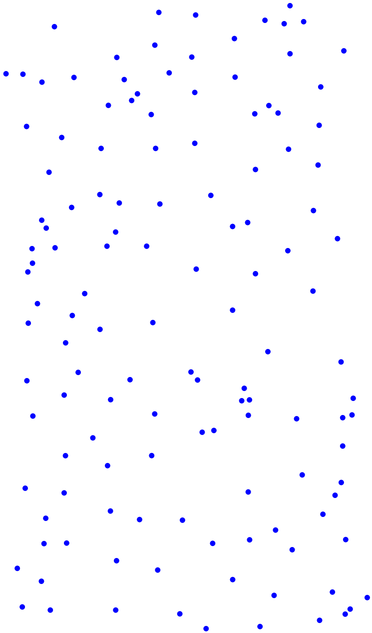
Surface law and charge rigidity for the Coulomb gas on $\mathbb{Z}^d$

Christophe Garban

Université Lyon 1 and Université de Genève

Joint works with *Avelio Sepúlveda* (Universidad de Chile)
and with *David Dereudre* (Université de Lille)

Motivation: Gibbs point processes in $\mathbb{R}^d$



1) Poisson Point Process

2) β-ensemble in $\mathbb{R}^2$

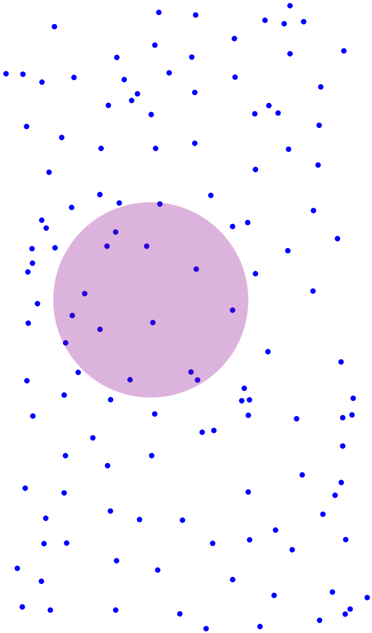
$$\mathbb{P}(\{z_i\}_i) \text{“} \propto \text{”} \prod_{i \neq j} |z_i - z_j|^\beta$$

(N.B. $\beta = 2 \ \equiv$ Ginibre ensemble)

3) Coulomb gas on $\mathbb{R}^3$

$$\mathbb{P}(\{z_i\}_i) \text{“} \propto \text{”} \exp[-\beta \sum_{i \neq j} \frac{1}{\|i-j\|}]$$

Motivation: Gibbs point processes in $\mathbb{R}^d$



A) Hyperuniformity

$\mathrm{Var}[\#\text{points in} A] = o(|A|)$

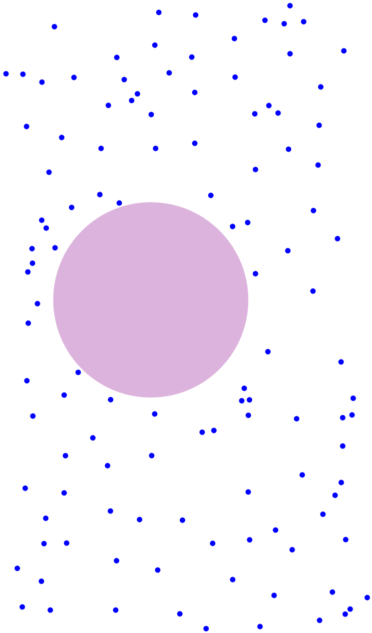
If furthermore
$\mathrm{Var}[\#\text{points}] \asymp |\partial A|$:
"surface law"

B) fluctuations

(CLT / Large deviations)

C) Number rigidity

Motivation: Gibbs point processes in $\mathbb{R}^d$



A) Hyperuniformity

$$\mathrm{Var}[\#\text{points in} A] = o(|A|)$$

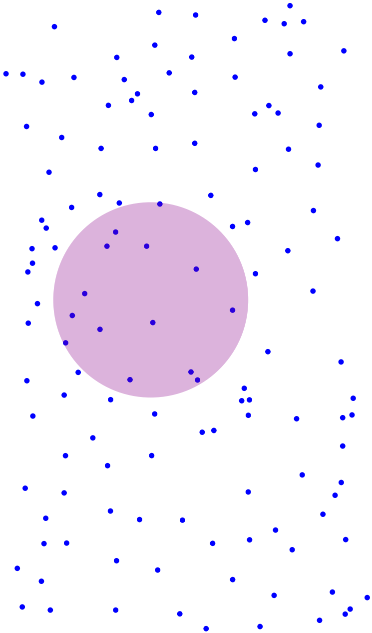
If furthermore
$\mathrm{Var}[\#\text{points}] \asymp |\partial A|$:
"surface law"

B) fluctuations

(CLT / Large deviations)

C) Number rigidity

Motivation: Gibbs point processes in $\mathbb{R}^d$



1) Poisson Point Process

$\mathrm{Var}[\#\text{points}] = \lambda|A|$ Not HY
Not Rigid

2) β-ensemble in $\mathbb{R}^2$

$\beta = 2$ **(Ginibre)** Surface law and rigid

Gosh - Peres 2017

$d = 1, \beta > 0$: Hyperuniform and rigid

Chhaibi - Najnudel 2018
Dereudre-Hardy-Leblé-Maida 2018

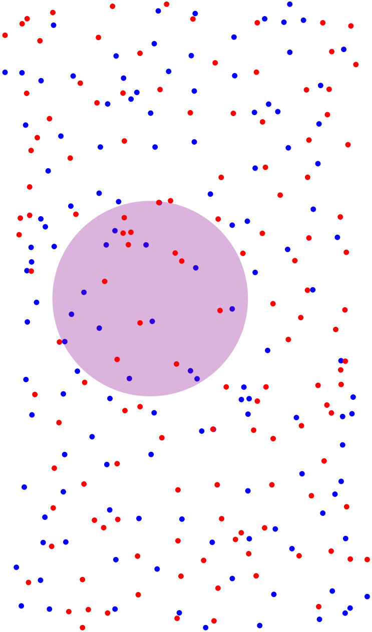
3) Coulomb gas on $\mathbb{R}^3$

$\mathrm{Var}[\#\text{points in } B_R] = O(R^4)$

Serfaty-Armstrong 2021

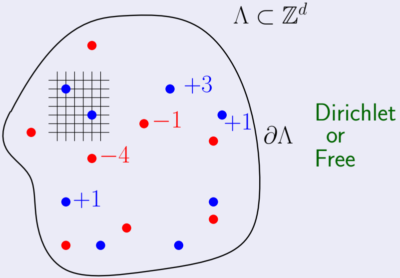
$\mathrm{Var}^{\textbf{Hierarch}}[\#\text{points in } B_R]$
$= O(R^2 \log R)$ **Chatterjee 2019**

Motivation: Gibbs point processes in $\mathbb{R}^d$



Coulomb gas on $\mathbb{Z}^d$

Definition.



