SCALING RELATIONS for 2D PERCOLATION

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7/2/2023

Scaling relations for 2D percolation

joint work with I. Manolescu

I Motivation :

60s Widom scaling hypothesis.

Widom / Fisher

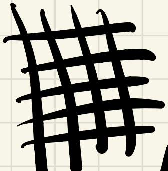
$$\frac{2-\alpha}{d} = \nu = \frac{2\beta}{d-2+\eta}$$

$$2-\eta = d\,\frac{\delta-1}{\delta+1} = \frac{\gamma}{\nu}$$

1987 Kesten 2D Bernoulli perco.

II Result :

Fortuin - Kasteleyn percolation



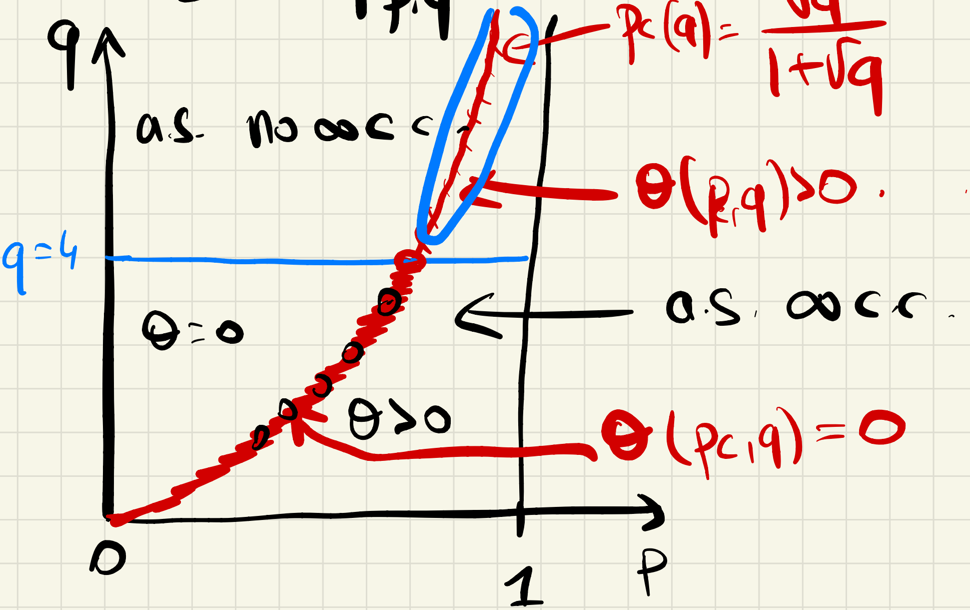
Λ

$$\phi_{\Lambda,p,q}(\omega) = \frac{1}{Z}\, p^{|\omega|}(1-p)^{|E_\Lambda|-|\omega|}\, q^{k(\omega)}$$

$p \in [0,1]$, $q > 0$

$k(\omega)$: # connected comp.

$\Lambda \subset \mathbb{Z}^2 \quad \phi_{p,q}$



$$\theta(p,q) = \phi_{p,q}[0 \text{ is in } \infty \text{ c.c}]$$

$$\boxed{q \in [1,4]}$$

$$f''(p) = \frac{1}{|p-p_c|^{\alpha+o(1)}}$$ $f(p)$ free energy

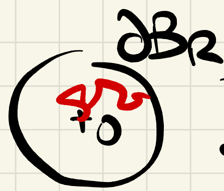
$$\theta(p) = (p-p_c)_+^{\beta+o(1)}$$

$$\chi(p) = \frac{1}{|p-p_c|^{\gamma+o(1)}}$$ $\chi(p) = \sum \phi[0 \leftrightarrow x]$

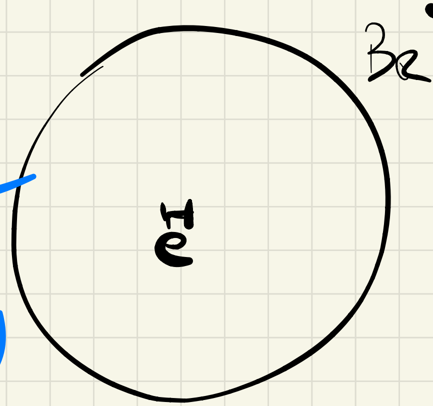
$$\xi(p) = \frac{1}{|p-p_c|^{\nu+o(1)}}$$ $\xi(p)$ correlation length

$$\phi_{p_c,q}[0 \leftrightarrow x] = \frac{1}{|x|^{d-2+\eta+o(1)}}$$

<u>2 other players</u>:

$\pi_1(p,R) = \phi_{p,q}[$  $]$

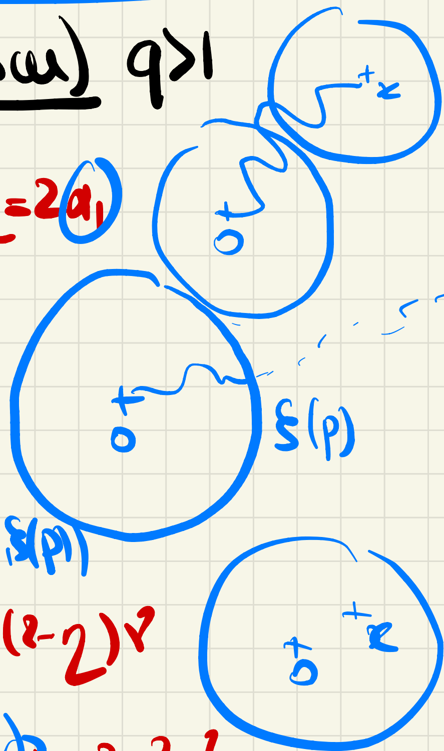
$\Delta(p,R) = \phi_{p,q}[w_e \mid$ all edges outside B_R are open$]$
$- \phi_{p,q}[w_e \mid$ ______________ closed$]$



$$\pi_1(p_c,R) = \frac{1}{R^{\alpha_1+o(1)}}$$

$$\Delta(p_c,R) = \frac{1}{R^{\iota+o(1)}}$$

Theorem (DC - Manolescu) $q>1$



$$\phi_{p_c}[0 \leftrightarrow x] \asymp \pi_1(p_c,|x|)^2 \qquad \eta = 2\alpha_1$$

$$\theta(p) \asymp \pi_1(p_c, \xi(p)) \qquad \beta = \alpha_1 \nu$$

comparison between $\pi_1(p,\xi(p))$, $\pi_1(p_c,\xi(p))$

$$\chi(p) \asymp \xi(p)^2 \pi_1(p_c,\xi(p))^2 \qquad \gamma = (2-\eta)\nu$$

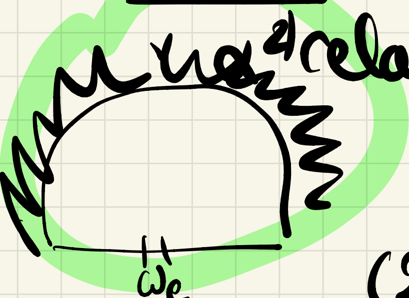
$$f''(p) \asymp \sum_{\ell \le \xi(p)} \ell\, \Delta(p_c,\ell)^2 \qquad \alpha = 2-2\nu$$

$$\Delta(p_c,\xi(p))\,\xi(p)^2 (p-p_c) \asymp 1 \qquad (2-\alpha_4)\nu = 1$$

generalized Kesten's relation

Comments 1) similar relation for δ

2) relation only valid when $\nu \le 1$ $(q \ge 2)$

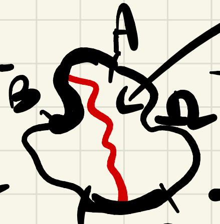


3) Kesten $q=1$

$$(2-\alpha_4)\nu = 1$$

III Stability below the correlation length

Theorem: $\exists\, c, C > 0$ s.t. $\forall L \le \xi(p)$

$\Big|\phi_p\Big[$ 

$\Big] - \phi_{p_c}\Big[$ 

$\Omega \Big| \le C\left(\frac{L}{\xi(p)}\right)^c$

$\# \; 1/L$

ξ easy to prove $\left|\frac{\pi_1(p,L)}{\pi_1(p_c,L)} - 1\right| \le C\left(\frac{L}{\xi(p)}\right)^c$

$\xi(p) \asymp L(p)$ characteristic length.

$p > p_c$.

$L(p) = \max\{L$ s.t. $\phi_p\Big[$ 

$\Big] \le \frac{99}{100}\}$.