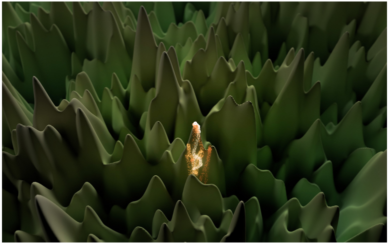
Weyl law in Liouville quantum gravity

Nathanaël Berestycki
Universität Wien
(joint with Mo-Dick Wong, Durham)



Roma, Feb 2023

Universality in Condensed Matter and Statistical Mechanics

Outline

(1) Introduction

Motivation, Crash introduction to LQG,
Definition of Liouville Brownian motion

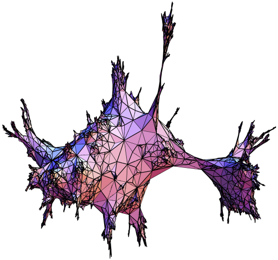
(2) Statement of results:

Asymptotic Weyl law for spectrum
Heat kernel short-time asymptotics

(3) Problems, conjectures

Motivation

What does a random surface look like?



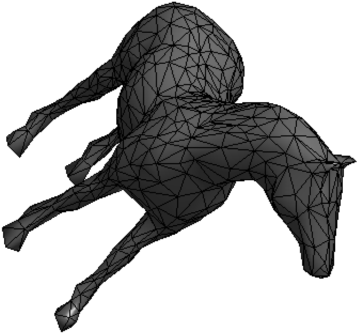
... and what does it mean?

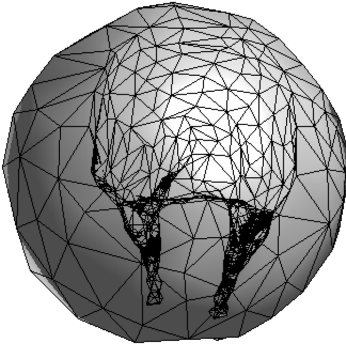
Mathematically: theory emerged in 2000s

Motivation

Liouville Quantum Gravity, or Liouville conformal field theory
2d Feynmann path integrals (Polyakov, 80s)

Isothermal coordinates





Conformal and metric structures encoded by conformal factor:

$$\rho(z) \geq 0, z \in D$$

and metric takes form

$$ds^2 = \rho(z)(dx^2 + dy^2).$$

DDK Ansatz (cf. Duplantier–Sheffield)

Metric of the form

$$ds^2 = e^{\gamma h}(dx^2 + dy^2).$$

where $\gamma \in \mathbb{R} \leftrightarrow$ **central charge**; h is the Gaussian free field.

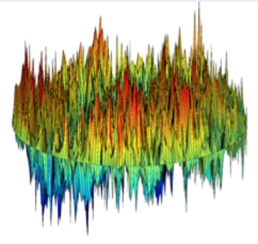
Gaussian free field

A random "function" $D \to \mathbb{C}$

$$(h, f) \sim \mathcal{N}(0, \sigma^2);$$

$$\sigma^2 = \iint G(x, y) f(x) f(y) dxdy$$

where G = Green function on D.



Definitions?

That gives $h(z)$, but what about $e^{\gamma h(z)}$?

Three objects:

A Riemannian **metric** on sphere $\mathbb{S}^2$ or domain $D \subset \mathbb{C}$	$e^{\gamma h(z)}(dx^2 + dy^2)$
A volume **measure**	$e^{\gamma h(z)} dz$
A **diffusion** (Brownian motion on surface)	$dZ_t = e^{-\frac{\gamma}{2} h(Z_s)} dB_s.$

What is known

Three objects:

A (Riemannian) **metric** on the sphere $\mathbb{S}^2$ or domain $D \subset \mathbb{C}$	$e^{\frac{\gamma}{d_\gamma} h(z)}(dx^2 + dy^2)$	V
A volume **measure**	$\mu_\gamma(dz) = e^{\gamma h(z)} dz$	V
A **diffusion** (Brownian motion on surface)	$dZ_t = e^{-\frac{\gamma}{2} h(Z_s)} dB_s.$	V

Remark

All this holds for **any** Gaussian, log-correlated field in every dimension $d \geq 1$, *except* for the metric.