$\Lambda \subset \mathbb{Z}^d$
$q \in \mathbb{Z}^\Lambda$

$$\mathbb{P}_\beta[q] \propto \exp\left(-\frac{\beta}{2}\langle q, (-\Delta)^{-1} q\rangle\right)$$
$$\propto \exp\left(-\frac{\beta}{2}\sum_{i,j} q_i q_j G_\Lambda(i,j)\right)$$

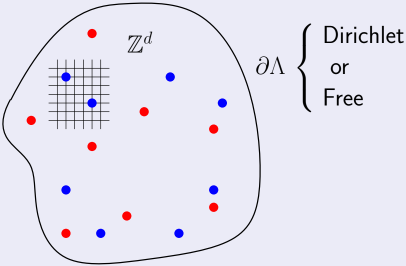
Claim

Dirichlet and *free* boundary conditions lead to (translation invariant) infinite volume limits. And, at least when $d = 2$, they coincide!

$$\mathrm{Var}_\beta\Big[\sum_{x \in B_R} q_x\Big]\,?$$

Hard core Coulomb gas on $\mathbb{Z}^d$

Definition.



$\Lambda \subset \mathbb{Z}^d$

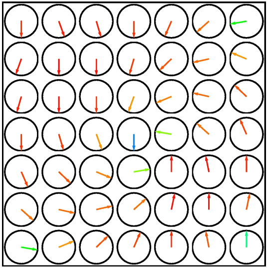
$z > 0$ (the *activity*)

$q \in \{-1, 0, 1\}^\Lambda$

$$\mathbb{P}_\beta[q] \propto z^{\|q\|_1} \exp(-\tfrac{\beta}{2}\langle q, (-\Delta)^{-1} q\rangle)$$

$$\mathrm{Var}_{\beta,z}[\sum_{x \in B_R} q_x]\,?$$

Villain model

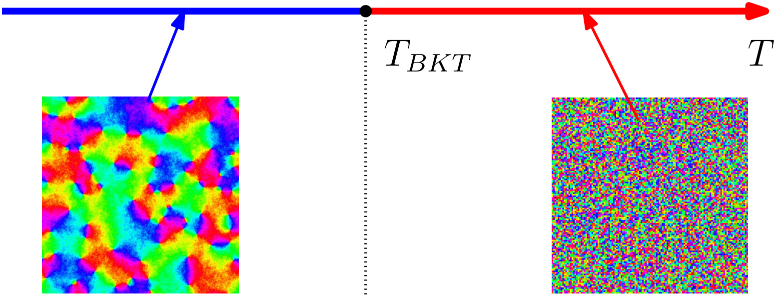


$$\mathbb{P}_\beta\big[(\theta_x)\big] \propto \prod_{i\sim j} \sum_m \exp\Big(-\frac{\beta}{2}(\theta_i - \theta_j + 2\pi m)^2\Big)$$

Link Villain $\leftrightarrow$ Coulomb:

$$\boxed{Z_\beta^{\text{Villain}} = Z_\beta^{\text{GFF}} \times Z_\beta^{\text{Coulomb}}}$$

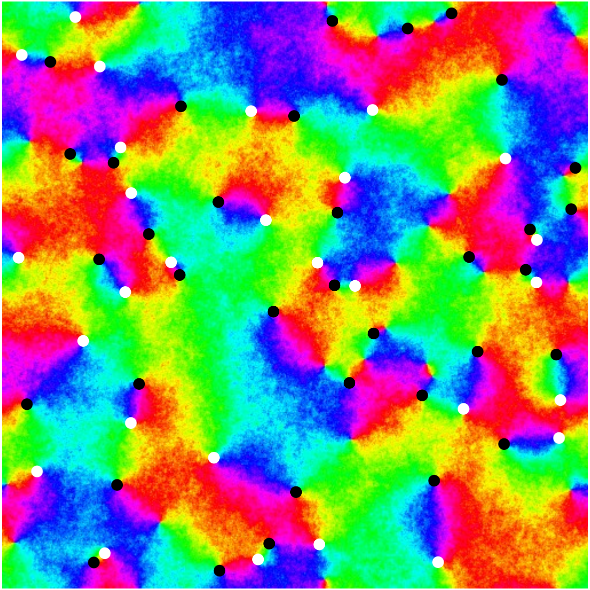
BKT transition



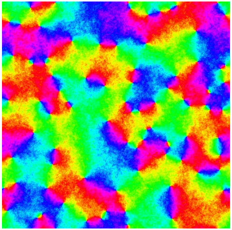
$\langle \sigma_x \sigma_y \rangle_\beta \approx \frac{1}{|x-y|^{\alpha(T)}}$

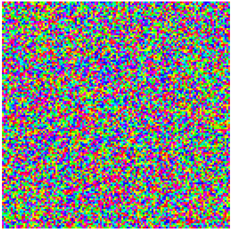
$\langle \sigma_x \sigma_y \rangle_\beta \approx e^{-c(T)\,|x-y|}$

Vortices of Villain



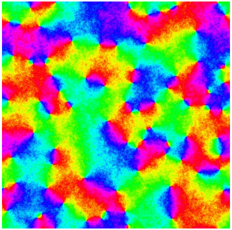
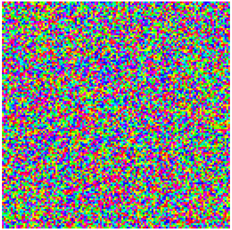
Vortex fluctuations





1. **Kosterlitz-Thouless** predicted in 1978 a transition from *surface law* to *area law* at T_{BKT}.

