Periodic striped states in Ising models with dipolar interactions

Davide Fermi

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joint work with
A. Giuliani (Università Roma Tre)

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Outline of the talk

1. Motivations and background
   - Emergence of periodic patterns and striped configurations.
   - Ising models with competing interactions.
2. Main results
   - The variational class with straight domain walls.
   - Characterization of the GS.
3. Sketch of the proof

Reference

[1] D.F., A. Giuliani, in "*The Physics and Mathematics of Elliott Lieb. The 90th Anniversary Volume I*" (2022).

1. Motivations and background

Modulated phases

- A large number of physical, chemical and biological systems exhibit the spontaneous formation of regular patterns, such as compact droplet-like domains and striped periodic structures of uniform phase.

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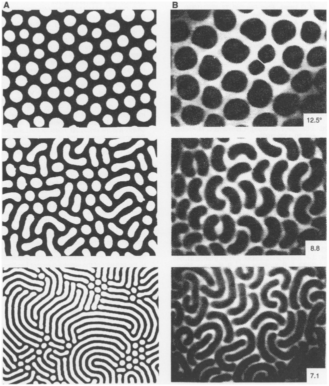
- polymer suspensions (Ohta-Kawasaki model);
- **micromagnets and magnetic films**;
- ferrofluids and liquid crystals;
- superconductors;
- anysotropic electron gases and QHE systems;
- ...



magnetic garnet | Langmuir films
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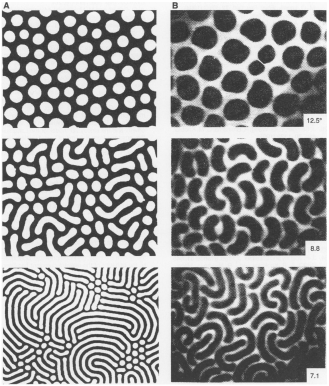

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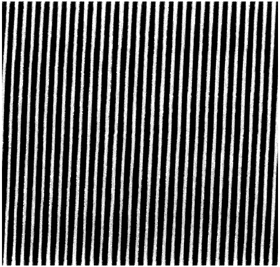
- The similarity of patterns suggests a common underlying mechanism, namely the competition between short-range and long-range forces.

  A fundamental understanding is still missing: main difficulties related to symmetry breaking.

Stripes in 2D ferromagnetic layers



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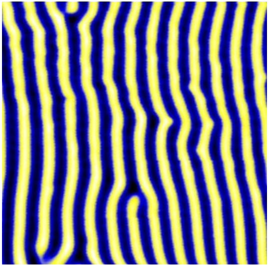

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- Main focus of this talk: low temperature physics of thin magnetic films.

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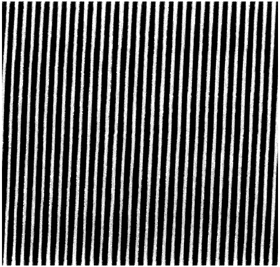

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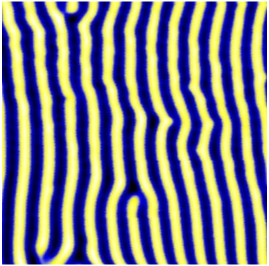

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Present physical understanding: the formation of periodic arrays of stripes is driven by the competition of short-range exchange ferromagnetic interactions, favoring a homogeneous ordered state, and long-range dipole-dipole forces, opposing the ordering on the sample scale.

Large experimental evidence and several numerical results, nevertheless only *few rigorous theoretical results* are available.

Frustrated ferromagnets

- One of the simplest way to describe the formation of periodic phases is to consider an Ising-type energy functional on the $\mathbb{Z}^d$−lattice of the form

$$\mathcal{H}(\boldsymbol{\sigma}) = -\,J \sum_{\langle \boldsymbol{x},\boldsymbol{y}\rangle} (\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{y}} - 1) + \sum_{\{\boldsymbol{x},\boldsymbol{y}\}} \frac{\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{y}} - 1}{|\boldsymbol{x}-\boldsymbol{y}|^p}\,.$$

  - $\boldsymbol{\sigma} \equiv (\sigma_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathbb{Z}^d} \in \{\pm 1\}^{\mathbb{Z}^d}$ is a generic spin configuration.
  - $J > 0$ is the FM coupling constant.
  - $\langle \boldsymbol{x},\boldsymbol{y}\rangle$ is a nearest-neighbor pair; $\{\boldsymbol{x},\boldsymbol{y}\}$ is any pair of distinct sites.
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- Frustation induced by competition between:
  - exchange interaction, favoring homogeneous FM states $(\sigma_{\boldsymbol{x}} = +1$ or $\sigma_{\boldsymbol{x}} = -1\ \forall \boldsymbol{x})$;
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Conjecture: *periodic striped states of optimal width are exact infinite volume ground states* (depending on J, breaking translation symmetry).

