Adiabatic evolution of low temperature many-body systems

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arxiv:2211.16836

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Supported by ERC StG MamBoQ (PI: M. PORTA)

Rome, 8 February 2023

Outline

- Background: adiabatic theorems
- Our setting: interacting Fermi particles on a lattice

  Dynamics with *weak* and *slowly varying* perturbations.
- Main result: convergent expansion for time-evolved local observables, especially at *small temperature*. Implications: zero-temperature adiabatic theorem
- Sketch of the proof: wick rotation, decay of Euclidean correlations
- Conclusion and next steps

The adiabatic theorem

- Basic setting: time-ependent Hamiltonian $H(s)$, $s \in [-1, 0]$ with unique ground state φ_s (energy $E(s)$); spectral gap for all s,

$$\inf_{s \in [0,1]} \operatorname{dist}(E(s), \sigma(H(s)) \setminus \{E(s)\}) = \delta > 0$$

- Let $\eta > 0$ and consider the quantum dynamics, for $t \in [-1/\eta, 0]$,

$$i\,\partial_t \psi(t) = H(\eta t)\,\psi(t), \qquad \psi(-1/\eta) = \varphi_{-1}$$

- [Born&Fock; Kato] Suppose $\|\dot{H}(s)\|$ is finite. Then as $\eta \to 0^+$ the dynamics follows the instantaneous ground state:

$$\|\psi(t) - \langle \psi(t), \varphi_{\eta t} \rangle \varphi_{\eta t}\| \leqslant C\eta \qquad \text{for all } t \in [-1/\eta, 0].$$

- Basic result in quantum dynamics with many applications and extensions.

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- Basic result in quantum dynamics with many applications and extensions.

- For many-body systems the basic version is unsatisfying; for instance in a quantum spin system or lattice Fermion model on $\Lambda_L = \mathbb{Z}_L^d$. Typical Hamiltonian

$$\mathcal{H}(s) = \sum_{X \subset \Lambda_L} \Phi_X(s),$$

with only Φ_X individually bounded. Typically $\|\dot{\mathcal{H}}(s)\| \sim L^d$, and previous bound becomes useless for $\eta L^d \gtrsim 1$.

- Recent results, especially [Bachman, De Roeck & Fraas 2017] for spin systems: norm bounds are too strong, so look at local topology: for $\mathcal{O}_X$ a bounded local operator,

$$|\langle \psi(t), \mathcal{O}_X \psi(t) \rangle - \langle \varphi_{\eta t}, \mathcal{O}_X \varphi_{\eta t} \rangle| \leqslant C \eta,$$

uniformly in L (if the gap is uniform).

- Linear response [BdRF17]; extends to Fermions [Monaco & Teufel 2017],…

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Positive temperature?

- What about $T>0$? Density matrix starting with $\rho_{\beta,L}=\frac{1}{Z_{\beta,L}}e^{-\beta\mathcal{H}(-1)}$ and evolved by Schrödinger equation

$$i\,\partial_t\rho(t)=[\mathcal{H}(\eta t),\rho(t)],\qquad \rho(-1/\eta)=\rho_{\beta,L}\ .$$

- Let $\langle\cdot\rangle_{\eta t}$ be the Gibbs state of $\mathcal{H}(\eta t)$ (instantaneous Gibbs state). (When) is

$$|\mathrm{Tr}\,\mathcal{O}_X\,\rho(t)-\langle\mathcal{O}_X\rangle_{\eta t}|\quad \text{small as } \eta\to 0^+\ ?\qquad\qquad (*)$$

- Abou Salem-Fröhlich 05; Jaksic-Pillet 14: adiabatic theorem for $\eta\to 0^+$ at fixed L, under suitable ergodicity hyp. (based on [Avron-Elgart 98])
- Jakšić-Pillet-Tauber 22: (*) for $\eta\to 0^+$ *after* $L\to\infty$ at fixed β implies that the specific entropy of $\langle\cdot\rangle_s$ is *constant* in s. No-go theorem?
- More modestly, is there an η-dependent but L-indep. range of β where (*) holds? In particular: $T\to 0$ after $L\to\infty$? *Today's talk.*

Lattice Fermions

- We consider interacting Fermions on a discrete d-dimensional torus or box $\Gamma_L := \mathbb{Z}_L^d$. Thermydamic limit ($L \to \infty$) only via uniform estimates.

  With a finite numer M of internal degrees of freedom (spin, particle species, unit cell,...) labeled by $S_M = \{1, \ldots, M\}$ this gives configuration space $\Lambda_L := \Gamma_L \times S_M$.