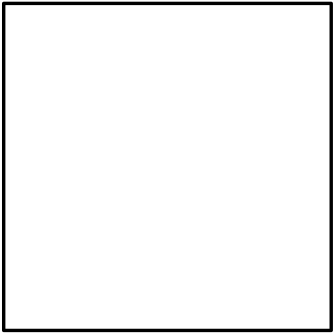
Vortex fluctuations

1. **Kosterlitz-Thouless** predicted in 1978 a transition from *surface law* to *area law* at T_{BKT}.
2. Soon after, using *sum-rules* heuristics, **Beijeren-Felderhof, Gruber-Lugrin-Martin, Martin-Yalcin, Dhar, etc. (early 80's)** predicted the opposite.
3. The *sum-rules* heuristics is the basis of predictions/rigorous results for many other point processes starting with **Lebowitz , Martin , Jancovici-Lebowitz-Manificat, Gosh-Lebowitz, ... etc.**

sum-rules heuristics

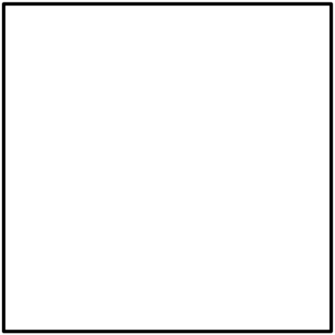
∂B_R



$$\mathrm{Var}_\beta[\textstyle\sum_{x \in B_R} q_x] = \sum_{x,y \in B_R} \langle q_x q_y \rangle_\beta$$

sum-rules heuristics

∂B_R

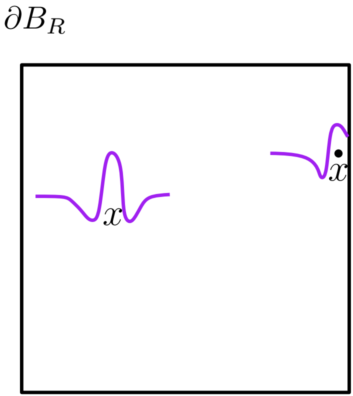


$\mathrm{Var}_\beta[\sum_{x \in B_R} q_x] = \sum_{x,y \in B_R} \langle q_x q_y \rangle_\beta$

Neutrality of the Coulomb gas

$$\approx\Rightarrow \quad \sum_{x \in B_R} \langle q_0 q_x \rangle_\beta = 0$$

sum-rules heuristics

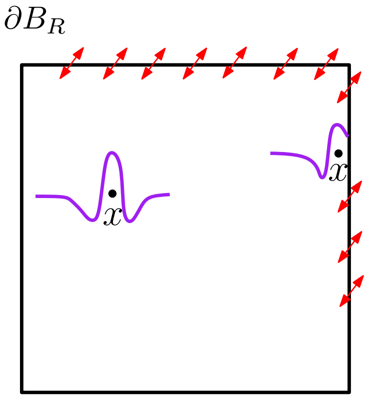


$$\mathrm{Var}_\beta[\sum_{x\in B_R} q_x] = \sum_{x,y\in B_R} \langle q_x q_y\rangle_\beta$$

Neutrality of the Coulomb gas

$$\approx\Rightarrow \quad \sum_{x\in B_R}\langle q_0 q_x\rangle_\beta = 0$$

sum-rules heuristics



$$\mathrm{Var}[\textstyle\sum_{x\in B_R} q_x] = \sum_{x,y\in B_R}\langle q_x q_y\rangle$$

Neutrality of the Coulomb gas

$$\approx\Rightarrow \quad \sum_{x\in B_R}\langle q_0 q_x\rangle = 0$$

Proof of *surface law* at **high temperature**

When β is small, one can turn the sum-rules heuristics into a rigorous proof.

1. **Proof 1.** Using *Debye screening* at small activity z, Brydges 1978, Brydges-Federbush 1980 for $d \geq 3$ and Yang 1987 ($d = 2$) imply

$$\boxed{\langle q_x q_y \rangle_{z,\beta} \leq e^{-c|x-y|}} \overset{\text{sum-rules}}{\approx \Rightarrow} \text{ surface law}$$

2. **Proof 2.** In $d \geq 2$, for the Villain-gas (not hard-core), a direct proof using a **local sampling algorithm** for Coulomb gas G., Sepúlveda 2020.

3. **Proof 3.** In $d = 2$ using the *cable graph XY*.

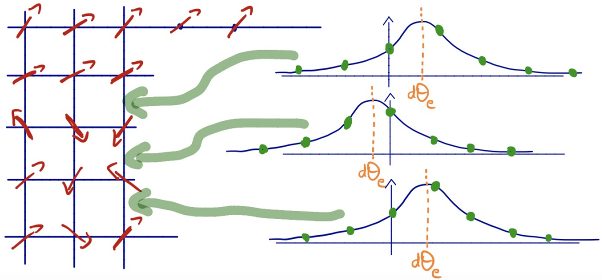
Local algorithm to sample the Coulomb gas in $d = 2$

$(\theta_x, m_e) \longrightarrow$

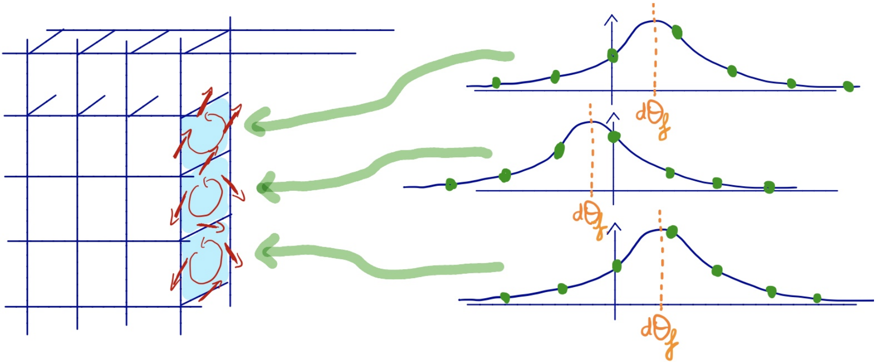
1. $\theta \sim \mathbb{P}^{\text{Villain}}$
2. $(m_e)_{e \in E} \sim \mathcal{L}(m \mid \theta)$
3. $q := dm$

$\longrightarrow q \sim \mathbb{P}_\beta^{\text{COULOMB}}$

$\theta \sim \mathbb{P}_\beta^{\text{Vil}} \longrightarrow (m_e) \sim \mathbb{P}(\cdot \mid \theta)$