GS characterization

- **Key element**: Reflection Positivity (RP) of AF interaction

$$\sum_{\substack{x_1 \geqslant 1,\, y_1 \leqslant 0 \\ \boldsymbol{x}_\parallel,\, \boldsymbol{y}_\parallel \in \mathbb{Z}^{d-1}}} \frac{\overline{s_{x_1,\boldsymbol{x}_\parallel}}\, s_{-y_1+1,\boldsymbol{y}_\parallel}}{|\boldsymbol{x}-\boldsymbol{y}|^p} > 0\,,$$

for all $p > 0$, $d \geqslant 1$.
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- Limiting regimes

  $J \gtrsim 0$: GS = AF state for $1 \leqslant d \leqslant 3$, $p > 0$ [Fröhlich-Israel-Lieb-Simon 1978, by RP];
  $J = +\infty$: GS = homogeneous FM state;
  $J \lesssim +\infty$: GS = homogeneous FM state for $\boldsymbol{p > d+1}$
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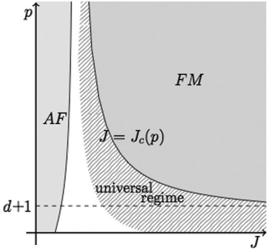
- For $\boldsymbol{p>d+1}$ there is a transition line at

$$J = J_c(p) := \sum_{\boldsymbol{x}\in\mathbb{Z}^d,\, x_1\geqslant 1} \frac{|x_1|}{|\boldsymbol{x}|^p} < \infty$$

For $J=J_c(p)$ the surface energy of an isolated infinite straight domain wall vanishes.

$J>J_c(p)$: GS $=$ homogeneous FM state (via Peierls' droplet argument);

$J<J_c(p)$: *expected* GS $=$ periodic pattern (energy lowered by antiphase boundaries).



[Giuliani-Lebowitz-Lieb 2011]

- Numerical evidence and variational computations in specific classes suggest

GS = periodic striped states.

Minimizing energy-per-site in thermodynamic limit ⇒ stripes of optimal width

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- $d = 1, p > 1$: GS = identical segments of optimal length, with alternating spin signs

  ⇒ for $d \geqslant 2$, $p > d$ striped GS are periodic.

  [Giuliani-Lebowitz-Lieb 2006]

  $d \geqslant 2,\ p > d$: upper & lower bounds for the GS energy-per-site, coinciding at dominant order.

  [Giuliani-Lebowitz-Lieb 2006, 2007, 2011]

  $d \geqslant 2,\ p > 2d$: upper & lower bounds for the GS energy-per-site, coinciding at next-to-leading order.

  [Giuliani-Lieb-Seiringer 2013]

  $d \geqslant 2,\ p > 2d,\ J \lesssim J_c(p)$: unique infinite-vol. GS = optimal periodic striped states.

  [Giuliani-Seiringer 2016]

  Methods: RP, Peierls' argument, corners as excitations, localization estimates.

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- Results on analogous continuous versions: periodicity of 1D minimizers, computation of GS energy at leading-order, minimizers in variational classes of periodic configurations, ...

  [Acerbi, Conti, Daneri, Fusco, Giuliani, Kerschbaum, Morini, Müller, Muratov, Otto, Runa, Serfaty, ...]

Dipolar ferromagnets: $d = 2$, $p = 3$

- Relevant setting for the physics of thin magnetic films.
  Conjecture: periodic striped GS for $J \gg 1$.
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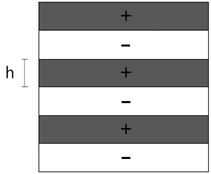
- Partial analysis in [MacIsaac-Whitehead-Robinson-De'Bell, Phys.Rev.B 1995]