Possible programme:

What do classical theorems of geometry/analysis become in Liouville quantum gravity?

This talk:

Can you hear the shape of LQG? (Mark Kac 1966)
Do eigenvalues determine the shape of a domain?
Here: $\rightarrow$ Weyl law

Gaussian multiplicative chaos

Let $h =$ centered Gaussian field in $\mathbb{R}^d$, covariance

$$K(x,y) = -\log(|x-y|) + g(x,y)$$

g continuous, bounded. Set $h_\varepsilon = h \star \theta_\varepsilon$ regularisation at scale ε, where $\theta =$ convolution kernel.

Theorem (Kahane (85), Duplantier–Sheffield (2010), B. (2017), Shamov (2017))

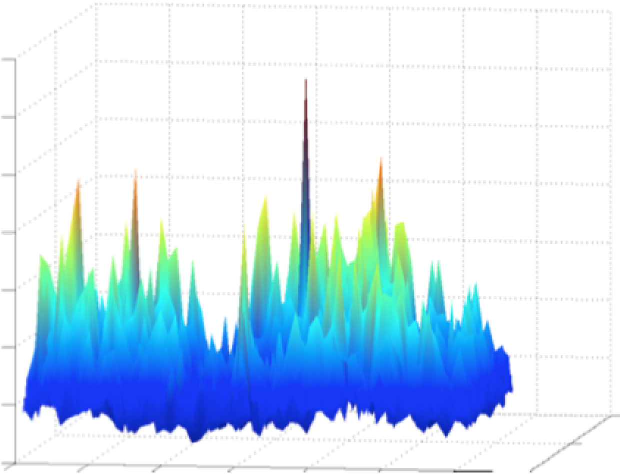
Define:

$$\mu_\gamma(S) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_S e^{\gamma h_\varepsilon(z)} dz$$

exists in proba. μ_γ is called **Liouville measure** */ GMC associated to h.*
Furthermore, $\mu_\gamma \neq 0$ iff $\gamma < \sqrt{2d}$.

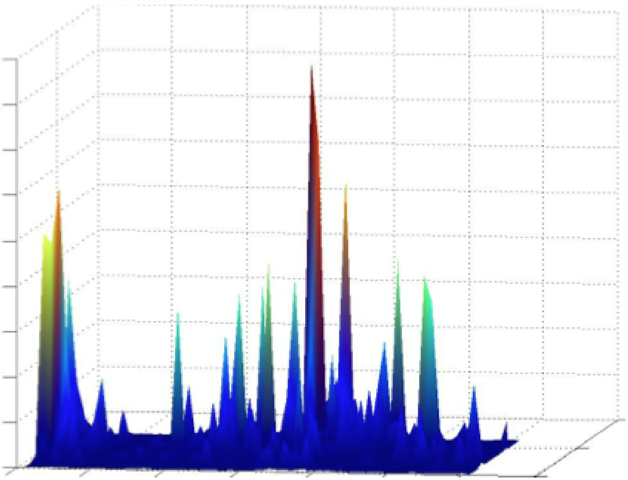
Intuitively, $\mu_\gamma =$ uniform measure, in isothermal coordinates.

Visualisation of Liouville measure



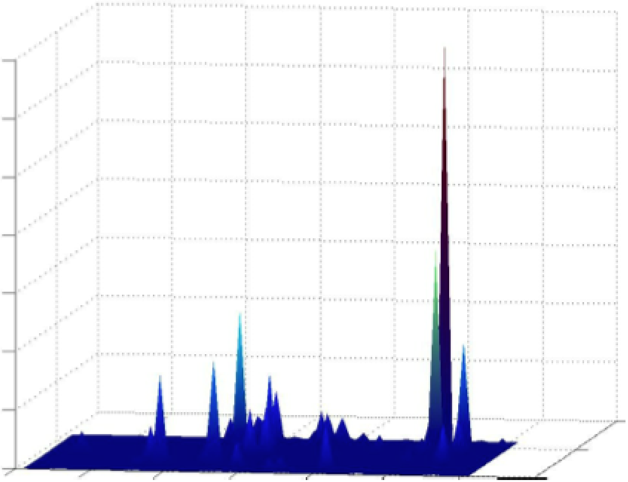
$\gamma = 0.2$

Visualisation of Liouville measure



$\gamma = 1$

Visualisation of Liouville measure



$\gamma = 1.8$

Properties of Liouville measure

Some properties

- μ_γ non zero iff $\gamma < \sqrt{2d}$.
- No atoms: no point carries positive mass.
- "Universality": limit independent of regularisation.
- Conformal covariance.
- multifractal spectrum...