Local algorithm to sample the Coulomb gas in $d = 3$

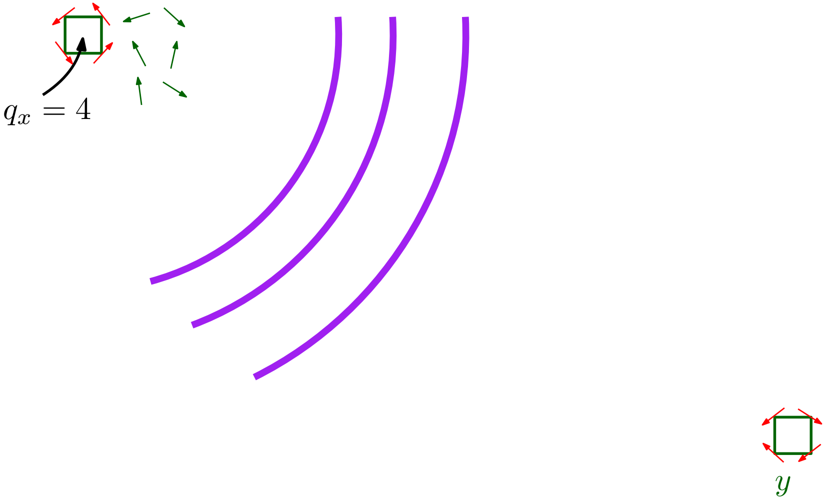


Proof 2: local algorithm implies Debye screening

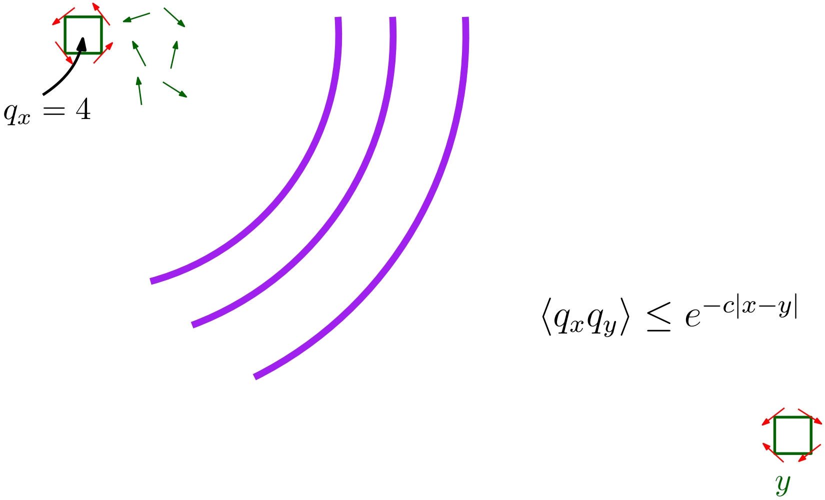
Proof 2: local algorithm implies Debye screening

Proof 2: local algorithm implies Debye screening

Proof 2: local algorithm implies Debye screening

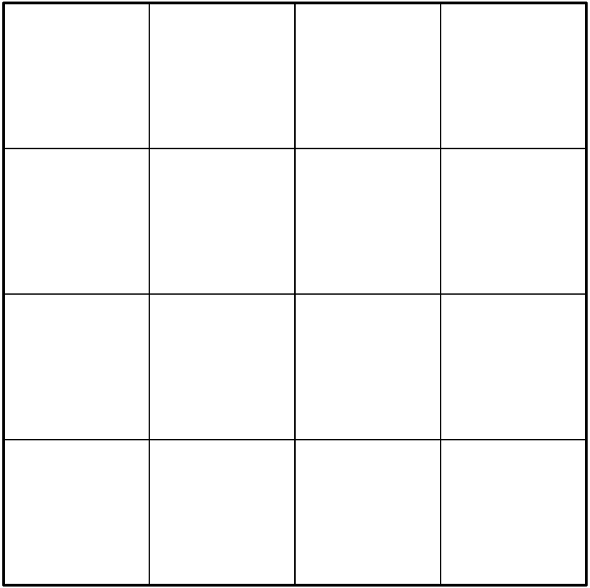


Proof 2: local algorithm implies Debye screening

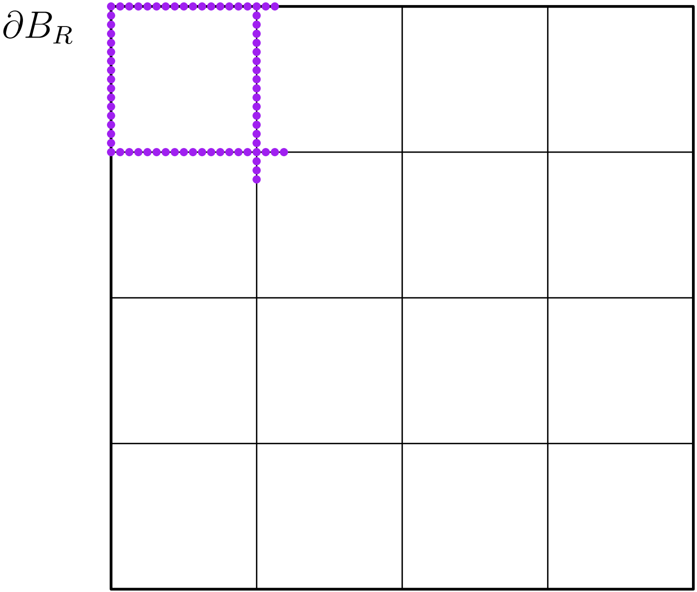


Proof 3. Area law using the cable graph XY (still high T)

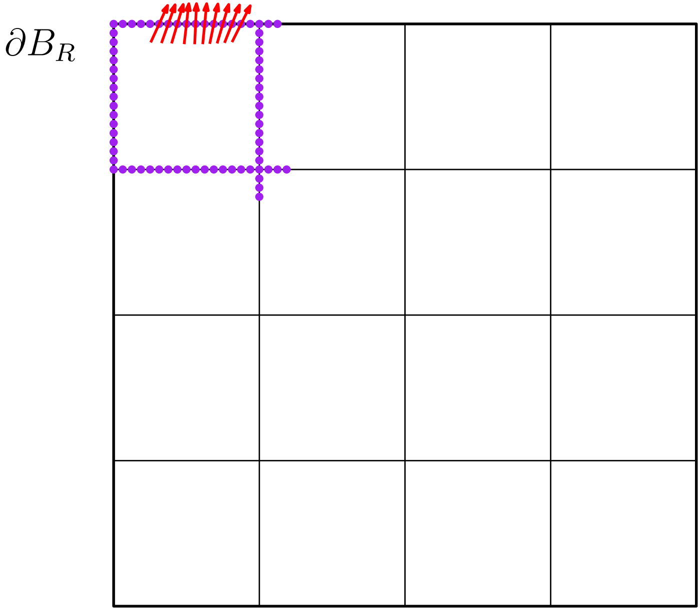
∂B_R



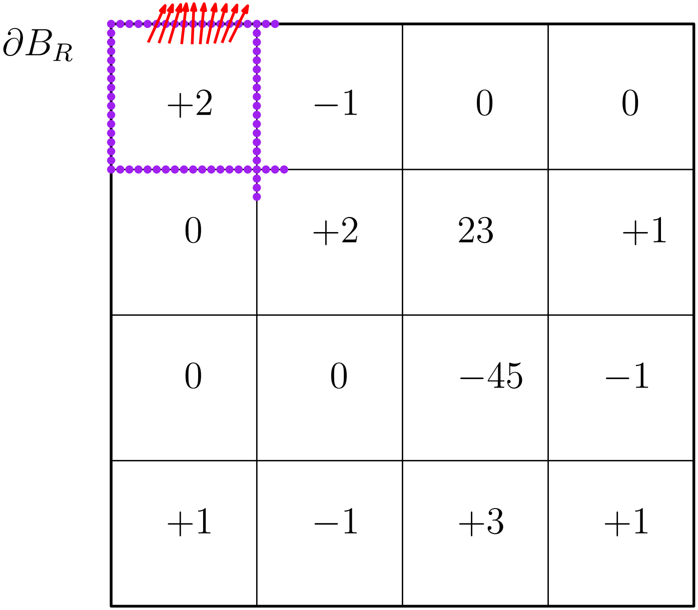
Proof 3. Area law using the cable graph XY (still high T)