Periodic stripes

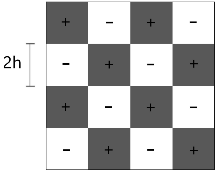


$$\sigma_{\boldsymbol{x}} = (-1)^{\lfloor x_2/h \rfloor};$$

$$\mathcal{E}_s(h) = \frac{2}{h}\left[J - a_s \log h - b_s + \mathcal{O}(h^{-1})\right],$$

$$a_s = 2\,, \qquad b_s = 2.276\ldots\,.$$

Square checkerboards



$$\sigma_{\boldsymbol{x}} = (-1)^{\lfloor x_1/2h \rfloor + \lfloor x_2/2h \rfloor};$$

$$\mathcal{E}_c(h) = \frac{2}{h}\left[J - a_c \log h - b_c + \mathcal{O}(h^{-1})\right],$$

$$a_c = 2\,, \qquad b_c = 0.352\ldots\,.$$

- $a_s = a_c$ and $b_s > b_c$ $\Rightarrow$ $\mathcal{E}_s(h) < \mathcal{E}_c(h)$ for h large.
- Optimal stripes width $= h_* \sim (e^{1-\frac{a_s}{2}})\, e^{J/2}$, with $J \gg 1$.
- Results derived by sheer numerical computations (no conceptual insight).

2. Main results

Preliminaries

- Consider the 2D torus $\Lambda_L \equiv \mathbb{Z}^2/L\,\mathbb{Z}^2$ of side $L \gg 1$ and the Hamiltonian

$$\mathcal{H}_L(\boldsymbol{\sigma}) = -\frac{J}{2}\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Lambda_L\\|\boldsymbol{x}-\boldsymbol{y}|=1}}(\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{y}}-1) + \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{y}\in\Lambda_L}\sum_{\boldsymbol{m}\in\mathbb{Z}^2}\frac{\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{y}}-1}{|\boldsymbol{x}-\boldsymbol{y}+L\boldsymbol{m}|^3}\,,$$

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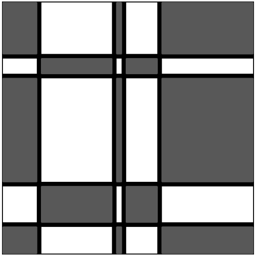
$$\boldsymbol{\sigma} \in \{\pm 1\}^{|\Lambda_L|} \sim (L\text{- periodic spin configurations on } \mathbb{Z}^2).$$

- Consider the variational class Ω_L of L - periodic spin configurations on $\mathbb{Z}^2$ s.t. the union of Peierls' contours consist solely of horizontal and/or vertical straight lines.



Remark: $\boldsymbol{\sigma} \in \Omega_L$ = modulated, generally aperiodic, configurations with spin signs alternating in checkerboard-like pattern.

Theorem (F. - Giuliani 2022)

There exists $J_0 > 0$ such that, for any $J \geqslant J_0$ and any L integer multiple of $2h_*$, the only minimizers of $\mathcal{H}_L$ within the variational class Ω_L are periodic striped configurations of optimal width h_*.

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The optimal length $h_* \equiv h_*(J)$ is derived minimizing w.r.t. h the *energy per site* of periodic striped configurations $\boldsymbol{\sigma}_s(h)$ of generic half-period $h \geqslant 1$:

$$\mathcal{E}_s(h) \equiv \mathcal{E}(\boldsymbol{\sigma}_s(h)) := \frac{1}{(nL)^2}\, \mathcal{H}_{nL}\big(\boldsymbol{\sigma}_s(h)\big) \qquad \big(\text{independent of } n \in \mathbb{N}\big)\,.$$

Caveat: for a.e. $J > 0$ the minimizer $h_* \equiv h_*(J)$ of $\mathcal{E}_s(h)$ over $\mathbb{N}$ is unique; for exceptional values of J, two contiguous minimizers $h_*(J)$ and $h_*(J)+1$.

[Giuliani-Lebowitz-Lieb 2006]

Key ideas

- Analytic expression for the energy per site $\mathcal{E}_s(h)$ of periodic striped states, identifying the optimal stripes width.

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- Using *Block Reflection Positivity* to prove that minimizers of $\mathcal{H}_L$ within Ω_L are periodic checkerboard states $\boldsymbol{\sigma}_c(h_1, h_2)$, consisting of $h_1 \times h_2$ tiles with alternating spin signs.

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  $\Rightarrow$ *Restrictions on the variational class.*

- Proving that $\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1, h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_s(h_*)\big)$, by exhibiting different "*spin flips*" which strictly decrease the energy per site.

