Response solutions for forced systems with large dissipation and arbitrary frequency vectors

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Abstract

We study the behaviour of one-dimensional strongly dissipative systems subject to a quasi-periodic force. In particular we are interested in the existence of response solutions, that is quasi-periodic solutions having the same frequency vector as the forcing term. Earlier results available in the literature show that, when the dissipation is large enough and a suitable function involving the forcing has a simple zero, response solutions can be proved to exist and to be attractive provided some Diophantine condition is assumed on the frequency vector. In this paper we show that the results extend to the case of arbitrary frequency vectors.

1 Introduction

Consider the singular ordinary differential equation in $\mathbb{R}$

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, g(x) = \varepsilon\, f(\boldsymbol{\omega} t), \tag{1.1}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter and $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^d$, with $d \in \mathbb{N}$. The functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{T}^d \to \mathbb{R}$ are assumed to be real analytic. In particular, the function $t \mapsto f(\boldsymbol{\omega} t)$ is quasi-periodic in t and, under the regularity assumptions we made, we can take its Fourier expansion

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}},$$

with the Fourier coefficients $f_{\boldsymbol{\nu}}$ decaying exponentially in $\boldsymbol{\nu}$.

For motivations and physical applications we refer to [30, 10, 17, 13] and references cited therein. For a brief overview of related results on the existence of quasi-periodic solutions with frequency vectors not satisfying any Diophantine condition we refer to Section 1.2 below. Here we confine ourselves to recall that the equation (1.1) with $\varepsilon > 0$ describes a one-dimensional system in the presence of dissipation and subject to an autonomous force g and an additional quasi-periodic forcing term f. The inverse of the *perturbation* *parameter* ε plays the role of the damping coefficient, so that a small value for ε corresponds to large dissipation – on the contrary, no smallness condition is assumed on the forces f and g acting on the system. The vector $\boldsymbol{\omega}$ is the *frequency vector* of the forcing term. A *response solution* to (1.1) is a quasi-periodic solution with the same frequency vector $\boldsymbol{\omega}$ as the forcing.

1.1 Main results

If $\varepsilon = 0$, for any constant $c \in \mathbb{R}$, $x = c$ is a solution to (1.1). The problem we want to address is whether it is possible to choose the constant c so that, for ε small enough, the equation (1.1) admits a response solution which tends to c as ε tends to zero.

Without any assumptions on the functions f and g, the answer in general is negative. More precisely, let us consider the following non-degeneracy condition.

Hypothesis 1. *Let f and g be the functions in* (1.1). *The function $g(x) - f_{\mathbf{0}}$ has a simple zero c_0.*

If the hypothesis is not satisfied one can provide counterexamples showing that response solutions may fail to exist [18]. On the contrary, if the hypothesis holds, we shall prove that a response solution always exists, without requiring any further condition on the frequency vector $\boldsymbol{\omega}$.

This generalises previous results available in the literature and gives a positive answer to a question raised in [13]. Indeed in [20, 21, 17, 18, 7, 13] some condition was assumed on $\boldsymbol{\omega}$. We briefly recall the results. If we set $\alpha_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n\right\}$ and define

$$\varepsilon_n(\boldsymbol{\omega}) := \frac{1}{2^n}\log\frac{1}{\alpha_n(\boldsymbol{\omega})}, \qquad \mathfrak{B}(\boldsymbol{\omega}) := \sum_{n=0}^{\infty}\varepsilon_n(\boldsymbol{\omega}),$$

we say that $\boldsymbol{\omega} \in \mathbb{R}^d$ satisfies

- the *standard Diophantine condition* if there exists two positive constants γ_0 and $\tau > 0$ such that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| > \gamma_0|\boldsymbol{\nu}|^{-\tau}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{\mathbf{0}\}$,
- the *Bryuno condition* if the sequence $\varepsilon_n(\boldsymbol{\omega})$ is summable, i.e. $\mathfrak{B}(\boldsymbol{\omega}) < \infty$ [6].

In [20] response solutions were proved to exist by assuming the vector $\boldsymbol{\omega}$ to satisfy the standard Diophantine condition and the result was then extended to vectors satisfying the Bryuno condition in [21]. An even weaker condition was considered in [7]. More precisely, if one requires a condition which does not depend on the function f, one needs $\varepsilon_n(\boldsymbol{\omega})$ to go to 0 as $n \to \infty$; for fixed f, one can allow vectors $\boldsymbol{\omega}$ such that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|^{-1} \le \alpha\, \mathrm{e}^{\eta|\boldsymbol{\nu}|}$ $\forall \boldsymbol{\nu} \in \mathbb{Z}^d_*$, for some $\alpha > 0$ and $\eta < \xi$, with ξ being the width of the strip of analyticity of f (see Section 2 for notations) – we refer to [12, Sect. IV] for a more detailed comparison with the literature. The results were extended further to the case in which the zero c_0 of the function $g(x) - f_{\mathbf{0}}$ is of odd order $\mathfrak{n} > 1$, in [17, 18] for summable $\varepsilon_n(\boldsymbol{\omega})$ and in [13] for $\varepsilon_n(\boldsymbol{\omega})$ converging to zero.

In this paper, still assuming Hypothesis 1, we remove the condition on the frequency vector and we only require that the components of $\boldsymbol{\omega}$ are rationally independent, that is $\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0 \; \forall \boldsymbol{\nu} \in \mathbb{Z}^d_*$. This can be done without any loss of generality, since, if this is not the case, f can be expressed as a quasi-periodic function with frequency vector $\boldsymbol{\omega}' \in \mathbb{R}^{d'}$, for some integer $d' < d$, with rationally independent components. Thus, we shall prove the following result.

Theorem 1. *Consider the ordinary differential equation* (1.1) *and assume Hypothesis 1. For any frequency vector* $\boldsymbol{\omega} \in \mathbb{R}^d$*, there exists* $\varepsilon_0 > 0$ *such that for all* $|\varepsilon| < \varepsilon_0$ *there is at least one quasi-periodic solution* $x_0(t) = c_0 + X(\boldsymbol{\omega} t, \varepsilon)$ *to* (1.1)*, such that* $X(\boldsymbol{\psi}, \varepsilon)$ *is analytic in* $\boldsymbol{\psi}$ *and goes to* 0 *as* $\varepsilon \to 0$*. If* $g'(c_0) > 0$ *the solution is locally attractive in the plane* $(x, \dot{x})$ *and hence unique in a neighbourhood of* $(c_0, 0)$*.*

The equation (1.1) is a special case of the more general

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, h(x, \boldsymbol{\omega} t) = 0, \tag{1.2}$$

where $h : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ is a real analytic. If we take the Fourier expansion of $\boldsymbol{\psi} \mapsto h(x, \boldsymbol{\psi})$ by writing

$$h(x, \boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i} \boldsymbol{\nu} \cdot \boldsymbol{\psi}} h_{\boldsymbol{\nu}}(x),$$

a natural counterpart of Hypothesis 1 for (1.2) is the following,

Hypothesis 2. *Let* h *be the function in* (1.2)*. The function* $h_{\mathbf{0}}(x)$ *has a simple zero* c_0*.*

Then the following result generalises Theorem 1.