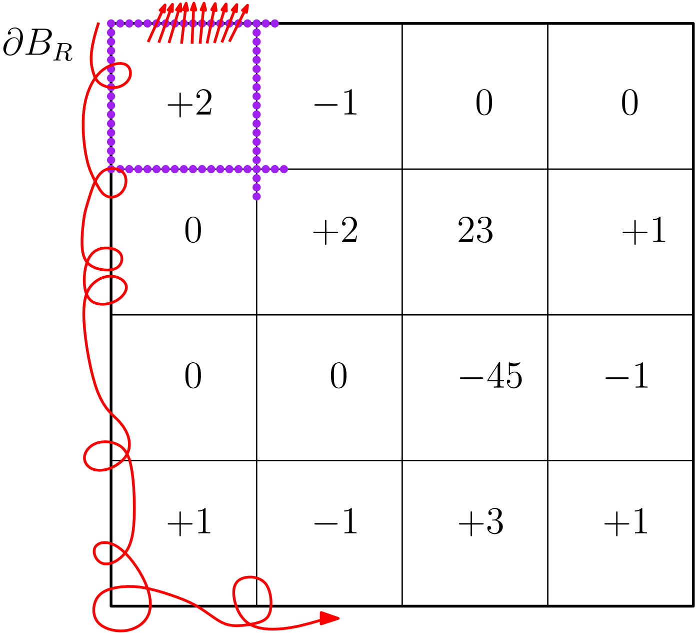
Proof 3. Area law using the cable graph XY (still high T)



Proof 3. Area law using the cable graph XY (still high T)



Proof 3. Area law using the cable graph XY (still high T)



Surface area at any temperature

Theorem 1 G.-Sepúlveda 2023+

For any $d \geq 2$, any $\beta > 0$ (both for the $\mathbb{Z}$-case or hard-core $z < 1$), there exists constants $c, C > 0$,

$$c|\partial A| \;\leq \operatorname{Var}_{(z,)\beta}[\sum_{x \in A} q_x] \;\leq\; C|\partial A|$$

Surface area at any temperature

Theorem 1 G.-Sepúlveda 2023+

For any $d \geq 2$, any $\beta > 0$ (both for the $\mathbb{Z}$-case or hard-core $z < 1$), there exists constants $c, C > 0$,

$$c|\partial A| \; \leq \; \mathrm{Var}_{(z,)\beta}[\sum_{x \in A} q_x] \; \leq \; C|\partial A|$$

Proof relies on the following very useful identity

Lemma

For any $f : \Lambda \to \mathbb{C}$.

$$\mathbb{E}_\beta^{Coul}\left[e^{\langle q,f\rangle}\right] = \mathbb{E}_\beta^{GFF}\left[e^{\langle \Delta\phi,f\rangle}\right]\mathbb{E}_{\frac{1}{\beta}}^{IV-GFF}\left[e^{i\frac{1}{\beta}\langle \Delta\psi,f\rangle}\right] \qquad (\star)$$

Proof: $\approx$ Poisson summation formula / Sine-Gordon transformation type.

Lemma $\Rightarrow$ Theorem 1

$$\mathbb{E}_\beta^{Coul}\left[e^{\alpha\langle q,f\rangle}\right] = \mathbb{E}_\beta^{GFF}\left[e^{\alpha\langle\Delta\phi,f\rangle}\right]\mathbb{E}_{\frac{1}{\beta}}^{IV-GFF}\left[e^{i\frac{\alpha}{\beta}\langle\Delta\psi,f\rangle}\right] \qquad (\star)$$

$$\alpha \searrow 0 \Rightarrow \quad \mathrm{Var}_\beta^{Coul}[\langle q,f\rangle] = \mathrm{Var}_\beta^{GFF}[\langle\Delta\phi,f\rangle] - \mathrm{Var}_\beta^{IV-GFF}[\tfrac{1}{\beta}\langle\Delta\psi,f\rangle]$$

Lemma $\Rightarrow$ Theorem 1

$$\mathbb{E}_{\beta}^{Coul}\left[e^{\alpha\langle q,f\rangle}\right] = \mathbb{E}_{\beta}^{GFF}\left[e^{\alpha\langle \Delta\phi,f\rangle}\right]\mathbb{E}_{\frac{1}{\beta}}^{IV-GFF}\left[e^{i\frac{\alpha}{\beta}\langle\Delta\psi,f\rangle}\right] \qquad (\star)$$

$$\alpha \searrow 0 \Rightarrow \quad \mathrm{Var}_{\beta}^{Coul}[\langle q,f\rangle] = \mathrm{Var}_{\beta}^{GFF}[\langle\Delta\phi,f\rangle] - \mathrm{Var}_{\beta}^{IV-GFF}[\tfrac{1}{\beta}\langle\Delta\psi,f\rangle]$$

In particular applied to $f := 1_A$ gives

$$\begin{aligned}
\mathrm{Var}_{\beta}^{Coul}[\langle q,1_A\rangle] &\leq \mathrm{Var}_{\beta}^{GFF}[\langle\Delta\phi,1_A\rangle] \\
&= \mathrm{Var}_{\beta}^{GFF}[\langle\phi,\Delta 1_A\rangle] \\
&= C(\beta)\langle\Delta 1_A,(-\Delta)^{-1}\Delta 1_A\rangle \\
&= C(\beta)\langle 1_A,(-\Delta)1_A\rangle \\
&\leq c_d C(\beta)|\partial A| \quad \square
\end{aligned}$$

Corollary

$$\mathrm{Var}_{\beta}^{Coul}[\langle q,f\rangle] \leq O(\|\nabla f\|_2^2)$$

1. Gives another direct proof for **Debye screening** (straightforward only for the $\mathbb{Z}$-valued case).
   Exponential decay, control on $Z_{x,y}^{\alpha,-\alpha}/Z$.
2. **Large deviations** for charge fluctuations ? Say, for example in $d \geq 2$,

$$\mathbb{P}\big[|\sum_{x\in B_R} q_x| > R^d\big]\,?$$

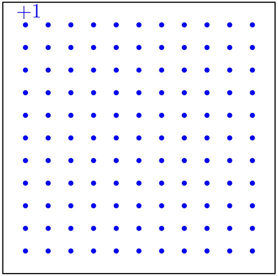
Some other direct applications of the identity ⋆