- Let $\mathcal{F}_L$ be the Fermionic Fock space over Λ_L with creation/annihilation operators satisfying

$$\{a_{\mathbf{x}}^+, a_{\mathbf{y}}^+\} = \{a_{\mathbf{x}}^-, a_{\mathbf{y}}^-\} = 0, \qquad \{a_{\mathbf{x}}^+, a_{\mathbf{y}}^-\} = \delta_{\mathbf{x},\mathbf{y}} \qquad (\mathbf{x},\mathbf{y} \in \Lambda_L)$$

- For each $X \subset \Lambda_L$ let $\mathcal{A}_X$ denote complex polynomials made out of $a_{\mathbf{x}}^{\pm}$ for $\mathbf{x} \in X$, and $\mathcal{A}_X^{\mathcal{N}}$ those commuting with $\mathcal{N} = \sum_{\mathbf{x} \in \Lambda_L} a_{\mathbf{x}}^+ a_{\mathbf{x}}^-$ (almost always consider the latter).

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Interaction, Hamiltonian, Gibbs state

- Finite range, particle conserving interactions associated with Hamiltonians over Λ_L:

$$\mathcal{H} = \sum_{X \subset \Lambda_L} \Phi_X \qquad \text{with } \Phi_X = \Phi_X^*,$$

also $[\Phi_X, \mathcal{N}] = 0$ (particles conserved); also assume $\|\Phi_X\|$ bounded and finite range uniformly in L. Typical example:

$$\mathcal{H} = \sum_{\mathbf{x},\mathbf{y} \in \Lambda_L} a^+_{\mathbf{x}} h(\mathbf{x}, \boldsymbol{y}) a^-_{\mathbf{y}} + \sum_{\mathbf{x},\mathbf{y} \in \Lambda_L} a^+_{\mathbf{x}} a^+_{\mathbf{y}} v(\mathbf{x}; \boldsymbol{y}) a^-_{\mathbf{y}} a^-_{\mathbf{x}}$$

with h, v bounded and finite-range.

- Other properties not too important so e.g. standard boundary conditions allowed

- Grand-canonical Gibbs state: for $\mathcal{O}_X \in \mathcal{A}_X$,

$$\langle \mathcal{O}_X \rangle_{\beta,\mu,L} = \mathrm{Tr}_{\mathcal{F}_L} \mathcal{O}_X \rho_{\beta,\mu,L} \qquad \text{with} \qquad \rho_{\beta,\mu,L} = \frac{e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathrm{Tr}_{\mathcal{F}_L} e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}$$

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Real and imaginary time evolutions

- Heisenberg evolution associated with $\mathcal{H}$ is

$$\tau_t(\mathcal{O}_X) = e^{i\mathcal{H}t}\,\mathcal{O}\,e^{-i\mathcal{H}t}.$$

- Later helpful to denote Euclidean evolution

$$\gamma_s(\mathcal{O}_X) = e^{s(\mathcal{H}-\mu\mathcal{N})}\,\mathcal{O}_X\,e^{-s(\mathcal{H}-\mu\mathcal{N})};$$

  clearly, if $[\mathcal{O}_X, \mathcal{N}] = 0$ (i.e. $\mathcal{O}_X \in \mathcal{A}_X^{\mathcal{N}}$)

$$\gamma_s(\mathcal{O}_X) = \tau_{-is}(\mathcal{O}_X) \qquad \text{for all } t \in \mathbb{C}.$$

- Including the chemical potential in γ_s makes the Gibbs state a Kubo-Martin-Schwinger (KMS) state:

$$\langle \gamma_s(\mathcal{O}_1)\,\gamma_t(\mathcal{O}_2)\rangle_{\beta,\mu,L} = \langle \gamma_{t+\beta}(\mathcal{O}_2)\,\gamma_s(\mathcal{O}_1)\rangle_{\beta,\mu,L}$$

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Correlation functions

- Connected/truncated correlation function or cumulant defined by

$$\langle \mathcal{O}_1; \mathcal{O}_2 \rangle_{\beta,\mu,L} := \langle \mathcal{O}_1\, \mathcal{O}_2 \rangle_{\beta,\mu,L} - \langle \mathcal{O}_1 \rangle_{\beta,\mu,L} \langle \mathcal{O}_2 \rangle_{\beta,\mu,L};$$

more generally n-point version ($\mathcal{O}_j$ even in $a^{\pm}$)

$$\langle \mathcal{O}_1; \mathcal{O}_2; \cdots; \mathcal{O}_n \rangle := \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \log \langle \exp(\lambda_1 \mathcal{O}_1 + \cdots + \lambda_n \mathcal{O}_n) \rangle \Big|_{\lambda_1 = \cdots = 0}$$

- When do these decay in space/time? Space decay from Lieb-Robinson bound for finite range bounded interactions:

$$\| [\tau_t(\mathcal{O}_X), \mathcal{O}_Y] \| \leqslant C\, e^{vt - c\, \mathrm{dist}(X,Y)}.$$

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Time ordered Euclidean correlations

- Define Euclidean time-ordering

$$\mathbf{T}\,\gamma_{t_1}(\mathcal{O}_1)\cdots\gamma_{t_n}(\mathcal{O}_n) := \sum_{\pi\in\Pi_n} \mathbb{1}(t_{\pi(1)} \geqslant \cdots \geqslant t_{\pi(n)})\,\gamma_{t_{\pi(1)}}(\mathcal{O}_{\pi(1)})\cdots\gamma_{t_{\pi(n)}}(\mathcal{O}_{\pi(n)})$$