Theorem 2. *Consider the ordinary differential equation* (1.2) *and assume Hypothesis 2. For any frequency vector* $\boldsymbol{\omega} \in \mathbb{R}^d$*, there exists* $\varepsilon_0 > 0$ *such that for all* $|\varepsilon| < \varepsilon_0$ *there is at least one quasi-periodic solution* $x_0(t) = c_0 + X(\boldsymbol{\omega} t, \varepsilon)$ *to* (1.2)*, such that* $X(\boldsymbol{\psi}, \varepsilon)$ *is analytic in* $\boldsymbol{\psi}$ *and goes to* 0 *as* $\varepsilon \to 0$*.*

The two theorems provide information about the behaviour of a one-dimensional system in the presence of large dissipation and subject to a quasi-periodic force, as described by (1.2) – or by (1.1) as a particular case. The existence of a response solution depends on the zeroes of the function $h_{\mathbf{0}}(x)$. If the function either has no zero or has a zero of even order, the results in [18] show that in general no response solution exists. On the contrary, if the function has a zero of order $\mathfrak{n} = 1$, then the system always admits a response solution, without any assumption on the frequency vector of the forcing term, provided the dissipation is large enough. Such a result is obviously stronger than the results of the papers quoted above, since no condition is assumed on $\boldsymbol{\omega}$. On the other hand, as it will emerge from the analysis of the next sections, in general a smaller value is obtained for the estimate of ε_0: this means that the closer $\boldsymbol{\omega}$ to a resonance, the larger dissipation is needed for the response solution to exist. Moreover, without any assumption on $\boldsymbol{\omega}$, less information is obtained about the regularity in ε of the response solution. In fact, the stronger non-resonance condition on $\boldsymbol{\omega}$, the more regularity is obtained on the dependence of the solution on ε. For instance, if either $d = 1$ (periodic case) or $d = 2$ and $\boldsymbol{\omega}$ satisfies a standard Diophantine condition with exponent $\tau = 1$, the solution turns out to be Borel-summable in ε [21]. Under the non-resonance condition considered in [13], the response solution is found to be C^{∞} in ε – and analytic in a suitable domain with boundary tangent to the origin [7, 12]. On the contrary, if we do not assume any condition on $\boldsymbol{\omega}$, in general no more than a continuous dependence on ε can be obtained. Also the analyticity properties in the complex plane are weaker with respect to the cases considered in [7, 12], since we can prove only analyticity in ε in conical domains; see the end of Section 3 for more details.

We conclude with a few remarks.

1. As mentioned above, if $\boldsymbol{\omega}$ is such that the sequence $\varepsilon_n(\boldsymbol{\omega})$ converges to zero, the existence of a response solution to (1.1) can be proved under the weaker assumption that order $\mathfrak{n}$ of the zero of the function $g(x) - f_{\mathbf{0}}$ is odd [13]. Unfortunately, the proof of theorem 1 given in Section 2 does not extend to the case $\mathfrak{n} > 1$.
2. Another issue that deserves further investigation is the uniqueness and stability of the quasi-periodic solution. In the case of Theorem 1, under the assumption that $g'(c_0) > 0$, asymptotic stability and hence uniqueness follow from the same argument as given in [2, Section 5].
3. In this paper we have explicitly considered the one-dimensional case; however we expect the results to carry over into the case in which $x \in \mathbb{R}^n$ and f and g are vector-valued real analytic function.

The rest of the paper is organised as follows. We shall give the proof of Theorem 1 in Section 2 and of Theorem 2 in Section 3. Of course we could have confined ourselves to Theorem 2, since Theorem 1 is included as a particular case. The reason why we have stated apart Theorem 1 is that, as we shall see, the proofs in the forthcoming sections are in increasing order of difficulty. Thus, it may be helpful to start considering first Theorem 1, where the proof is easier, before tackling the more general case where further technical intricacies arise. Moreover, the equation (1.1) studied in Theorem 1 is the one usually considered in the literature.

The proof of the theorems will be performed by introducing an auxiliary parameter μ, eventually to be put equal to 1, and looking for a formal power series expansion of the solution in terms of μ. Therefater, the series will be proved to be convergent by relying on a diagrammatic representation of the coefficients and showing that the radius of convergence is greater than 1 provide ε is taken small enough: hence $\mu = 1$ is allowed. Note that the series is not a power series in ε. However, notwithstanding the solution, as already observed above, is not even expected to be differentiable in ε, we still are able to use convergent power series expansions.

1.2 Related results on quasi-periodic solutions

Periodic, quasi-periodic and almost periodic solutions in singularly perturbed systems have been widely studied in the literature; we refer to [4, 15] for an introduction to almost periodic functions. The ordinary differential equation (1.1), as well as (1.2), can be considered as a particular case of the system

$$\dot{x} = f(t, x, y, \varepsilon), \qquad \varepsilon\,\dot{y} = g(t, x, y, \varepsilon), \tag{1.3}$$

where, more generally, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, for some $n, m \in \mathbb{N}$, and the functions f, g are almost periodic in time t. By assuming that an almost periodic solution exists for the unperturbed system

$$\dot{x} = f(t, x, y, 0), \qquad 0 = g(t, x, y, 0), \tag{1.4}$$

then, under suitable non-degeneracy conditions on the vector field, an almost periodic is shown to exist nearby; see for instance [16, 1, 31, 30, 14] in the periodic case and [24, 8, 28] in the almost periodic case. Note that the non-degeneracy conditions are not satisfied by our equation. Indeed, while the quoted results can be seen as results on the persistence of central manifolds under suitable hypothesis of stability or hyperbolicity or exponential dichotomy (for instance attractive limit cycles in van de Pol-like systems), theorems 1 and 2 are about the bifurcation of quasi-periodic solutions from a fixed point.

Almost periodic solutions were also studied in non-singularly perturbed linear and non-linear systems, once more assuming exponential dichotomy on the unperturbed linearised system; see for instance [23, 8, 14, 25, 32].

In the different context of conservative systems, nonlinear Duffing equations with almost periodic forcing were studied in [3, 4, 5] with variational techniques. In particular, almost periodic solutions with the same frequency vector as the forcing were proved to exist without assuming any smallness condition on the forcing and any Diophantine condition on its frequency vector. The latter issue marks a remarkable difference with respect KAM-like results, where restrictive conditions are imposed on the frequency vector. Note, however, then, even in the perturbative regime (small forcing), the solutions do not describe KAM invariant curves, as they bifurcate from a stable fixed point which they reduce to in the absence of the forcing – a situation which has some analogies with the kind of solutions we discuss in this paper.

In all the aforementioned papers, uniqueness of the the solution is proved as well. However, the hypotheses on the equations assumed in the papers ensure that the solutions are bounded for all times – a property that in the case of equation (1.1) holds if $g'(c_0) > 0$. Under such further assumption, as noted in the second remark at the end of Section 1.1, also the quasi-periodic solution to (1.1) turns out to be unique (in fact locally attractive).

2 Proof of Theorem 1

Henceforth, we assume $\boldsymbol{\omega}$ to be non-resonant, that is $\boldsymbol{\omega}\cdot\boldsymbol{\nu}\neq 0$ for all $\boldsymbol{\nu}\in\mathbb{Z}^d_*=\mathbb{Z}^d\setminus\{\mathbf{0}\}$. As noted in Section 1 this is not restrictive. We shall prove that there exists a response solution $x_0(t)$ to (1.1) such that $(x_0(t),\dot{x}(t))$ describes a curve in a neighbourhood of $(c_0,0)$. As already observed in the second remark at the end of Section 1.1, stability and uniqueness can be proved exactly as in [2].

Let us denote by Σ_ξ the strip of $\mathbb{T}^d$ of width ξ and by $\Delta(c_0,\rho)$ the disk of center c_0 and radius ρ in the complex plane. By the assumptions on f and g, for any $c_0\in\mathbb{R}$ there exist $\xi_0>0$ and $\rho_0>0$ such that $\boldsymbol{\psi}\mapsto f(\boldsymbol{\psi})$ is analytic in Σ_{ξ_0} and $x\mapsto g(x)$ is analytic in $\Delta(c_0,\rho_0)$. Then for all $\xi<\xi_0$ and all $\rho<\rho_0$ one has

$$f(\boldsymbol{\psi})=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}f_{\boldsymbol{\nu}},\qquad |f_{\boldsymbol{\nu}}|\leq\Phi\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|},\tag{2.1a}$$

$$g(x)=g(c_0)+\sum_{p=\mathfrak{n}}^{\infty}a_p(x-c_0)^p,\qquad a_p:=\frac{1}{p!}\frac{\mathrm{d}^pg}{\mathrm{d}x^p}(c_0),\qquad |a_p|\leq\Gamma\,\rho^{-p},\tag{2.1b}$$

where Φ is the maximum of $f(\boldsymbol{\psi})$ for $\boldsymbol{\psi}\in\Sigma_\xi$ and Γ is the maximum of $g(x)$ for $x\in\Delta(c_0,\rho)$.

