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PHASE TRANSITION IN THE IDS OF THE ANDERSON MODEL ARISING FROM A SUPERSYMMERIC SIGMA MODEL

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Anderson model (random Schrödinger)

electronic transport and wave prop. in disordered syst

on $\mathbb{Z}^d$

$$H = -\Delta + \lambda V \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$$

- $-\Delta$ = lattice Laplacian
- $(Vf)_x = V_x f_x$ $\quad \{V_x\}_{x \in \mathbb{Z}^d}$ random var
- $\lambda > 0$ parameter

H is a random operator : $H : D \longrightarrow \ell^2(\mathbb{Z}^d)$ $\quad D = \{ f \in \ell^2(\mathbb{Z}^d)$ with finite supp.$\}$

if $\left.\begin{cases} d\mathbb{P}(V) \text{ translation invariant} \\ \mathbb{E}[V_0^2] < \infty \end{cases}\right\} \Rightarrow \begin{cases} \bullet\ H \text{ ergodic} \\ \bullet\ H \text{ admits a self-adj extension} \\ \bullet\ \sigma(H) \subset \mathbb{R} \text{ is deterministic} \end{cases}$

spectral regimes

$\lambda = 0$ (no dis.) $\to$ $H = -\Delta$ a.c. spectrum, extended eigenvectors

$\lambda \gg 1$ (large dis.) $\to$ H '$\approx$' λV pp spectrum, localized eigenvectors

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proved: pp spectrum, local. eigenv. $\begin{cases} \text{on the whole } \sigma(H) \text{ if } d=1 \text{ or } d\geq 2 \text{ and } \lambda \gg 1 \\ \text{at the edge of } \sigma(H)\ \forall \lambda\ \forall d \end{cases}$

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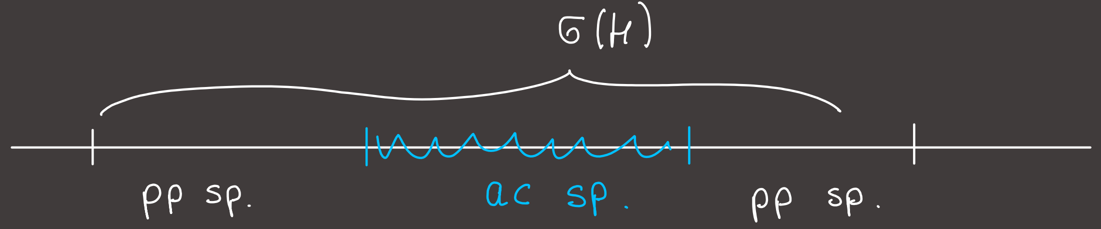
major open conjecture: in $d \geq 3$ phase transition between ac and pp spectrum:

at <u>weak disorder</u> ($\lambda \ll 1$) $\exists\, I \subset \sigma(H)$ with

$$\sigma \cap I = \sigma_{ac}$$
$$\sigma \setminus I = \sigma_{pp}$$



Integrated density of states (IDS) $\to$ the most natural observable

$N(E,H) :=$ average number of eigenvalues $\leq E$ per unit volume

$$= \lim_{L\to\infty} \mathbb{E}\left[N(E, H^{\#}_{\Lambda_L})\right] \qquad \Lambda_L = [-L,L]^d \cap \mathbb{Z}^d, \quad H^{\#}_{\Lambda_L} = (H|_{\Lambda_L})^{\#} \leftarrow \text{bound. cond.}$$

$$N(E, H_{\Lambda_L}) := \frac{\#\,\text{eigenval.} \leq E}{|\Lambda_L|} = \frac{\operatorname{tr} \mathbb{1}_{(-\infty,E]}(H^{\#}_{\Lambda_L})}{|\Lambda_L|}$$

Case 1: the V_x are indipendent identically distributed

- universality at the edge of the spectrum $\rightarrow$ Lipschitz tail

$$N(E,H) = c_1 e^{-\frac{c_2}{(E-E_0)^{d/2}}} \qquad E \sim E_0$$

c_1, c_2 depend on $d\rho(V_x)$

compare with $H = -\Delta \Rightarrow N(E,-\Delta) \sim (E-E_0)^{d/2}$ $\quad E_0 = 0$

heuristic arg: assume $E_0 = 0$, $V_x \geq 0$ as, $d\mu(V_x) = \rho(V_x)\,dV_x$ $\|\rho\|_\infty < \infty$.

lowest eig. $\lambda_1(H) < E \ll 1$ if $V_x \ll 1$ $\forall x$ in a set Λ with $|\Lambda| \sim E^{-d/2}$

$$V \text{ iid} \Rightarrow \mathbb{P}(V_x \ll 1\ \forall x \subset \Lambda) \underset{\varepsilon<1}{\leq} \varepsilon^{|\Lambda|} = e^{-|\Lambda| \ln \frac{1}{\varepsilon}} = e^{-\frac{c}{E^{d/2}}}$$

- information on the spectral type ?