1. Gives another direct proof for **Debye screening** (straightforward only for the $\mathbb{Z}$-valued case).
   Exponential decay, control on $Z^{\alpha,-\alpha}_{x,y}/Z$.
2. **Large deviations** for charge fluctuations ? Say, for example in $d \geq 2$,

$$\mathbb{P}\Big[|\sum_{x \in B_R} q_x| > R^d\Big] \,?$$

Corollary

$\mathbb{P}\big[|\sum_{x \in B_R} q_x| > R^d\big] \approx e^{-cR^{d+1}}$



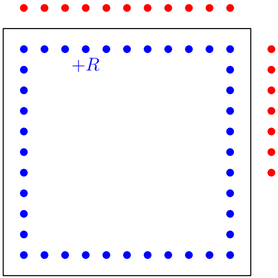
Some other direct applications of the identity ⋆

1. Gives another direct proof for **Debye screening** (straightforward only for the $\mathbb{Z}$-valued case).
   Exponential decay, control on $Z^{\alpha,-\alpha}_{x,y}/Z$.
2. **Large deviations** for charge fluctuations ? Say, for example in $d \geq 2$,

$$\mathbb{P}\Big[|\sum_{x\in B_R} q_x| > R^d\Big]\,?$$

Corollary

$\mathbb{P}\Big[|\sum_{x\in B_R} q_x| > R^d\Big] \approx e^{-cR^{d+1}}$



CLT for the scaling limit of the **electric potential**

Setup. Consider Ω a domain in $\mathbb{R}^d, d \geq 2$.
Approximate into $\Omega_\eta := \eta\mathbb{Z}^d \cap \Omega$.
Look at the Coulomb gas of particles q_η in Ω_η.

CLT for the scaling limit of the **electric potential**

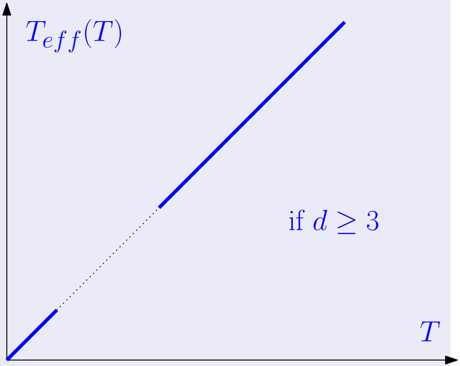
Setup. Consider Ω a domain in $\mathbb{R}^d$, $d \geq 2$.
Approximate into $\Omega_\eta := \eta\mathbb{Z}^d \cap \Omega$.
Look at the Coulomb gas of particles q_η in Ω_η.

Theorem 2 (G.-Sepúlveda 2023+)

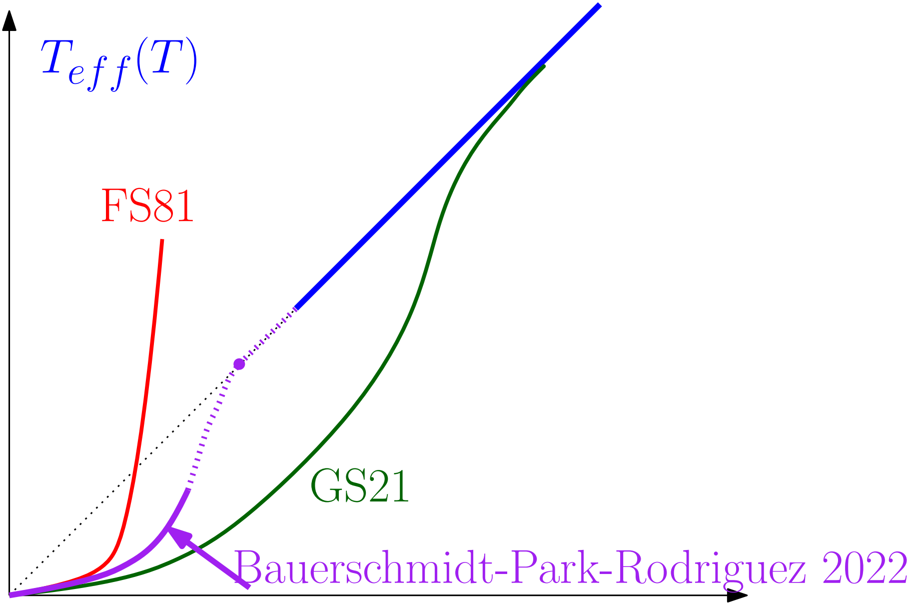
$\forall d \geq 2$, "$\forall T$", $\exists T_{eff}(T)$