3. Sketch of the proof

Energy per site of periodic striped states

Lemma. For periodic striped states $\boldsymbol{\sigma}_s(h)$, the energy per site fulfills

$$\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) = \frac{2}{h}\left[J - 2\log h - \alpha_s + \mathcal{O}\big(h^{-1}\big)\right], \qquad \text{for } h \to +\infty\,,$$

$$\alpha_s := 2\left(1 + \gamma_{\mathsf{EM}} - \log(\pi/2) + 4\pi \textstyle\sum_{\ell,n=1}^{\infty} \ell\, K_1(2\pi\ell n)\right) = 2.276\ldots$$

$$\big(\,\gamma_{\mathsf{EM}} = 0.577\ldots = \text{Euler-Mascheroni const.}, \quad K_1 = \text{mod. Bessel funct.}\big)\,.$$

- Stripes of minimum energy have optimal width $(J \to +\infty)$

$$h_*(J) = c_*\, e^{J/2}\big(1 + \mathcal{O}(e^{-J/4})\big)\,, \qquad c_* := e^{1-\frac{\alpha_s}{2}} = 0.871\ldots\,.$$

- Derived by explicit computations, Poisson summation, Riemann sum approx.
- Compatible with [MacIsaac *et al.* 1995] $+$ analytic expression for α_s.

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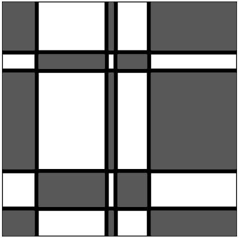
Use RP in the form of chessboard estimate with open boundary conditions [Giuliani-Lebowitz-Lieb 2007], once in the horizontal and once in the vertical direction, to infer that

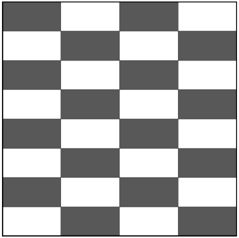
$$\mathcal{H}_L(\boldsymbol{\sigma}) \geqslant \sum_T |T| \; \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1(T), h_2(T))\big) \,,$$

(sum over tiles T forming the state $\boldsymbol{\sigma} \in \Omega_L$)



$\boldsymbol{\sigma} \in \Omega_L$

$\rightarrow$



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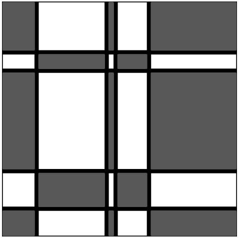

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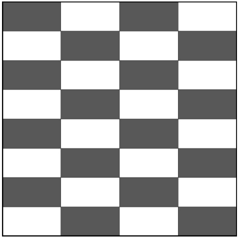

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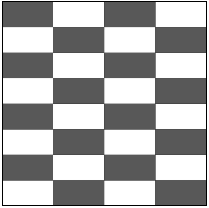
$\Rightarrow$ Sufficient to prove $\mathcal{E}(\boldsymbol{\sigma}_c(h_1, h_2)) > \mathcal{E}(\boldsymbol{\sigma}_s(h_*))$ for $h_1 \geqslant h_2$, $(h_1, h_2) \neq (\infty, h_*)$.

◎ To apply BRP it is crucial that Peierls' boundaries are only straight lines.

I - Excluding thin tiles

For h_2 too small, energy is lowered by removing 2 adjacent horizontal walls.



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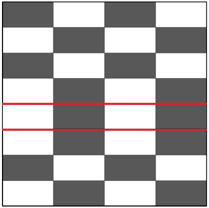
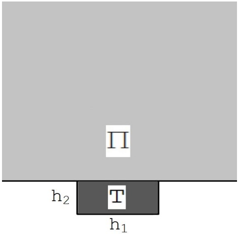


$\boldsymbol{\sigma}_I$

> 4

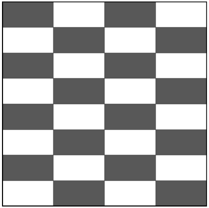


$$\mathcal{E}\Big(\boldsymbol{\sigma}_c(h_1,h_2)\Big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_I) > \frac{4}{L}\left(J - \frac{1}{h_1}\sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in\Pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right)$$

$$> \frac{4}{L}\Big(J - 2\log h_2 - \alpha_s - 1 - 2\log(\pi/2)\Big) \overset{!}{>} 0\,.$$

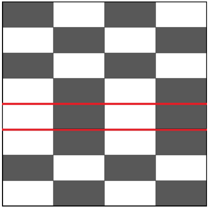
I - Excluding thin tiles

For h_2 too small, energy is lowered by removing 2 adjacent horizontal walls.



