while the case of polynomials of higher order (Theorem 2) is new. In the latter case the only request on the harmonics is that the corresponding wave numbers have to be close enough to each other and that larger is their number the larger are their values.

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