Liouville Brownian motion

Recall: $h_\varepsilon(z)$ = regularised field at scale ε.
Then ε-regularised Liouville Brownian motion:

$$d\mathbf{Z}_s^\varepsilon = \varepsilon^{\gamma^2/4} e^{-\frac{\gamma}{2} h_\varepsilon(\mathbf{Z}_s)} dB_s$$

In other words,

$$\mathbf{Z}_s^\varepsilon = B_{\phi_\varepsilon^{-1}(t)}; \qquad \phi_\varepsilon(t) = \varepsilon^{\gamma^2/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} ds$$

Theorem (B. 2015, Garban–Rhodes–Vargas 2016)

$\lim_{\varepsilon \to 0} \mathbf{Z}_s^\varepsilon$ *exists (i.e.,* $\lim_{\varepsilon \to 0} \phi_\varepsilon(t)$ *exists). The limit is called* **Liouville Brownian motion**.

Liouville Brownian motion

Properties

- Continuous; does not stay stuck
- Can be started from all points simultaneously $\to$ **Feller semigroup** (Garban–Rhodes–Vargas)
- Liouville measure μ_γ is a.s. **invariant** (in infinite volume say).
- Spends all its time in a set of measure zero ($\mathcal{T}_\gamma =$ **"γ-thick points"**).
- Can even compute $\dim(\{t : Z_t \in \mathcal{T}_\alpha\})$ (B., Jackson)
- for $\gamma > \sqrt{d}$, the trajectory has zero derivative at almost every time (Jackson).
- Spectral dimension $d_s = 2$ a.s. (Rhodes–Vargas, conjectured by Ambjørn).
- ...

Spectrum of Liouville Brownian motion

Infinitesimal generator is hard to describe but Green function is straightforward:

Liouville Green function

Fix bounded domain D. Given h a GFF in D, we have

$$\mathbf{G}_D(x, dy) = G_D(x, y)\mu_\gamma(dy).$$

This is a.s. a finite measure if $\gamma < \sqrt{2d}$.

Definition of eigenvalues

Say $\boldsymbol{\lambda}_n > 0$ is an eigenvalue of LBM if $\exists f_n \in \mathbf{L}^2(\mu_\gamma)$ such that

$$\mathbf{G}_D f_n = \frac{1}{\boldsymbol{\lambda}_n} f_n$$

Heat kernel

Theorem (Maillard–Rhodes–Vargas–Zeitouni '16)

There a.s. exists a **jointly continuous** function $\mathbf{p}_t(x, y)$ such that for all nonnegative function f, for all $x \in D$,

$$\mathbb{E}_x[f(\mathbf{Z}_t)] = \int_D f(y)\mathbf{p}_t(x, y)\mu_\gamma(dy).$$

(jointly measurable version + semigroup property: earlier in Garban, Rhodes and Vargas (2014)).

Spectrum of Liouville Brownian motion

Lemma (Maillard–Rhodes–Vargas–Zeitouni '16)

A.s. there exists an ONB of eigenfunctions $(f_n)_{n\geq 1}$ for $\mathbf{L}^2(\mu_\gamma)$ and $\boldsymbol{\lambda}_n \uparrow \infty$ a.s.

Spectral representation of heat kernel:

Theorem (Maillard–Rhodes–Vargas–Zeitouni '16)

$$\mathbf{p}_t(x, y) = \sum_{n\geq 1} e^{-\boldsymbol{\lambda}_n t} f_n(x) f_n(y),$$

a.s. (both sides are jointly continuous)

Weyl law in standard Euclidean geometry

Theorem (Weyl 1912, Courant 1922...)