Let us rewrite (1.1) as

$$\varepsilon\ddot{x}+\dot{x}+\varepsilon\,g(c_0)+\varepsilon\,a\,(x-c_0)+\varepsilon\,G(x)=\varepsilon\,f(\boldsymbol{\omega}t),\tag{2.2}$$

where $a:=a_1\neq 0$ by Hypothesis 1, and

$$G(x):=g(x)-g(c_0)-a\,(x-c_0)=\sum_{p=2}^{\infty}a_p(x-c_0)^p,$$

We look for a quasi-periodic solution to (2.2), that is a solution of the form

$$x(t,\varepsilon)=c_0+\zeta+u(\boldsymbol{\omega}t,\varepsilon,\zeta),\qquad u(\boldsymbol{\psi},\varepsilon,\zeta)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}u_{\boldsymbol{\nu}},\tag{2.3}$$

where ζ is a parameter that has to be fixed eventually and $\boldsymbol{\psi}\mapsto u(\boldsymbol{\psi},\varepsilon,\zeta)$ is a zero-average quasi-periodic function, with Fourier coefficients depending on both ε and ζ. Thus, we can write (2.2) in Fourier space. If we set

$$D(\varepsilon,s):=-\varepsilon s^2+\mathrm{i}s+\varepsilon\,a,\tag{2.4}$$

we obtain the following equations:

$$D(\varepsilon,\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,u_{\boldsymbol{\nu}}=-\varepsilon\,[G(c_0+\zeta+u)]_{\boldsymbol{\nu}}+\varepsilon\,f_{\boldsymbol{\nu}},\qquad \boldsymbol{\nu}\neq\mathbf{0},\tag{2.5a}$$

$$\varepsilon\,a\,\zeta=-\varepsilon\,[G(c_0+\zeta+u)]_{\mathbf{0}}\,,\tag{2.5b}$$

where we have used that $g(c_0) = f_{\mathbf{0}}$ by Hypothesis 1. By the notation $[G(c_0+\zeta+u)]_{\boldsymbol{\nu}}$ we mean that we first write u according to (2.3), then expand $G(c_0+\zeta+u)$ in Fourier series in $\boldsymbol{\psi}$ and finally keep the Fourier coefficient with index $\boldsymbol{\nu}$. The splitting into two sets of equations is typical of the Lyapunov-Schmidt reduction [9, 22, 27]; we call (2.5a) the *range equation* and (2.5b) the *bifurcation equation*.

In order to solve (2.5), we proceed as follows. We first ignore (2.5b) and look for a solution to (2.5a), depending on the parameter ζ. If we are able to do this, then we pass to (2.5b) and try to fix ζ in such a way to make such an equation to be satisfied.

2.1 The range equation

We start by studying (2.5a) only and considering ζ as a free parameter, close enough to 0. We introduce the auxiliary parameter μ by modifying (2.5a) into

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u_{\boldsymbol{\nu}} = -\mu\, [\varepsilon G(c_0+\mu\zeta+u)]_{\boldsymbol{\nu}} + \mu\varepsilon\, f_{\boldsymbol{\nu}}, \qquad \boldsymbol{\nu}\neq\mathbf{0}, \tag{2.6}$$

and look for a quasi-periodic solution to (2.6) in the form of a power series in μ,

$$u(\boldsymbol{\omega}t, \varepsilon, \zeta, \mu) = \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mu^k \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{(k)}_{\boldsymbol{\nu}}. \tag{2.7}$$

We shall show that there exists $\mu_0 > 0$ such that, for all ζ small enough, there exists a solution of the form (2.7), analytic in μ for $|\mu| < \mu_0$. The original equation (2.5a) is recovered when $\mu = 1$, so we need $\mu_0 > 1$. The argument below is a variant of that given in [13], to which we refer for more details about the construction described hereafter.

By inserting (2.7) into (2.6) we obtain a recursive definition for the coefficients $u^{(k)}_{\boldsymbol{\nu}}$. To simplify the notations, we set $u^{(1)}_{\mathbf{0}} = \zeta$ and $u^{(k)}_{\mathbf{0}} = 0$ $\forall k\geq 2$. Then, one has, formally,

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u^{(1)}_{\boldsymbol{\nu}} = \varepsilon\, f_{\boldsymbol{\nu}} \tag{2.8a}$$

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u^{(k)}_{\boldsymbol{\nu}} = -\varepsilon \sum_{p=2}^{\infty} a_p \sum_{\substack{k_1,\ldots,k_p\geq 1\\ k_1+\ldots+k_p=k-1}} \sum_{\substack{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_p\in\mathbb{Z}^d\\ \boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\boldsymbol{\nu}}} u^{(k_1)}_{\boldsymbol{\nu}_1}\ldots u^{(k_p)}_{\boldsymbol{\nu}_p}, \qquad k\geq 2, \tag{2.8b}$$

where we recall that $\boldsymbol{\nu}\neq\mathbf{0}$. Here and henceforth the sums over the empty set are meant as zero. In particular (see Remark 2.1 in [13]) for $k = 2$ one has $u^{(2)}_{\boldsymbol{\nu}} = 0$ $\forall\boldsymbol{\nu}\in\mathbb{Z}^d_*$.

By iterating (2.8b) one obtains a diagrammatic representation of the coefficients $u^{(k)}_{\boldsymbol{\nu}}$ in terms of trees. The construction is very similar to that in [13]; see also [19] for a review on the tree formalism.

A *rooted tree* θ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (root line). All the points in θ except the root are called *nodes*. The orientation of the lines in θ induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes v and w, we shall write $w \prec v$ every time v is along the path (of lines) which connects w to the root; we shall write $w \prec \ell$ if $w \preceq v$, where v is the unique node that the line ℓ exits. For any node v denote by p_v the number of lines entering v.

Given a rooted tree θ we denote by $N(\theta)$ the set of nodes, by $E(\theta)$ the set of *end nodes*, i.e. nodes v with $p_v = 0$, by $V(\theta)$ the set of *internal nodes*, i.e. nodes v with $p_v \geq 1$, and by $L(\theta)$ the set of lines; by definition $N(\theta) = E(\theta) \amalg V(\theta)$. If, for any discrete set A, we denote by $|A|$ its cardinality, we define the *order* of θ as $k(\theta) := |N(\theta)|$.

We associate with each end node $v \in E(\theta)$ a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d$. We split $E(\theta) = E_0(\theta) \amalg E_1(\theta)$, with $E_0(\theta) = \{v \in E(\theta) : \boldsymbol{\nu}_v = \mathbf{0}\}$ and $E_1(\theta) = \{v \in E(\theta) : \boldsymbol{\nu}_v \neq \mathbf{0}\}$. With each line $\ell \in L(\theta)$ we associate a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$ with the constraint

$$\boldsymbol{\nu}_\ell = \sum_{\substack{w \in E(\theta) \\ w \prec \ell}} \boldsymbol{\nu}_w.$$

Finally we impose the constraints that

- $p_v \geq 2 \; \forall v \in V(\theta)$,
- $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ for any line ℓ exiting a node in $V(\theta)$.

We call *equivalent* two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. In the following we shall consider only inequivalent labelled rooted trees, and we shall call them trees *tout court*, for simplicity.