→ Lipschitz tail ⟹ pp spectrum, exponentially loc. eigenf. at the edge of $\sigma(H)$

→ $N(E)$ cannot see the phase trans in $\sigma(H)$

$V_x \to g(V_x)$ analytic ⟹ $E \to N(E)$ analytic [Constantinescu - Fröhlich - Spencer 84]

'regular' 'regular'

Case 2 : the V_x are not independent

- linear dep $V_x \sim \sum_{y \in \mathbb{Z}^d} \omega_y f(x-y)$ $\{\omega_y\}_{y \in \mathbb{Z}^d}$ indep id distr.

  -> phase trans between Lipschitz and non Lip.
  (classical - quantum transition) [Leschke-Warzel PRL 2004
  Najar JSP 2008]

- our model -> V_x indip at distance 2
  -> no Lipschitz tail!
  -> phase trans between $E \leftrightarrow \sqrt{E}$ in $d \geq 3$
  -> the transition has the same dependence on dimension and strength of disorder as the (conjectured) ac/pp sp. trans

The model on $\mathbb{Z}^d$:

$(\beta f)_x = \beta_x f_x \qquad \{\beta_x\}_{x \in \mathbb{Z}^d}$ 1-dep. positive r.v.

$H_\beta = 2\beta - P_W \qquad (P_W)_{ij} = W_{ij} P_{ij} \qquad P_{ij} = \delta_{|i-j|=1} \qquad W_{ij} > 0$ edge weight

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$\mathbb{P}(\beta)$: finite volume marginal $\qquad \Lambda \subset \mathbb{Z}^d$ finite

$$\boxed{d\nu_\Lambda^W(\beta) = \mathbb{1}_{H_{\beta,\Lambda}>0} \frac{e^{-\frac{1}{2}\left(\langle 1, H_{\beta\Lambda} 1\rangle + \langle \eta, H_{\beta\Lambda}^{-1} \eta\rangle - 2\langle \eta, 1\rangle\right)}}{\sqrt{\det H_{\beta\Lambda}}} \left(\frac{2}{\pi}\right)^{\frac{|\Lambda|}{2}} \prod_{j\in\Lambda} d\beta_j}$$

$\langle f, g\rangle := \sum_{x\in\Lambda} f_x g_x \qquad H_{\beta,\Lambda} = H_\beta|_\Lambda \in \mathbb{R}^{\Lambda\times\Lambda} \qquad \eta_x = \sum_{\substack{y\notin\Lambda \\ y\sim x}} W_{xy} \to$ boundary cond.

Remarks

- exact formula for the Laplace transf

$$\mathbb{E}_\Lambda\left[e^{-\langle f, \beta\rangle} \right] = \prod_{i\sim j} e^{-W_{ij}\left(\sqrt{1+f_i}\sqrt{1+f_j}-1\right)} \qquad \prod_j \frac{e^{-\eta_j\left(\sqrt{1+f_j}-1\right)}}{\sqrt{1+f_j}}$$

$\Rightarrow$ β indep. at dist. 2: $\mathbb{E}\left[e^{-f_i\beta_i - f_{i+2}\beta_{i+2}} \right] = \dfrac{e^{-\tilde\eta_j\left(\sqrt{1+f_j}-1\right)}}{\sqrt{1+f_j}} \quad \dfrac{e^{-\tilde\eta_{j+2}\left(\sqrt{1+f_{j+2}}-1\right)}}{\sqrt{1+f_{j+2}}}$

$$\tilde\eta_j = \sum_{k\in\mathbb{Z}^d\ k\sim j} W_{jk}$$

- $2\beta - P_W > 0 \Rightarrow$ if $\beta_i \sim 0 \Rightarrow$ other β large assume $W_{xy} = W > 0 \;\; \forall \; x \sim y \;\Rightarrow$

- H_β ergodic and $\sigma(H_\beta) \subset [0, \infty)$
- $H_\beta = 2\beta - WP = W\left(\frac{2\beta}{W} - P\right) = W\Big(\underbrace{\frac{2\beta}{W} - 2d}_{\lambda V} - \Delta\Big)$ ← Anderson model

$\Rightarrow$ we consider $\boxed{\frac{H_\beta}{W} = \frac{2\beta}{W} - P}$

- $\mathbb{E}\left[\frac{2\beta}{W}\right] = 2d + \frac{1}{W}$ $\begin{cases} \gg 1 & \text{if } W \ll 1 \\ \sim 2d & \text{if } W \gg 1 \end{cases}$
- $\mathrm{Var}\left[\frac{2\beta}{W}\right] = \frac{2d}{W} + \frac{2}{W^2}$ $\begin{cases} \gg 1 & \text{if } W \ll 1 \\ \ll 1 & \text{if } W \gg 1 \end{cases}$