Let $N(\lambda) = \#$ eigenvalues $\leq \lambda$ for $-\Delta$ in $D \subset \mathbb{R}^d$. Then

$$\frac{N(\lambda)}{\lambda^{d/2}} \to c_d \,\mathrm{Leb}(D), \qquad \lambda \to \infty$$

(explicit $c_d \in (0, \infty)$).

What is the analogue in LQG?

Which d shall we even use?

Weyl law in LQG

Now take $d = 2$, $h =$ GFF in bounded domain D.

Theorem (B.–Wong, '23+)

Let $0 < \gamma < 2$. Let $\mathbf{N}(\lambda) = \#\{n : \boldsymbol{\lambda}_n \leq \lambda\}$. Then in probability,

$$\frac{\mathbf{N}(\lambda)}{\lambda} \to c(\alpha)\mu_\gamma(D),$$

where $\alpha = \frac{2}{\gamma} - \frac{\gamma}{2} > 0$, and

$$c(\alpha) := \frac{1}{\pi}\left\{\mathbb{E}\left[\int_0^\infty \mathcal{I}\left(e^{\gamma(B_t - \alpha t)}\right) dt\right] + \mathbb{E}\left[\int_0^\infty \mathcal{I}\left(e^{-\gamma \mathcal{B}_t^\alpha}\right) dt\right]\right\}$$

$\mathcal{I}(x) = xe^{-x}$, $\mathcal{B}^\alpha$ a BM with drift α conditioned to stay positive.

Concisely, $c(\alpha) = \frac{1}{\pi}\mathbb{E}[\int_{-\infty}^\infty \mathcal{I}(e^{\gamma C(t)})dt]$,
$C(t)$ a γ-**quantum cone** (Sheffield).

Heat kernel asymptotics

Remark: $d = 2 \to$ spectral dimension.

Theorem. (B.–Wong '23+)

For any open set $A \subset D$,

$$t \int_A \mathbf{p}_t(x,x)\mu_\gamma(dx) \xrightarrow[t \to 0^+]{} c(\alpha)\mu_\gamma(A)$$

in probability.

This improves on Rhodes–Vargas who proved the "spectral dimension" $d_s = 2$ (predicted by Ambjørn 1990s):

$$\mathbf{p}_t(x,x) = t^{-1+o(1)}.$$

Pointwise behaviour of heat kernel?

Q: Is it the case that

$$\mathbf{p}_t(x,x) \sim \frac{c(\alpha)}{t}$$

in probability as $t \to 0$ for $\mu_\gamma -$ a.e. x ?

Pros and cons:

$\ominus$ Dependence on x very sensitive – multifractal inhomogeneities in local behaviour of h.

$\oplus$ Restrict to x "typical". LBM paths also spend all their time on typical points.

Local fluctuations

However, when we average over the Gaussian free field:

Theorem (B.–Wong '23+)

$$\lambda \int_0^\infty e^{-\lambda t} t \mathbf{p}_t(x,x) dt \xrightarrow[\lambda \to \infty]{d} R_\gamma$$

a nondegenerate random variable.

By a probabilistic Tauberian theorem, this shows

$$t\mathbf{p}_t(x,x) \not\to c(\alpha)$$

as $t \to 0$. In fact, cannot converge (a.s. or in proba) to *any* RV.

Local fluctuations

Still averaging over h,

Conjecture (B.–Wong '23+)

$$t\mathbf{p}_t(x,x)dt \xrightarrow[t\to 0^+]{d} R_\gamma$$

in distribution.

Missing: Tauberian theorem for convergence in distribution!

Q: quenched fluctuations? We believe there are additional logarithmic fluctuations.

Finally...

Conjecture (B.–Wong '23+)

You CAN hear the shape of LQG, a.s. :
h is a.s. equal to a measurable function of $\{\boldsymbol{\lambda}_n\}_{n\geq 1}$.

In fact, $\{\boldsymbol{\lambda}_n\}_{n\geq 1}$ determines (D, h) up to equivalence

$$(D, h) \leftrightarrow (D', h \circ f^{-1} + Q \log |(f^{-1})'|),$$

$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, $f : D \to D'$ conformal map.