We associate with each node $v \in N(\theta)$ a *node factor*

$$F_v := \begin{cases} -\varepsilon\, a_{p_v}, & v \in V(\theta), \\ \varepsilon\, f_{\boldsymbol{\nu}_v}, & v \in E_1(\theta), \\ \zeta, & v \in E_0(\theta), \end{cases}$$

and with each line $\ell \in L(\theta)$ a *propagator*

$$\mathcal{G}_\ell := \begin{cases} 1/D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ 1, & \boldsymbol{\nu}_\ell = \mathbf{0}. \end{cases}$$

Finally we define the value of the tree θ as

$$\mathscr{V}(\theta) := \Bigg(\prod_{v \in N(\theta)} F_v \Bigg) \Bigg(\prod_{\ell \in L(\theta)} \mathcal{G}_\ell \Bigg). \tag{2.9}$$

It is not difficult to show that, with the notations above, the equations (2.8) are solved for all $k \in \mathbb{N}$, provided the coefficients $u_{\boldsymbol{\nu}}^{(k)}$ are defined as

$$u_{\boldsymbol{\nu}}^{(k)} = \sum_{\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta), \qquad \boldsymbol{\nu} \in \mathbb{Z}^d_*, \tag{2.10}$$

where $\mathcal{T}_{k,\boldsymbol{\nu}}$ is the set of non-equivalent trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line.

For ε small enough and all $s \in \mathbb{R}$, one has (see Lemma 2.2 in [13] for a proof)

$$|D(\varepsilon, s)| \geq \max\{|a\varepsilon|, |s|\}. \tag{2.11}$$

Moreover, by Lemma 2.3 in [13], for any tree θ one has $|E(\theta)| \geq |V(\theta)| + 1$ and, as a consequence,

$$|E(\theta)| \geq \frac{1}{2}\left(k(\theta) + 1\right). \tag{2.12}$$

Finally set

$$C_0 := \rho^{-1} \max\{\Gamma/|a|, \Phi, 1\}, \tag{2.13}$$

with ρ, Φ and Γ defined as in (2.1).

Lemma 2.1. *For any fixed $A \in (0, C_0)$ there exist $\bar{\varepsilon} > 0$ and $\bar{\zeta} > 0$ such that for any $k \geq 1$, any $\boldsymbol{\nu} \in \mathbb{Z}^d$ and any tree $\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}$ one has*

$$|\mathscr{V}(\theta)| \leq A_0 \, A^k \prod_{v \in E(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}$$

with A_0 a suitable positive constant, provided $|\varepsilon| < \bar{\varepsilon}$ and $|\zeta| < \bar{\zeta}$.

Proof. One bounds (2.9) as

$$\begin{aligned}|\mathscr{V}(\theta)| &\leq \left(\prod_{v\in V(\theta)} |\varepsilon\, a_{p_v}|\right)\left(\prod_{v\in E_1(\theta)} \frac{|\varepsilon\, f_{\boldsymbol{\nu}_v}|}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)|}\right)\left(\prod_{v\in E_0(\theta)} |\zeta|\right)\left(\prod_{v\in V(\theta)} \frac{1}{|a\varepsilon|}\right) \\ &\leq |\zeta|^{|E_0(\theta)|}\Gamma^{|V(\theta)|}\rho^{-(|N(\theta)|-1)}\Phi^{|E_1(\theta)|}|a|^{-|V(\theta)|}\left(\prod_{v\in E_1(\theta)} \frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)|}\right),\end{aligned}$$

where we have bounded $f_{\boldsymbol{\nu}_v}$ as in (2.1a) and used the bound $|D(\varepsilon, s)| \geq |a\varepsilon|$ for the propagators of the lines exiting the nodes $v \in V(\theta)$. For each end node $v \in E_1(\theta)$ we extract a factor $\mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}$, so that, if we define C_0 as in (2.13), we obtain

$$|\mathscr{V}(\theta)| \leq \rho\, C_0^k |\zeta|^{|E_0(\theta)|}\left(\prod_{v\in E_1(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}\right)\left(\prod_{v\in E_1(\theta)} \frac{|\varepsilon|\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)|}\right). \tag{2.14}$$

For any given $n_0 \in \mathbb{N}$ we have $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \geq \alpha_{n_0}(\boldsymbol{\omega})$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ such that $|\boldsymbol{\nu}| \leq 2^{n_0}$. Set $\delta = \delta(n_0) := \mathrm{e}^{-\xi 2^{n_0}/4}$. Let A be such that $0 < A < C_0$. We first fix n_0 such that $C_0^2\delta/|a| \leq A^2$, then we fix $\bar{\varepsilon}$ and $\bar{\zeta}$ by requiring that $C_0^2\bar{\varepsilon}/\alpha_{n_0}(\boldsymbol{\omega}) \leq A^2$ and $C_0^2\bar{\zeta} < A^2$.

By (2.11), in (2.14), for all $v \in E_1(\theta)$, we can bound $|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)| \geq \alpha_{n_0}(\boldsymbol{\omega})$ if $|\boldsymbol{\nu}_v| \leq 2^{n_0}$ and $|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)| \geq |\varepsilon a|$ if $|\boldsymbol{\nu}_v| > 2^{n_0}$. Thus, for all $v \in E_1(\theta)$, one has

$$\frac{|\varepsilon|\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v, \varepsilon)|} \leq \max\left\{\frac{\delta}{|a|}, \frac{|\varepsilon|}{\alpha_{n_0}(\boldsymbol{\omega})}\right\}, \tag{2.15}$$

so that in (2.14), if $\delta/|a| \geq |\varepsilon|/\alpha_{n_0}(\boldsymbol{\omega})$, we can bound

$$C_0^k|\zeta|^{|E_0(\theta)|}\left(\prod_{v\in E_1(\theta)}\frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v,\varepsilon)|}\right)\leq C_0^k\bar{\zeta}^{|E_0(\theta)|}\left(\frac{\delta}{|a|}\right)^{|E_1(\theta)|}\leq\frac{A}{C_0}A^k,$$

while, if $\delta/|\varepsilon a| < 1/\alpha_{n_0}(\boldsymbol{\omega})$, we can bound

$$C_0^k|\zeta|^{|E_0(\theta)|}\left(\prod_{v\in E(\theta)}\frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v,\varepsilon)|}\right)\leq C_0^k\bar{\zeta}^{|E_0(\theta)|}\left(\frac{|\varepsilon|}{\alpha_{n_0}(\boldsymbol{\omega})}\right)^{|E_1(\theta)|}\leq\frac{A}{C_0}A^k,$$

where we have used twice (2.12) with $k(\theta) = k$.

Summarising, for all $k \in \mathbb{N}$ and all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ we have obtained

$$|\mathscr{V}(\theta)| \leq A_0 A^k\left(\prod_{v\in E(\theta)}\mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}\right), \qquad A_0 := \rho\frac{A}{C_0}.$$

Therefore the assertion follows. ■

Lemma 2.2. *For any $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ one has*

$$\left|u^{(k)}_{\boldsymbol{\nu}}\right| \leq A_0 C^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2},$$

where ξ is as in (2.1a) and C is a positive constant proportional to A, with A_0 and A as in Lemma 2.1.