$$\eta^{1-\frac{d}{2}} \Delta^{-1} q_\eta \overset{law}{\to} ,$$

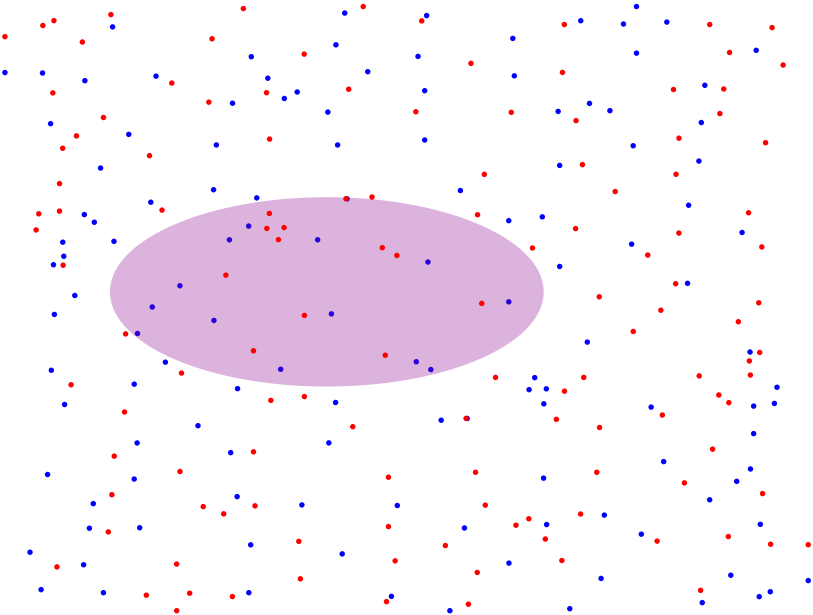
where ϕ is a $T_{eff}(T)$-GFF on Ω



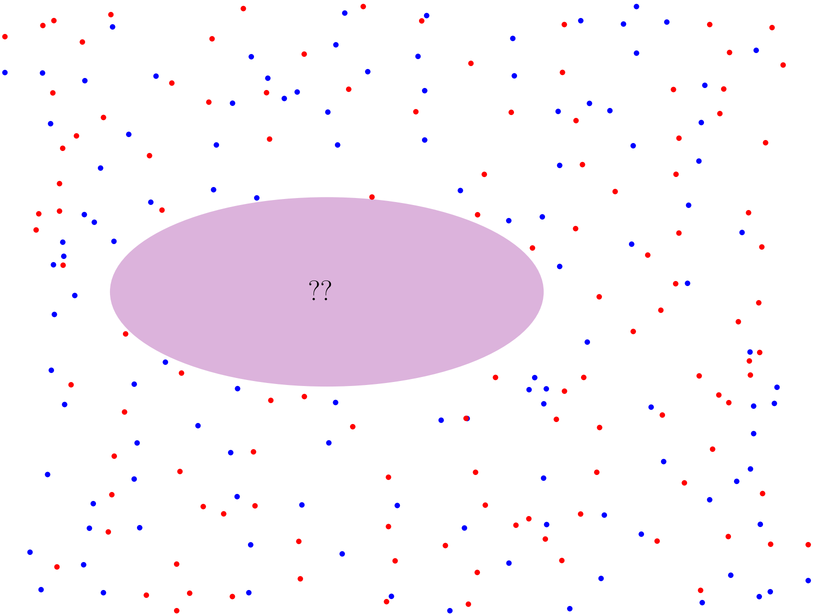
Plot of $T_{eff}(T)$ if $d = 2$



C) And now Rigidity ?



C) And now Rigidity ?



C) And now Rigidity ?

Theorem 3 Dereudre-G. (2023+)

- In $d = 2$.

C) And now Rigidity ?

Theorem 3 Dereudre-G. (2023+)

- In $d = 2$. The Villain-Coulomb gas as well as the hard-core gas ($z < 1$) are **rigid** at any β.

C) And now Rigidity ?

Theorem 3 Dereudre-G. (2023+)

- In $d = 2$. The Villain-Coulomb gas as well as the hard-core gas ($z < 1$) are **rigid** at any β.
- In $d \geq 3$.

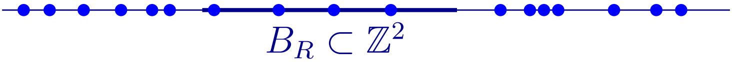
C) And now Rigidity ?

Theorem 3 Dereudre-G. (2023+)

- In $d = 2$. The Villain-Coulomb gas as well as the hard-core gas ($z < 1$) are **rigid** at any β.
- In $d \geq 3$. The Villain-Coulomb gas is **not rigid** at low β.

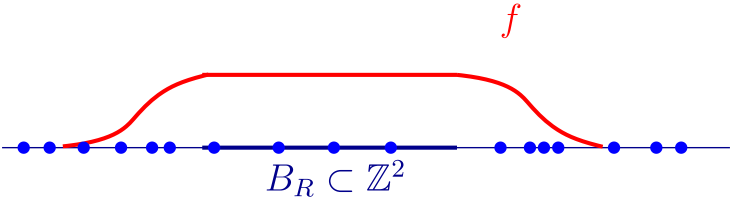
Proof of rigidity ($d = 2$, any T)

Gosh-Peres 2017



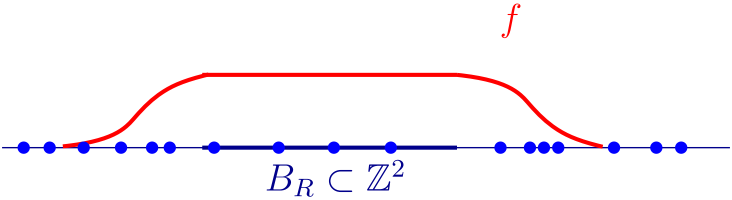
Proof of rigidity ($d = 2$, any T)

Gosh-Peres 2017



$\mathrm{Var}[\langle q, f \rangle] \leq \epsilon$

Gosh-Peres 2017



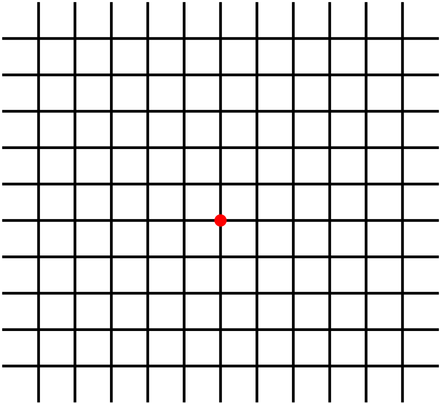
$\mathrm{Var}[\langle q, f\rangle] \leq \epsilon$

Recall $\mathrm{Var}_\beta[\langle q, f\rangle] \leq O(\|\nabla f\|_2^2)$

Proof of non-rigidity ($d \geq 3$, high T)

Searching for a trail of evidence in a maze

Arias-Castro- Candes- Helgason- Zeitouni (2008)

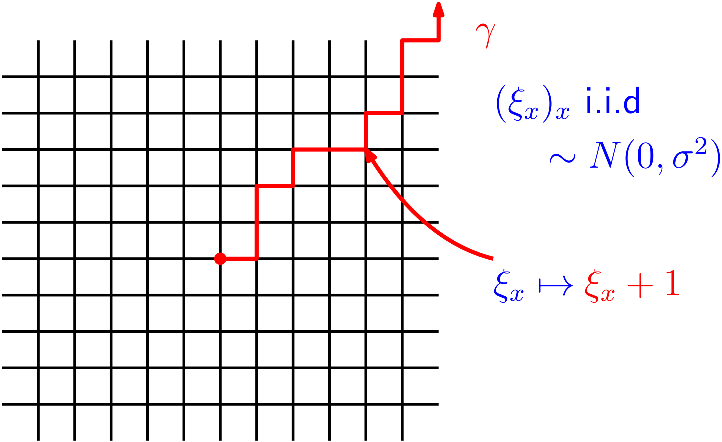


$(\xi_x)_x$ i.i.d

$\sim N(0, \sigma^2)$

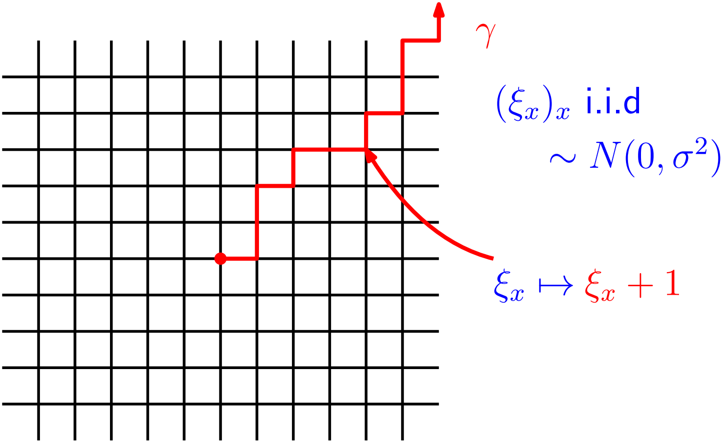
Proof of non-rigidity ($d \geq 3$, high T)