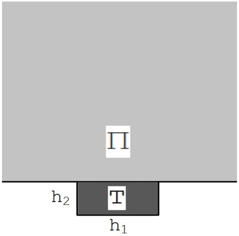
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_I$

> 4



$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_I) > \frac{4}{L}\Bigg(J - \frac{1}{h_1}\sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in\Pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\Bigg)$$
$$> \frac{4}{L}\Big(J - 2\log h_2 - \alpha_s - 1 - 2\log(\pi/2)\Big) \overset{!}{>} 0\,.$$

Using again BRP and Riemann sum approx., it follows

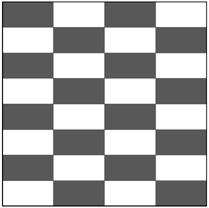
Lemma I. For all $J>0$,

$$h_1 \geqslant h_2\,, \quad 1 \leqslant h_2 \leqslant c_I e^{J/2}, \qquad c_I := \tfrac{2}{\pi}\,e^{-(\alpha_s+1)/2} = 0.123\ldots \quad \Rightarrow \quad \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,3h_2)\big).$$

(recall $h_* \sim c_* e^{J/2}$, $c_* = 0.871\ldots$)

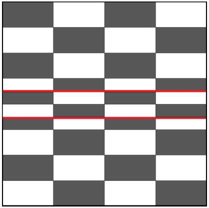
II - Excluding thick tiles

For h_2 too large, energy is lowered by adding 2 adjacent horizontal walls.



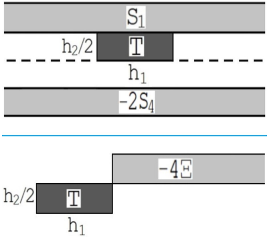
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_{II}$

$>$

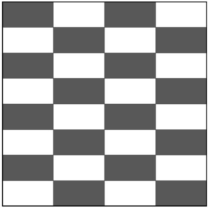


$$\mathcal{E}\Big(\boldsymbol{\sigma}_c(h_1, h_2)\Big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_{II}) > \frac{4}{L}\left[-J + \frac{1}{h_1}\sum_{\boldsymbol{x}\in T}\left(\sum_{\boldsymbol{y}\in S_1} - 2\sum_{\boldsymbol{y}\in S_4} - 4\sum_{\boldsymbol{y}\in \Xi}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right]$$

$$> \frac{4}{L}\left[-J + 2\log h_2 + \alpha_s - 2\log\left(\frac{128}{9\pi}\right) - \frac{8h_2}{h_1} + \mathcal{O}\left(h_2^{-1}\right)\right] \overset{!}{>} 0\,.$$

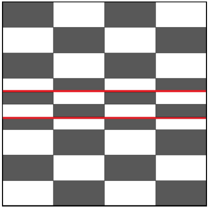
II - Excluding thick tiles

For h_2 too large, energy is lowered by adding 2 adjacent horizontal walls.



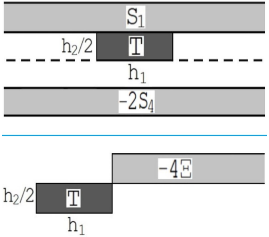
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_{II}$

$>$



$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_{II}) > \frac{4}{L}\left[-J + \frac{1}{h_1}\sum_{\boldsymbol{x}\in T}\Bigg(\sum_{\boldsymbol{y}\in S_1} - 2\sum_{\boldsymbol{y}\in S_4} - 4\sum_{\boldsymbol{y}\in \Xi}\Bigg)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right]$$

$$> \frac{4}{L}\left[-J + 2\log h_2 + \alpha_s - 2\log\left(\frac{128}{9\pi}\right) - \frac{8h_2}{h_1} + \mathcal{O}\big(h_2^{-1}\big)\right] \overset{!}{>} 0\,.$$

Using again BRP and Riemann sum approx., it follows

Lemma II. There exists $J_{II} \gg 1$ s.t. for any $0 < \delta \leqslant 1$, $J > J_{II}$,

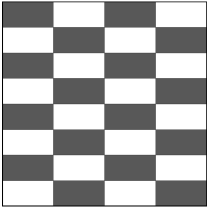
$$c_{II}(\delta)e^{J/2} \leqslant h_2 \leqslant \delta\, h_1\,,$$

$$c_{II}(\delta) := \frac{129}{9\pi}e^{-(\alpha_s/2)+4\delta} = (1.461\ldots)\,e^{4\delta} \quad \Rightarrow \quad \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2/2)\big).$$

(recall $h_* \sim c_* e^{J/2}$, $c_* = 0.871\ldots$)

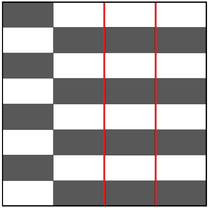
III - Excluding long tiles of almost-optimal width

If $h_2 \sim h_*$ and $h_1 \gg 1$, energy is lowered by removing 2 adjacent vertical walls.