Proof. To bound the coefficients $u^{(k)}_{\boldsymbol{\nu}}$ defined by (2.9) we use the bounds of Lemma 2.1 and sum over all trees in $\mathcal{T}_{k,\boldsymbol{\nu}}$. The sum over the Fourier labels $\{\boldsymbol{\nu}_v\}_{v\in E_1(\theta)}$ is performed by using the factors $\mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}$ associated with the end nodes in $E_1(\theta)$, and gives a bound $C_1^{|E_1(\theta)|}\mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}$, for some positive constant C_1. The sum over the other labels produces a factor $C_2^{|N(\theta)|}$, with C_2 a suitable positive constant. By taking $C = AC_1C_2$ the assertion follows. ■

Lemma 2.3. *For any $\boldsymbol{\omega} \in \mathbb{R}^d$ there exist $\bar{\varepsilon} > 0$ and $\bar{\zeta} > 0$ such that, for $\mu = 1$, $|\varepsilon| < \bar{\varepsilon}$ and $|\zeta| < \bar{\zeta}$ the series* (2.7) *converges to a function $u(\boldsymbol{\psi}, \varepsilon, \zeta) = u(\boldsymbol{\psi}, \varepsilon, \zeta, 1)$, which is analytic in $\boldsymbol{\psi}$ in a strip $\Sigma_{\xi'}$, with $\xi' < \xi/2$, and such that $u(\boldsymbol{\omega}t, \varepsilon, \zeta)$ solves* (2.5a).

Proof. In Lemma 2.1 we can fix A in such a way that $C \leq B$, for some constant $B < 1$. Then the series (2.7) converges provided $B\mu < 1$, which allow $\mu = 1$. Furthermore, the function (2.7) solves (2.6) order by order by construction. Since the series converges uniformly, then it is also a solution *tout court* to (2.6) with $\mu = 1$ and hence of (2.5a). Analyticity in $\boldsymbol{\psi} \in \Sigma_{\xi'}$ for any $\xi' < \xi/2$ follows from the bound on the Fourier coefficients given by Lemma 2.2. ■

2.2 The bifurcation equation

Continuity of the function $\varepsilon \mapsto u(\boldsymbol{\psi},\varepsilon,\zeta)$ holds trivially for $\varepsilon > 0$. On the contrary, continuity at $\varepsilon = 0$ requires some discussion. Indeed, that $u(\boldsymbol{\psi},\varepsilon,\zeta)$ tends to 0 as $\varepsilon \to 0$ does not follow from Lemma 2.3, since the constants A and A_0 do not tend to 0 as $\varepsilon \to 0$. However, continuity at $\varepsilon = 0$ can be proved by following the same lines as in the proof of Lemma 2.2, up to some minor changes.

Lemma 2.4. *Let $\bar\varepsilon$ and $\bar\zeta$ be as in Lemma 2.3. For any $|\zeta| < \bar\zeta$, the function $u(\boldsymbol{\psi},\varepsilon,\zeta)$ in Lemma 2.3 is continuous in $\varepsilon \in [0,\bar\varepsilon)$. In particular, it tends to 0 as $\varepsilon \to 0$.*

Proof. Let ζ be such that $|\zeta| < \bar\zeta$. As already noted, continuity in ε is obvious for $\varepsilon > 0$. Set

$$F(\varepsilon,\zeta) = \|u(\cdot,\varepsilon,\zeta)\|_\infty := \sup\{u(\boldsymbol{\psi},\varepsilon,\zeta) : \boldsymbol{\psi} \in \Sigma_{\xi'}\},$$

with ξ' as in Lemma 2.3. Since $F(0,\zeta) = 0$, we have only to prove that $F(\varepsilon,\zeta) \to 0$ as $\varepsilon \to 0$, that is that for all $\eta > 0$ there exists $\delta > 0$ such that $0 < \varepsilon < \delta$ implies $|F(\varepsilon,\zeta)| < \eta$.

Let $\bar\varepsilon$ be as in Lemma 2.3. The series in (2.7) with $\mu = 1$ can be written as

$$u(\boldsymbol{\psi},\varepsilon,\zeta) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u_{\boldsymbol{\nu}}, \qquad u_{\boldsymbol{\nu}} := \sum_{k=1}^{\infty} u_{\boldsymbol{\nu}}^{(k)},$$

so that

$$F(\varepsilon,\zeta) \le \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \big|u_{\boldsymbol{\nu}}^{(k)}\big| \, \mathrm{e}^{\xi'|\boldsymbol{\nu}|}.$$

By reasoning as in the proof of Lemma 2.1 – see in particular (2.14) –, we can bound

$$\begin{aligned}
\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \big|u_{\boldsymbol{\nu}}^{(k)}\big| \, \mathrm{e}^{\xi'|\boldsymbol{\nu}|} &\le \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \sum_{\theta\in\mathcal{T}_{\boldsymbol{\nu},k}} \big|\mathscr{V}(\theta)\big| \, \mathrm{e}^{\xi'|\boldsymbol{\nu}|} \le \rho\, C_0^k \sum_{\theta\in\mathcal{T}_{\boldsymbol{\nu},k}} |\zeta|^{|E_0(\theta)|} \prod_{v\in E_1(\theta)} \sum_{\boldsymbol{\nu}_v\in\mathbb{Z}^d} \frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v,\varepsilon)|} \\
&\le \rho\, C_0 A^{-1} C^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon)|},
\end{aligned}$$

where we have used the fact that $|E_1(\theta)| \ge 1$ (since $\boldsymbol{\nu} \ne \mathbf{0}$) and the bound (2.15) for all the end nodes $v \in E_1(\theta)$ but one. Therefore we obtain, for any $N \in \mathbb{N}$,

$$F(\varepsilon,\zeta) \le \frac{\rho\, C_0\, A^{-1}}{1-C} \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^d \\ |\boldsymbol{\nu}|\le N}} \frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon)|} + D_0 \mathrm{e}^{-\xi N/8}, \qquad D_0 := \frac{\rho\, C_0\, A^{-1}}{1-C} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{-\xi|\boldsymbol{\nu}|/8}.$$

Fix $\eta > 0$. Choose N such that $D_0 \mathrm{e}^{-\xi N/8} < \eta/2$. If we define

$$r_N := \min\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le N\},$$

we can bound

$$\frac{\rho\, C_0\, A^{-1}}{1-C} \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^d \\ |\boldsymbol{\nu}|\le N}} \frac{|\varepsilon|\, \mathrm{e}^{-\xi|\boldsymbol{\nu}|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon)|} \le \frac{|\varepsilon|\, D_0}{r_N}.$$

Thus, for δ small enough and $0<\varepsilon<\delta$, we have $|\varepsilon|D_0/r_N < \eta/2$ and hence $|F(\varepsilon)|<\eta$. ■

Note that we are not able to prove more than continuity for the function $\varepsilon \mapsto u(\boldsymbol{\psi},\varepsilon,\zeta)$. Indeed, $\partial_\varepsilon u(\boldsymbol{\psi},\varepsilon,\zeta)$ is well defined for $\varepsilon>0$, but the argument used in proving Lemma 2.4 does not allow us to obtain its boundedness as $\varepsilon\to 0$.

Lemma 2.5. *Let $\bar\varepsilon$ and $\bar\zeta$ be as in Lemma 2.2 and let $u=u(\boldsymbol{\omega}t,\varepsilon,\zeta)$ be as in Lemma 2.3. There exist neighbourhoods $U\subset(-\bar\varepsilon,\bar\varepsilon)$ and $V\subset(-\bar\zeta,\bar\zeta)$ and a function $\zeta: U\to V$ such that for all $\varepsilon\in U$ the equation* (2.5b) *holds for $\zeta=\zeta(\varepsilon)$. Moreover the function $\varepsilon\mapsto\zeta(\varepsilon)$ is continuous in U and $\zeta(\varepsilon)$ is the only solution to* (2.5b) *in V.*