Searching for a trail of evidence in a maze
Arias-Castro- Candes- Helgason- Zeitouni (2008)



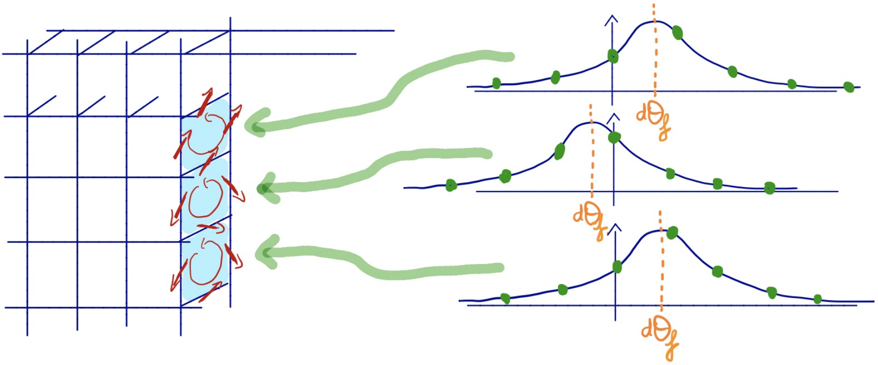
Proof of non-rigidity ($d \geq 3$, high T)

Searching for a trail of evidence in a maze
Arias-Castro- Candes- Helgason- Zeitouni (2008)

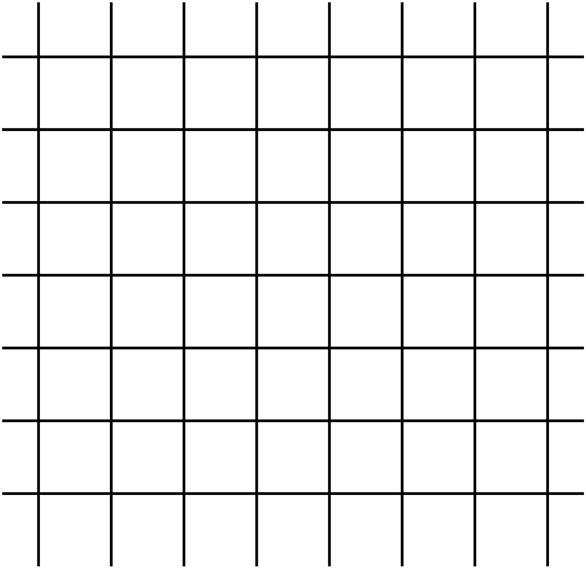


THM. If $d \geq 3$, by choosing $\gamma \sim$ *unpredictable paths* of **Benjamini-Pemantle-Peres 1998**, this shift is **undistinguishable**.

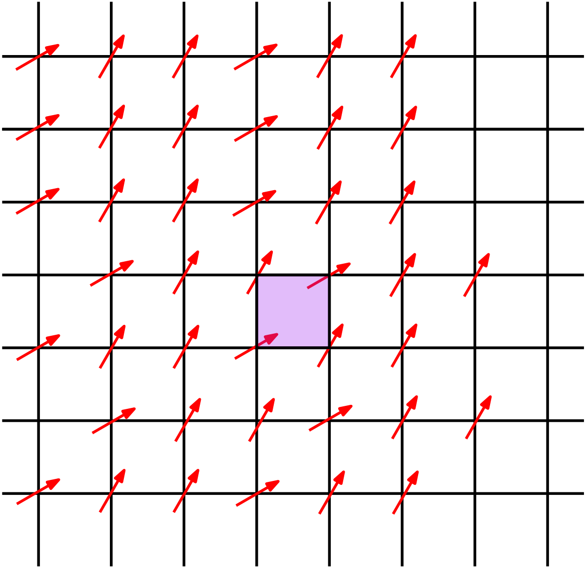
Proof of non-rigidity ($d \geq 3$, high T)



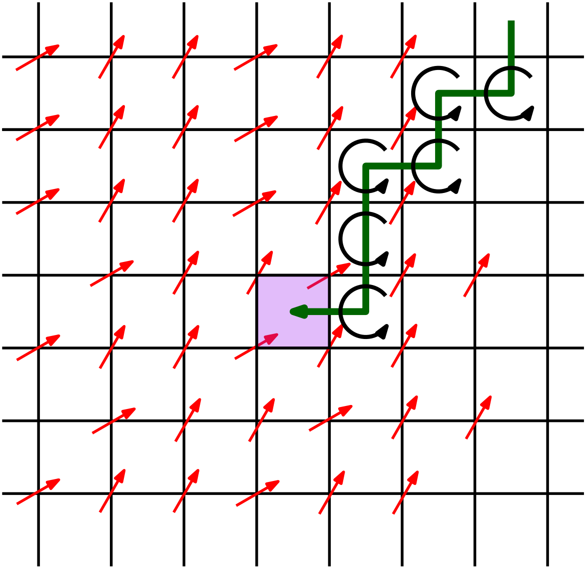
Proof of non-rigidity ($d \geq 3$, high T)



Proof of non-rigidity ($d \geq 3$, high T)



Proof of non-rigidity ($d \geq 3$, high T)



Concluding remarks

- Soft proof for *surface law* at any β and any $d \geq 2$ for the (hard-core or not) Coulomb gas on $\mathbb{Z}^d$.
- Retrospectively, one may wonder what is $\langle q_x q_y \rangle$ at high β in $d = 2$ to still guarantee *surface law* ?

$$\langle q_x q_y \rangle_\beta \approx \langle \Delta\Psi_x \Delta\Psi_y \rangle_{1/\beta} = (?) \frac{1}{|x-y|^4}$$

- **Rigidity** in $d = 3$ at low temperature ?
- Gives a new perspective for the Coulomb gas with *insulating walls* (free boundary conditions). In particular, at least for the Villain gas, it seems to contradict the absence of screening along the boundary discussed in Federbush-Kennedy 1985.