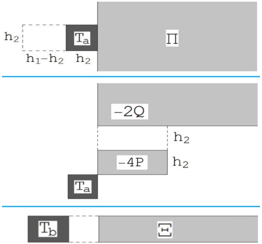
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_{III}$

$>$

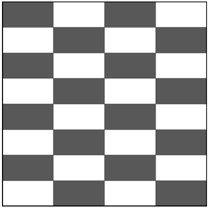


$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_{III})$$

$$> \frac{4}{L}\left[J - \frac{1}{h_2}\left(\sum_{\boldsymbol{x}\in T_a}\sum_{\boldsymbol{y}\in\Pi} + \sum_{\boldsymbol{x}\in T_b}\sum_{\boldsymbol{y}\in\Xi} - 4\sum_{\boldsymbol{x}\in T_a}\sum_{\boldsymbol{y}\in P} - 2\sum_{\boldsymbol{x}\in T_a}\sum_{\boldsymbol{y}\in Q}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right]$$

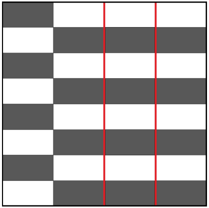
III - Excluding long tiles of almost-optimal width

If $h_2 \sim h_*$ and $h_1 \gg 1$, energy is lowered by removing 2 adjacent vertical walls.



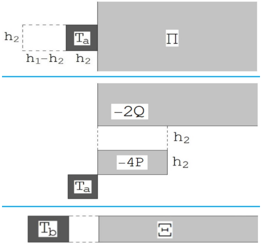
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_{III}$

$>$

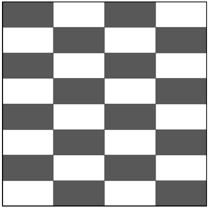


$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1, h_2)\big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_{III})$$

$$> \frac{4}{L}\left[J - 2\log h_2 - \alpha_s - 2\log(\pi/2) + 4 - \frac{1}{2}\,\frac{h_2}{h_1} - 2\left(\frac{h_2}{h_1}\right)^2\right] \overset{!}{>} 0\,.$$

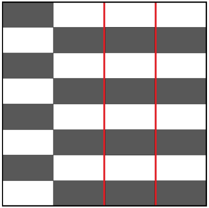
III - Excluding long tiles of almost-optimal width

If $h_2 \sim h_*$ and $h_1 \gg 1$, energy is lowered by removing 2 adjacent vertical walls.



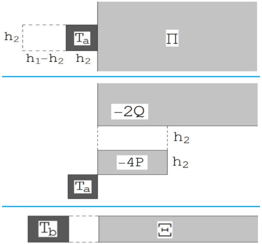
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_{III}$

$>$



$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1, h_2)\big) - \frac{1}{L^2}\,\mathcal{H}_L(\boldsymbol{\sigma}_{III})$$

$$> \frac{4}{L}\left[J - 2\log h_2 - \alpha_s - 2\log(\pi/2) + 4 - \frac{1}{2}\,\frac{h_2}{h_1} - 2\left(\frac{h_2}{h_1}\right)^2\right] \overset{!}{>} 0\,.$$

Using again BRP and Riemann sum approx., it follows

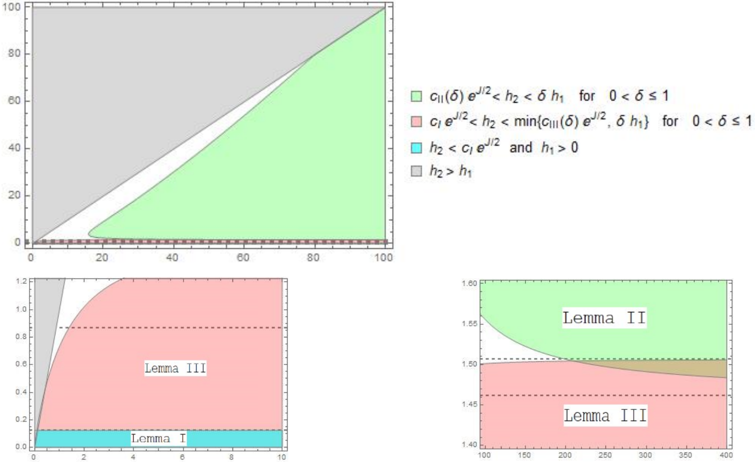
Lemma III. There exists $J_{III} \gg 1$ s.t. for any $0 < \delta \leqslant 1$, $J > J_{III}$,