Proof. Write (2.5b) as

$$H(\zeta,\varepsilon) := a\,\zeta + [G(c_0+\zeta+u)]_{\mathbf{0}} = 0.$$

By construction, the function $u(\boldsymbol{\psi},\varepsilon,\zeta)$ is analytic on ζ in a neighbourhood of the origin. Therefore, $H(\zeta,\varepsilon)$ is analytic in ζ and, by Lemma 2.3, is continuous in ε. One has

$$H(0,0)=0, \qquad \frac{\partial}{\partial\zeta}H(0,0) = a \neq 0,$$

so that we can apply the implicit function theorem, in the version of Loomis and Sternberg [26], so as to conclude that there exist neighbourhoods $U\subset(-\bar\varepsilon,\bar\varepsilon)$ and $V\subset(-\bar\zeta,\bar\zeta)$ such that for all $\varepsilon\in U$ one can find a unique value $\zeta(\varepsilon)\in V$, depending continuously on ε, such that $H(\zeta(\varepsilon),\varepsilon)=0$. ■

Lemma 2.6. *Let $u(\boldsymbol{\psi},\varepsilon,\zeta)$ and $\zeta(\varepsilon)$ be as in Lemma 2.3 and in Lemma 2.5, respectively. There exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ the function $x(t,\varepsilon) = c_0+\zeta(\varepsilon)+u(\boldsymbol{\omega}t,\varepsilon,\zeta(\varepsilon))$ solves* (2.2)*. Moreover $x(t,\varepsilon)\to c_0$ as $\varepsilon\to 0$.*

Proof. The result follows immediately from Lemma 2.3 and Lemma 2.5. ■

3 Proof of Theorem 2

Let the function $h(\boldsymbol{\psi},x)$ in (1.2) be analytic on the domain $\Sigma_\xi\times\Delta(c_0,\rho_0)$, with the same notations as in Section 1. We expand

$$h(\boldsymbol{\psi},x) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} h_{\boldsymbol{\nu}}(x)\,\mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}, \qquad h_{\boldsymbol{\nu}}(x) = \sum_{p=0}^{\infty} a_{\boldsymbol{\nu},p}\,(x-c_0)^p\,, \qquad a_{\boldsymbol{\nu},p} = \frac{1}{p!}\frac{\partial^p h_{\boldsymbol{\nu}}}{\partial x^p}(c_0),$$

so that, for any $\rho < \rho_0$,

$$|a_{\boldsymbol{\nu},p}| \le \Gamma \rho^{-p} \mathrm{e}^{-\xi|\boldsymbol{\nu}|}, \tag{3.1}$$

for a suitable positive constant Γ. Then we rewrite (1.2) as

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, a\,(x - c_0) \;+\; \varepsilon \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} h_{\boldsymbol{\nu}}(c_0)$$
$$+\; \varepsilon \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} a_{\boldsymbol{\nu},1}\, \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\,(x - c_0) + \varepsilon \sum_{p=2}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} a_{\boldsymbol{\nu},p}\, \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\,(x - c_0)^p = 0,$$

where we have used that $h_{\mathbf{0}}(c_0) = 0$ and $a := a_{\mathbf{0},1} \neq 0$ by Hypothesis 2 with $\mathfrak{n} = 1$.

We look for a solution of the form (2.3). By passing to Fourier space, setting

$$\alpha_1(\boldsymbol{\psi}) := \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} a_{\boldsymbol{\nu},1}\, \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}, \qquad \alpha_p(\boldsymbol{\psi}) := \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} a_{\boldsymbol{\nu},p}\, \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}},$$

and defining $D(\varepsilon, s)$ as in (2.4), we obtain the equations

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u_{\boldsymbol{\nu}} = -\varepsilon h_{\boldsymbol{\nu}}(c_0) - \varepsilon\,[\alpha_1\,(x - c_0)]_{\boldsymbol{\nu}} - \varepsilon \sum_{p=2}^{\infty} [\alpha_p\,(x - c_0)^p]_{\boldsymbol{\nu}}\,, \qquad \boldsymbol{\nu} \neq \mathbf{0}, \tag{3.2a}$$

$$\varepsilon\, a\, \zeta = -\,[\alpha_1\,(x - c_0)]_{\mathbf{0}} - \varepsilon \sum_{p=2}^{\infty} [\alpha_p\,(x - c_0)^p]_{\mathbf{0}}\,. \tag{3.2b}$$

By following the same strategy as in Section 2, we shall study first (3.2a) and look for a solution depending on the parameter ζ. Thereafter we shall fix ζ by requiring that (3.2b) is solved as well.

Instead of (3.2a) we consider the equation

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u_{\boldsymbol{\nu}} = -\mu\,\varepsilon h_{\boldsymbol{\nu}}(c_0) - \mu\,\varepsilon\,[\alpha_1\,(x - c_0)]_{\boldsymbol{\nu}} - \mu\,\varepsilon \sum_{p=2}^{\infty} [\alpha_p\,(x - c_0)^p]_{\boldsymbol{\nu}}\,, \qquad \boldsymbol{\nu} \neq \mathbf{0}, \tag{3.3}$$

and look for a quasi-periodic solution to (3.3) in the form of a power series in μ,

$$x(t, \varepsilon, \mu) = c_0 + \zeta + u(\boldsymbol{\omega}t, \varepsilon, \zeta, \mu), \qquad u(\boldsymbol{\omega}t, \varepsilon, \zeta, \mu) = \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mu^k \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{(k)}_{\boldsymbol{\nu}}. \tag{3.4}$$

We find the recursive equations

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u^{(1)}_{\boldsymbol{\nu}} = -\varepsilon\, h_{\boldsymbol{\nu}}(c_0) \tag{3.5a}$$

$$D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu})\, u^{(k)}_{\boldsymbol{\nu}} = -\varepsilon \sum_{\boldsymbol{\nu}_0\in\mathbb{Z}^d_*} a_{\boldsymbol{\nu}_0,1}\, u^{(k-1)}_{\boldsymbol{\nu}-\boldsymbol{\nu}_0}$$
$$-\,\varepsilon \sum_{p=2}^{\infty} \sum_{\substack{k_1,\ldots,k_p\ge 1\\ k_1+\ldots+k_p=k-1}} \sum_{\substack{\boldsymbol{\nu}_0,\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_p\in\mathbb{Z}^d\\ \boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\boldsymbol{\nu}}} a_{\boldsymbol{\nu}_0,p}\, u^{(k_1)}_{\boldsymbol{\nu}_1} \ldots u^{(k_p)}_{\boldsymbol{\nu}_p}, \qquad k \ge 2, \tag{3.5b}$$

where $\boldsymbol{\nu} \neq \mathbf{0}$ and we have set once more $u^{(1)}_{\mathbf{0}} = \zeta$ and $u^{(k)}_{\mathbf{0}} = 0 \ \forall k \geq 2$.

We have still a tree representation of the coefficients $u^{(k)}_{\boldsymbol{\nu}}$, with a few differences with respect to Section 2. Define the sets $N(\theta)$, $E(\theta)$, $E_0(\theta)$, $E_1(\theta)$, $V(\theta)$ and $L(\theta)$ as previously and call $k(\theta) := |N(\theta)|$ the order of θ. Now we split $V(\theta) = V_1(\theta) \amalg V_2(\theta)$, with $V_1(\theta) = \{v \in V(\theta) : p_v = 1\}$ and $V_2(\theta) = \{v \in E(\theta) : p_v \geq 2\}$. Contrary to Section 2, now in general $V_1(\theta) \neq \emptyset$.

We associate with each node $v \in N(\theta)$ a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d$ and with each line $\ell \in L(\theta)$ a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$ with the constraint

$$\boldsymbol{\nu}_\ell = \sum_{\substack{w \in N(\theta) \\ w \prec \ell}} \boldsymbol{\nu}_w .$$

Finally we impose the constraints that

- $\boldsymbol{\nu}_v \neq \mathbf{0} \ \forall v \in V_1(\theta)$,
- $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ for any line ℓ exiting a node in $V(\theta)$.