$\Rightarrow$ strong disorder: $W \ll 1$ $\quad \frac{H_\beta}{W} \approx 2\beta$

weak disorder: $W \gg 1$ $\quad \frac{H_\beta}{W} \sim 2d - P = -\Delta$

$\Rightarrow \lambda \equiv \frac{1}{W}$

Origin of the model: a certain random walk in a random environment

$\Lambda \subset \mathbb{Z}^d$ finite

generator of the random walk

$$-\Delta^{\omega(u)}_{ij} = \begin{cases} -\omega_{ij}(u) & i\sim j \\ \sum_{k\in\mathbb{Z}^d} \omega_{ik}(u) & i=j \\ 0 & \text{oth} \end{cases}$$

$$\omega_{ij}(u) = W_{ij} e^{u_i+u_j}$$

$(u_i)_{i\in\Lambda}$ random var. with prob $d\rho_\Lambda(u)$

$(u_i)_{i\notin\Lambda} \equiv 0$

$$e^{-\hat{u}} - \Delta^{\omega(u)} e^{-\hat{u}} = H_{\beta(u)}$$

$$\beta_j(u) = \sum_{\substack{k\in\mathbb{Z}^d \\ k\sim j}} W_{kj} e^{u_k-u_j} = \sum_{\substack{k\in\Lambda \\ k\sim j}} W_{kj} e^{u_k-u_j} + \eta_j e^{-u_j}$$

$$d\rho_\Lambda(u) \longrightarrow d\nu_\Lambda(\beta)$$

$$d\nu_\Lambda(u) = \prod_{i\sim j} e^{-W_{ij}\left[\operatorname{ch}(u_i-u_j)-1\right]} \prod_j e^{-\eta_j(\operatorname{ch} u_j-1)} \underbrace{\sqrt{\det H_{\beta(u)}}}_{\substack{\| \\ \sqrt{\det -\Delta^{\omega(u)}}\,\prod_j e^{-u_j}}} \prod_j du_j$$

- u are highly dependent
- not clear if $\exists$ infinite volume measure

$d\rho_\Lambda(u) =$

- mixing measure of a stochastic process with memory — VRJP
  (Werner) Sabot, Tarrès
- marginal of a supersymmetric spin model $H^{2|2}$ $\to$ toy model for quantum diffus.
  D., Spencer, Zirnbauer

VRJP / $H^{2|2}$

- positive recurr / exp localized
  - $\to$ in $d = 1$ $\forall\ W > 0$
  - $\to$ in $d \geq 2$ for $W << 1$ strong dis
- transient / delocaliz. $\to$ in $d \geq 3$ for $W >> 1$ weak dis

<u>localiz / deloc phase transition in $d \geq 3$</u>

[Zeng, Poudevigne, Rapenne, Kozma, Angel, Crawford, Bauerschmidt, Helmuth, Swan ...]

question: c.n we use these results to study $\sigma(H_\beta)$?

- pp spectrum + exp. loc. at large disorder [Collevecchio - Zeng EJP 2021]

- info on IDS [D., Rapenne, Rojas-Molina, Zeng 2022 preprint] no Lipschitz tail at large disorder or $d=1$

Thm

- $\exists \; W_c(d)$ st $\forall \; 0 < W < W_c(d)$ $\quad N(E, H_\beta) \geq c \dfrac{\sqrt{E}}{|\ln E|^{d/2}} \quad \forall \; 0 \leq E < E_0$

  where $c = c(W,d) > 0$, $E_0 = E_0(W,d,c) > 0$ $\quad \Rightarrow$ no Lips. tail.

- $\forall \; W > 0$ $\quad N(E) \leq C \sqrt{W} \sqrt{E} \quad \forall \; E > 0$

$\Longrightarrow$

$$C_2 \frac{\sqrt{E}}{|\ln E|^{d/2}} \leq N(E) \leq C_1 \sqrt{W} \sqrt{E} \quad \text{in} \quad \begin{cases} d = 1 \; \forall \; W > 0 \\ d \geq 2 \; \forall \; W < W_c \end{cases}$$

for E small

behavior at weak disorder

Thm

if $d \geq 3$ $W > W_0(d)$ $N(E) \leq C\, E \ll C\sqrt{W}\sqrt{E}$ $\forall E > 0$

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regularity

Thm

- $\forall W > 0$ $\mathbb{E}\left[N(E+\varepsilon, H_{\beta,\Lambda_L}) - N(E-\varepsilon, H_{\beta,\Lambda_L})\right] \leq C \sqrt{W}\sqrt{\varepsilon}$

  unif in $\Lambda \subset \mathbb{Z}^d$, E $\Longrightarrow$ N Hölder cont with exp $1/2$

- if $d \geq 3$ $W > W_0(d)$ $\mathbb{E}\left[N(E+\varepsilon, H_{\beta,\Lambda_L}) - N(E-\varepsilon, H_{\beta,\Lambda_L})\right] \leq C\sqrt{W}\,\varepsilon$

  $\Longrightarrow$ N Lipschitz cont.

open problems

- lower bound on IDS for $W \gg 1$ (we expect $N(E) \sim \begin{matrix} E & \text{in } d \geq 3 \\ \sqrt{E} & d = 2 \end{matrix}$)

- delocalization / ac spectrum for $d \geq 3$ $W \gg 1$