$$c_I e^{J/2} \leqslant h_2 \leqslant \min\left\{\delta\, h_1,\ c_{III}(\delta) e^{J/2}\right\},$$

$$c_{III}(\delta) := \tfrac{2}{\pi} e^{2-(\alpha_s/2)-\delta/4-\delta^2} = (1.507\ldots)e^{-\delta/4-\delta^2} \ \Rightarrow\ \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1, h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_c(3h_1, h_2)\big).$$

(recall $h_* \sim c_* e^{J/2}$, $c_* = 0.871\ldots$)

Remaining GS candidates



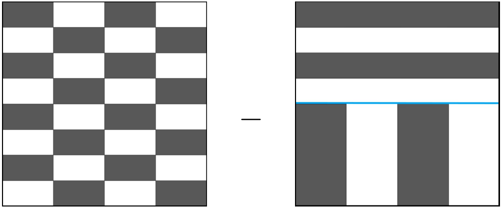
For $J \gg 1$, $\boldsymbol{\sigma}_c(h_1, h_2)$ is **not** a minimizer if (h_1, h_2) belongs to a colored region.

The only GS candidates remaining are $(h_1, h_2) = (+\infty, h)$ [*stripes*] and

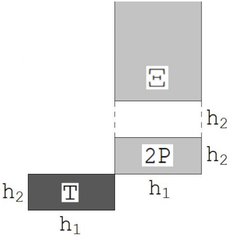
$$(h_1, h_2) = \left(\frac{h}{\lambda}, \frac{h}{1-\lambda}\right), \quad \text{for} \quad h \in [c_{\min}, c_{\max}]\, e^{J/2},\ \lambda \in [\lambda_{\min}, 1/2]\,.$$

Excluding tiles of bounded aspect ratio

For h_1/h_2 finite, compare checkerboard $\boldsymbol{\sigma}_c(h_1, h_2)$ with stripes $\boldsymbol{\sigma}_s(h_1), \boldsymbol{\sigma}_s(h_2)$.



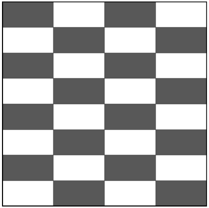
$\boldsymbol{\sigma}_c(h_1, h_2)$ $-$ $\boldsymbol{\sigma}_s(h_1) + \boldsymbol{\sigma}_s(h_2)$ > 4



$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1, h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) > \frac{4}{h_1 h_2}\left(2\sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in P} + \sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in \Xi}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,.$$

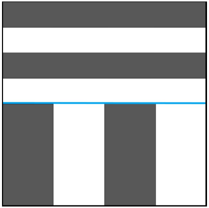
Excluding tiles of bounded aspect ratio

For h_1/h_2 finite, compare checkerboard $\boldsymbol{\sigma}_c(h_1, h_2)$ with stripes $\boldsymbol{\sigma}_s(h_1), \boldsymbol{\sigma}_s(h_2)$.



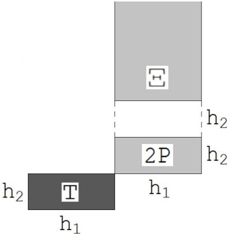
$\boldsymbol{\sigma}_c(h_1, h_2)$

$-$



$\boldsymbol{\sigma}_s(h_1) + \boldsymbol{\sigma}_s(h_2)$

> 4



$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) > \frac{4}{h_1 h_2}\left(2\sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in P} + \sum_{\boldsymbol{x}\in T}\sum_{\boldsymbol{y}\in \Xi}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,.$$

Fixing $h_1 = h/\lambda$, $h_2 = h/(1-\lambda)$, by Riemann sum approx. (with $h \sim c\, e^{J/2}$, $J \gg 1$) and by explicit estimate w.r.t. $\lambda \in [\lambda_{min}, 1/2]$ it follows