We associate with each node $v \in N(\theta)$ a *node factor*

$$F_v := \begin{cases} -\varepsilon \, a_{\boldsymbol{\nu}_v, p_v}, & v \in V(\theta), \\ -\varepsilon \, h_{\boldsymbol{\nu}_v}(c_0), & v \in E_1(\theta), \\ \zeta, & v \in E_0(\theta), \end{cases} \tag{3.6}$$

and with each line $\ell \in L(\theta)$ a *propagator*

$$\mathcal{G}_\ell := \begin{cases} 1/D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ 1, & \boldsymbol{\nu}_\ell = \mathbf{0}. \end{cases} \tag{3.7}$$

We define the value of the tree θ as (2.9) and write $u^{(k)}_{\boldsymbol{\nu}}$ as in (2.10), where $\mathcal{T}_{k,\boldsymbol{\nu}}$ denotes the set of non-equivalent trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line, constructed according to the new rules. Then the coefficients $u^{(k)}_{\boldsymbol{\nu}}$ solve formally (3.5).

In a tree θ we define a *chain* $\mathcal{C}$ as a subset of θ formed by a maximal connected set of nodes $v \in V(\theta)$ with $p_v = 1$ and by the lines exiting them. Therefore, if $V(\mathcal{C})$ and $L(\mathcal{C})$ denote the set of nodes and the set of lines of $\mathcal{C}$ and $V(\mathcal{C}) = \{v_1, v_2, \ldots, v_p\}$, with $v_1 \succ v_2 \succ \ldots \succ v_p$, then $L(\mathcal{C}) = \{\ell_1, \ell_2, \ldots, \ell_p\}$, where ℓ_i is the line exiting v_i, for $i = 1, \ldots, p$, and ℓ_i enters the node v_{i-1} for $i = 2, \ldots, p$. We call $p = |V(\mathcal{C})| = |L(\mathcal{C})|$ the *length* of the chain $\mathcal{C}$. Finally we define the *value* of the chain $\mathcal{C}$ as

$$\mathscr{V}(\mathcal{C}) := \Bigg(\prod_{v \in V(\mathcal{C})} F_v \Bigg) \Bigg(\prod_{\ell \in L(\mathcal{C})} \mathcal{G}_\ell \Bigg)$$

and denote by $\mathfrak{C}(\theta)$ the set of all the chains contained in θ. The main difference with respect to the trees considered in Section 2 is that, now, the trees may contain chains.

We define n_0 and δ as in Section 2. Define also $\beta := \max\{\delta, 2|\varepsilon a|/\alpha_{n_0}(\boldsymbol{\omega})\}$. Finally set

$$C_0 = \rho^{-1} \max\{\Gamma/|a|, 1\}, \tag{3.8}$$

with ρ and Γ defined as in (3.1).

Lemma 3.1. *Let $\mathcal{C}$ be a chain of length $p \geq 1$. Then*

$$|\mathscr{V}(\mathcal{C})| \leq C_0^p \beta^{(p-1)/2} \prod_{v \in V(\mathcal{C})} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}.$$

Proof. We proceed by induction on p. If $p = 1$ then $\mathcal{C}$ contains only one node v, so that

$$|\mathscr{V}(\mathcal{C})| \leq \frac{|\varepsilon|\Gamma\rho^{-1}\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|}}{|D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu}_v)|} \leq C_0 \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4},$$

where we have bounded $|D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu}_v)| \geq |\varepsilon a|$ by (2.11).

If $p \geq 2$, let $v_1 \succ v_2 \succ v_3 \succ \ldots \succ v_p$ be the nodes in $\mathcal{C}$ and let $\ell_1, \ell_2, \ell_3, \ldots, \ell_p$ be the lines exiting such nodes. The nodes $\{v_2, v_3, \ldots, v_p\}$ and the lines $\{\ell_2, \ell_3, \ldots, \ell_p\}$ form a chain $\mathcal{C}'$ of length $p-1$ and, if $p \geq 3$, the nodes $\{v_3, \ldots, v_p\}$ and the lines $\{\ell_3, \ldots, \ell_p\}$ form a chain $\mathcal{C}''$ of length $p-2$.

We assume that the bound holds up to $p-1$. If $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| \geq \alpha_{n_0}(\boldsymbol{\omega})/2$, then one has

$$|\mathscr{V}(\mathcal{C})| \leq \frac{|\varepsilon|\Gamma\rho^{-1}\mathrm{e}^{-\xi|\boldsymbol{\nu}_{v_1}|}}{|D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1})|} \left|\mathscr{V}(\mathcal{C}')\right| \leq C_0\beta \left(C_0^{p-1}\beta^{(p-2)/2}\right) \prod_{v \in V(\mathcal{C})} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4},$$

which yields the bound for p. If $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| < \alpha_{n_0}(\boldsymbol{\omega})/2$ and $p \geq 3$ we distinguish between two cases. If $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| \geq \alpha_{n_0}(\boldsymbol{\omega})/2$, we can bound

$$|\mathscr{V}(\mathcal{C})| \leq \frac{|\varepsilon|\Gamma\rho^{-1}\mathrm{e}^{-\xi|\boldsymbol{\nu}_{v_1}|}}{|D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1})|} \frac{|\varepsilon|\Gamma\rho^{-1}\mathrm{e}^{-\xi|\boldsymbol{\nu}_{v_2}|}}{|D(\varepsilon, \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2})|} \left|\mathscr{V}(\mathcal{C}'')\right| \leq C_0^2\beta \left(C_0^{p-2}\beta^{(p-3)/2}\right) \prod_{v \in V(\mathcal{C})} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4},$$

so that the bound follows once more.

If $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| < \alpha_{n_0}(\boldsymbol{\omega})/2$, then

$$\begin{aligned} |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{v_1}| &= |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{v_1} + \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2} - \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| \leq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{v_1} + \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| + |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| \\ &= |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| + |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2}| < \alpha_{n_0}(\boldsymbol{\omega}), \end{aligned}$$

so that, since $\boldsymbol{\nu}_{v_1} \neq \mathbf{0}$ and hence $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{v_1} \neq 0$, we conclude that $|\boldsymbol{\nu}_{v_1}| > 2^{n_0}$, which allows us to bound $\mathrm{e}^{-\xi|\boldsymbol{\nu}_{v_1}|} \leq \delta\mathrm{e}^{-3\xi|\boldsymbol{\nu}_{v_1}|/4}$. Therefore we obtain

$$|\mathscr{V}(\mathcal{C})| \leq C_0^2\delta \left|\mathscr{V}(\mathcal{C}'')\right| \leq C_0^2\delta \left(C_0^{p-2}\beta^{(p-3)/2}\right) \prod_{v \in V(\mathcal{C})} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4},$$

which gives the bound for p in this case too.

Finally if $p=2$ we can reason as in the case $p \geq 3$, the only difference being that the quantity $|\mathscr{V}(\mathcal{C}'')|$ has to replaced with 1 – since there is no further chain $\mathcal{C}''$ for $p=2$. Therefore we obtain $|\mathscr{V}(\mathcal{C})| \leq C_0^2\beta$, which is the desired bound. ∎

Lemma 3.2. *For any tree θ one has*

1. $|E(\theta)| \geq |V_2(\theta)| + 1$,
2. $|\mathfrak{C}(\theta)| \leq |E(\theta)| + |V_2(\theta)|$,
3. $|\mathfrak{C}(\theta)| \leq |V_1(\theta)|$,
4. $k(\theta) = |E(\theta)| + |V_1(\theta)| + |V_2(\theta)|$.