$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big)$$
$$> \frac{4\lambda(1-\lambda)}{h}\left[\frac{2\lambda}{3} - \log\lambda + 2 - \log\left(\frac{27}{16}\right) + \mathcal{O}\big(h^{-1}\log h\big)\right].$$

On the other side, using the explicit asymptotic expansion for $\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big)$, for $h_1 = h/\lambda$, $h_2 = h/(1-\lambda)$ and $h \sim c\, e^{J/2} \to +\infty$ we get

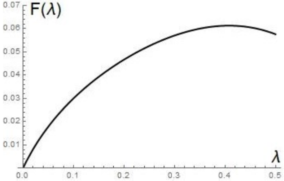
$$\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) \\ = -\frac{4}{h}\Big[\lambda \log\lambda + (1-\lambda)\log(1-\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big].$$

On the other side, using the explicit asymptotic expansion for $\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big)$, for $h_1 = h/\lambda$, $h_2 = h/(1-\lambda)$ and $h \sim c\, e^{J/2} \to +\infty$ we get

$$\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) = -\frac{4}{h}\Big[\lambda \log\lambda + (1-\lambda)\log(1-\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big].$$

Summing up

$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) > \frac{4\lambda(1-\lambda)}{h}\Big[F(\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big] > 0\,.$$



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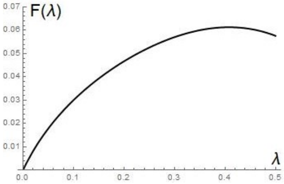
$$\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) = -\frac{4}{h}\Big[\lambda \log\lambda + (1-\lambda)\log(1-\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big].$$

Summing up

$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) > \frac{4\lambda(1-\lambda)}{h}\Big[F(\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big] > 0\,.$$



Lemma. There exists $J_{\min} \gg 1$ s.t. for $J > J_{\min}$, $h_1 = h/\lambda$ and $h_2 = h/(1-\lambda)$,

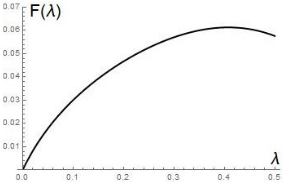
$$\begin{array}{c} c_{\min}\, e^{J/2} \leqslant h \leqslant c_{\max}\, e^{J/2}, \\ \lambda_{\min} \leqslant \lambda \leqslant 1/2 \end{array} \quad \Rightarrow \quad \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big)\,.$$

On the other side, using the explicit asymptotic expansion for $\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big)$, for $h_1 = h/\lambda$, $h_2 = h/(1-\lambda)$ and $h \sim c\, e^{J/2} \to +\infty$ we get

$$\mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_1)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h_2)\big) = -\frac{4}{h}\Big[\lambda \log\lambda + (1-\lambda)\log(1-\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big].$$

Summing up

$$\mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) - \mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big) > \frac{4\lambda(1-\lambda)}{h}\Big[F(\lambda) + \mathcal{O}\big(h^{-1}\log h\big)\Big] > 0\,.$$



Lemma. There exists $J_{\min} \gg 1$ s.t. for $J > J_{\min}$, $h_1 = h/\lambda$ and $h_2 = h/(1-\lambda)$,

$$\begin{matrix} c_{\min}\, e^{J/2} \leqslant h \leqslant c_{\max}\, e^{J/2}, \\ \lambda_{\min} \leqslant \lambda \leqslant 1/2 \end{matrix} \quad \Rightarrow \quad \mathcal{E}\big(\boldsymbol{\sigma}_c(h_1,h_2)\big) > \mathcal{E}\big(\boldsymbol{\sigma}_s(h)\big)\,.$$

So, periodic stripes corresp. to $(h_1,h_2) = (+\infty, h)$ are the only minimizers in Ω_L.
Conclude by optimization w.r.t. h. ■

Summary and outlook

- Periodic striped states (of optimal width) are the only energy minimizer in the class of aperiodic configurations with Peierls' boundaries consisting of straight domain walls.

- Proof derived by combination of chessboard estimates with quantitative *a priori* estimates, relying on suitable "spin flips".

Summary and outlook

- Periodic striped states (of optimal width) are the only energy minimizer in the class of aperiodic configurations with Peierls' boundaries consisting of straight domain walls.

- Proof derived by combination of chessboard estimates with quantitative *a priori* estimates, relying on suitable "spin flips".

- Future developments:
  - larger variational class (non-straight boundaries) for $d=2$, $p=3$;
  - different values of d, p (especially $d<p\leqslant 2d$);
  - continuum limit.

Thanks a lot for your attention!