Proof. Property 1 can be proved as the inequality $|E(\theta)| \geq |V(\theta)| + 1$ in Section 2. Property 2 is easily proved by noting that each chain has to be preceded by either an end node or a node $v \in V_2(\theta)$. Property 3 is trivial, since any chain has to contain at least one node $v \in V_1(\theta)$. Finally property 4 follows from the definition of order. ∎

A result analogous to Lemma 2.1 still holds. This can be proved as follows. For any tree θ we have

$$
\begin{aligned}
|\mathscr{V}(\theta)| &\leq \left(\prod_{v\in V_2(\theta)} \frac{|\varepsilon\, a_{p_v,\boldsymbol{\nu}_v}|}{|\varepsilon a|} \right) \left(\prod_{v\in E_1(\theta)} \frac{|\varepsilon\, h_{\boldsymbol{\nu}_v}(c_0)|}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v,\varepsilon)|} \right) \left(\prod_{v\in E_0(\theta)} |\zeta| \right) \left(\prod_{\mathcal{C}\in\mathfrak{C}(\theta)} |\mathscr{V}(\mathcal{C})| \right) \\
&\leq \rho\, C_0^k |\zeta|^{|E_0(\theta)|} \left(\prod_{v\in E_1(\theta)} \frac{|\varepsilon|\,\mathrm{e}^{-\xi|\boldsymbol{\nu}_v|/4}}{|D(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v,\varepsilon)|} \right) \left(\prod_{\mathcal{C}\in\mathfrak{C}(\theta)} \beta^{(|V(\mathcal{C})|-1)/2} \right) \left(\prod_{v\in V(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4} \right),
\end{aligned}
$$

so that

$$
|\mathscr{V}(\theta)| \leq \rho\, C_0^k \bar{\zeta}^{|E_0(\theta)|} \left(\max\left\{ \frac{\delta}{|a|}, \frac{\bar{\varepsilon}}{\alpha_{n_0(\boldsymbol{\omega})}} \right\} \right)^{|E_1(\theta)|} \beta^{(|V_1(\theta)|-|\mathfrak{C}(\theta)|)/2} \left(\prod_{v\in V(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4} \right)
$$

Therefore, for any constant $A \in (0, C_0)$, if we fix first n_0 and hence $\bar{\varepsilon}$ and $\bar{\zeta}$ so that for any $|\varepsilon| < \bar{\varepsilon}$ and any $|\zeta| < \bar{\zeta}$ one has

$$
C_0^4 \max\left\{ \bar{\zeta}, \frac{\delta}{|a|}, \frac{\bar{\varepsilon}}{\alpha_{n_0(\boldsymbol{\omega})}}, \beta \right\} < A^4, \tag{3.9}
$$

we obtain

$$
|\mathscr{V}(\theta)| \leq \rho\, C_0^k \left(\frac{A}{C_0} \right)^{4|E(\theta)|+2|V_1(\theta)|-2|\mathfrak{C}(\theta)|} \left(\prod_{v\in V(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4} \right). \tag{3.10}
$$

By using Lemma 3.2 we can bound

$$\begin{aligned}
4|E(\theta)| + 2|V_1(\theta)| - 2|\mathfrak{C}(\theta)| &= 2|E(\theta)| + 2|E(\theta)| + |V_1(\theta)| - |\mathfrak{C}(\theta)| + (|V_1(\theta)| - |\mathfrak{C}(\theta)|) \\
&\geq 2|E(\theta)| + 2\,(|V_2(\theta)| + 1) + |V_1(\theta)| - |\mathfrak{C}(\theta)| \\
&\geq |E(\theta)| + |V_2(\theta)| + |V_1(\theta)| + 2 + (|E(\theta)| + |V_2(\theta)| - |\mathfrak{C}(\theta)|) \\
&\geq |E(\theta)| + |V_2(\theta)| + |V_1(\theta)| + 2 = k(\theta) + 2,
\end{aligned}$$

which, inserted into (3.10), gives

$$|\mathscr{V}(\theta)| \leq \rho\, C_0^k \left(\frac{A}{C_0}\right)^{k+2} \left(\prod_{v\in V(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}\right) \leq A_0 A^k \left(\prod_{v\in V(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}\right), \qquad A_0 := \rho\frac{A^2}{C_0^2}.$$

From here on, we can reason as in Section 2 and we find that the coefficients $u_{\boldsymbol{\nu}}^{(k)}$ can be bounded as

$$\left|u_{\boldsymbol{\nu}}^{(k)}\right| \leq A_0 C^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2},$$

for a suitable constant C proportional to the constant A in (3.9). Thus, by choosing $\bar{\varepsilon}$ and $\bar{\zeta}$ small enough, we can make C such that $C \leq B < 1$, so that the series (3.4) converges for $\mu = 1$. Therefore the function $x(t,\varepsilon,\zeta)$ solves the range equation (3.2a) for any ζ small enough.

Both the proof of continuity in ε of the function $x(t,\varepsilon,\zeta)$ and the discussion of the bifurcation equation in order to fix the parameter ζ can be repeated exactly as in Section 2, so we omit the details.

We conclude this section with a few comments about the properties of analyticity in ε of the response solution. If we want to study the dependence of such a solution on ε in the complex plane, it is more convenient to write it not as in (2.3), but as

$$x(t,\varepsilon) = c_0 + u(\boldsymbol{\omega} t,\varepsilon,c_0), \qquad u(\boldsymbol{\psi},\varepsilon,c_0) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u_{\boldsymbol{\nu}},$$

so as to have only one parameter ε, instead of ε and ζ, and deal with both the range and bifurcation equations at the same time using the tree representation introduced in [13], to which we refer for more details. Note that, with such a tree representation, also lines with zero momentum may exit internal nodes: the propagator of a line ℓ with momentum $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ is still $1/D(\varepsilon,\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)$, whereas the propagator of a line with momentum $\boldsymbol{\nu}_\ell = \mathbf{0}$ is $1/a$. Then, if we allow ε to vary in the conical domain

$$\mathcal{D}(\lambda,\bar{\varepsilon}) = \{\varepsilon\in\mathbb{C} : |\mathrm{Re}\,\varepsilon| \geq \lambda|\mathrm{Im}\,\varepsilon|,\ |\varepsilon| < \bar{\varepsilon}\},$$

with a fixed $\lambda > 0$, it is not difficult to realise that, when one estimates the propagators of the lines with non-vanishing momentum, the bound (2.11) has to be replaced with $|D(\varepsilon,s)| \geq \lambda b \max\{|a\varepsilon|,|s|\}$, for a suitable constant $b > 0$. The propagators of the lines with vanishing momentum may be bounded trivially as $1/|a|$, so that they do not introduce any additional problems when bounding the product of the propagators. Therefore, the analysis performed above to prove the convergence of the series still applies, notwithstanding the slight changes in the tree representation, provided one takes $\bar{\varepsilon}$ small accordingly with λ. Indeed, redefining C_0 in (2.13) as $C_0 := \rho^{-1} \max\{\Gamma/\lambda|a|b, \Phi, 1\}$ for given $A \in (0, C_0)$, the parameter δ – and hence n_0 – has to be fixed so that $C_0^2\delta/\lambda|a|b \leq A^2$ (see Section 2 for details). This means that the series expansion in powers of μ converges provided $|\varepsilon| \leq \bar{\varepsilon}$, with $\bar{\varepsilon}$ fixed as a function of $\alpha_{n_0}(\boldsymbol{\omega})$ and n_0 chosen as a function of λ. However, since no assumption is made on $\boldsymbol{\omega}$, we can only say that $\alpha_{n_0}(\boldsymbol{\omega})$ is larger than some constant – fixed for fixed λ. The conclusion is that, for any given λ, we can only prove analyticity in the domain $\mathcal{D}(\lambda, \bar{\varepsilon})$, with $\bar{\varepsilon}$ small enough (in particular the smaller the parameter λ, the smaller the value of $\bar{\varepsilon}$), but we have no control on the shrinking of the domain with λ. So, in principle, we can estimate the domain of analyticity of the solution by taking the union of the domains $\mathcal{D}(\lambda, \bar{\varepsilon})$, by letting λ varying, say, in $(0, 1]$ – in analogy, for instance, to what was done in [12]. In practice, though, this is of no profit, since we have no information of how such a constant depends on λ.

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