- Time ordered cumulants $\langle \mathbf{T}\,\gamma_{t_1}(\mathcal{O}_{X_1});\cdots;\gamma_{t_n}(\mathcal{O}_{X_n})\rangle_{\beta,\mu,L}$ often decay in space and (Euclidean/imaginary) time; consequently used for perturbation theory of Gibbs states

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Driving the system

- For $t \leqslant 0$ we consider the time-dependent Hamiltonian

$$\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon\, g(\eta t)\mathcal{P},$$

 with $\eta > 0$ and $\mathcal{P} = \sum_{X \subset \Lambda_L} \Psi_X$, self-adjoint, finite-ranged, bounded. Interested in η small (slow) and ε small (weak) in different senses.

- More detail on "switch function" g soon, prototype is $g(\eta t) = e^{\eta t}$.
- Time-dependent density matrix

$$i\,\partial_t\,\rho(t) = [\mathcal{H}(\eta t), \rho(t)], \qquad \rho(-\infty) = \rho_{\beta,\mu,L}$$

 and time-dependent state given by expectation values $\mathrm{Tr}_{\mathcal{F}_L} \mathcal{O}\,\rho(t)$.

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- For $t \leqslant 0$ we consider the time-dependent Hamiltonian

$$\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon\, g(\eta t) \mathcal{P},$$

  with $\eta > 0$ and $\mathcal{P} = \sum_{X \subset \Lambda_L} \Psi_X$, self-adjoint, finite-ranged, bounded. Interested in η small (slow) and ε small (weak) in different senses.

- More detail on “switch function” g soon, prototype is $g(\eta t) = e^{\eta t}$.
- Time-dependent density matrix

$$i\, \partial_t\, \rho(t) = [\mathcal{H}(\eta t), \rho(t)], \qquad \rho(-\infty) = \rho_{\beta,\mu,L}$$

  and time-dependent state given by expectation values $\mathrm{Tr}_{\mathcal{F}_L} \mathcal{O}\, \rho(t)$.

- Dependence on ε, η? Comparison with Gibbs state of $\mathcal{H}(\eta t)$?

- Duhamel expansion for these dynamics:

$$\mathrm{Tr}_{\mathcal{F}_L}\mathcal{O}\,\rho(t) = \langle\mathcal{O}\rangle_{\beta,\mu,L} + \sum_{n=1}^{\infty}(-i\,\varepsilon)^n\int_* \mathrm{d}\underline{s}\left[\prod_{j=1}^{n} g(\eta\, s_j)\right] \times\langle[\cdots[\tau_t(\mathcal{O}),\tau_{s_1}(\mathcal{P})]\cdots,\tau_{s_n}(\mathcal{P})]\rangle_{\beta,\mu,L}$$

with the integral over $-\infty \leqslant s_n \leqslant \cdots \leqslant s_1 \leqslant t$.

- Decay of g and $\|\tau_s(\mathcal{P})\| = \|\mathcal{P}\| \leqslant C|\Lambda_L|$ let us bound nth term by

$$\frac{C^n|\varepsilon|^n}{n!}\frac{1}{\eta^n}|\Lambda_L|^n,$$

which is convergent for η, L fixed but useless for limits.

- Better bound using multicommutator Lieb-Robinson bounds [Bru-Pedra] uniform in L but not in η.

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Assumption on the switching function

Assumption S

The switching function can be expressed as

$$g(t) = \int_0^\infty e^{\xi t}\, h(\xi)\, \mathrm{d}\xi$$

for $h \in L^1([0,\infty))$ *satisfying*

$$\int_0^1 \frac{|h(\xi)|}{\xi^{d+2}}\, \mathrm{d}\xi < \infty, \qquad \int_1^\infty \xi\, |h(\xi)|\, \mathrm{d}\xi < \infty$$

or a finite signed measure (e.g. sum of Dirac δ*s) with* $|h(\xi)|\mathrm{d}\xi$ *replaced by the total variation.*

Examples: $g(t) = e^t$, $g(t) = (t-a)^{-n}$ with $n \geqslant d+4$ and $a > 0$.

Commentary on Assumption S

- g always analytic on LHP; so decay for $t \to -\infty$ (even $\mathrm{Re}\, t \to -\infty$ off $\mathbb{R}$) but unbounded support
- No particular restrictions on $g^{(n)}(0)$
- We will use this to approximate $g(\eta t)$ by

$$g_{\beta,\eta}(t) = \sum_{\omega} \tilde{h}_{\beta}(\omega)\, e^{\omega t}$$

  with the sum over *positive* integer multiples of $2\pi / \beta$; then as well as analyticity and decay this is

  - periodic $g_{\beta,\eta}(t + i\,\beta) = g_{\beta,\eta}(t)$, cf. KMS condition, Matsubara frequencies
  - oscillating in imaginary time, with

$$\int_0^{\beta} g_{\beta,\eta}(t + i\,s)\, \mathrm{d}s = 0$$

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First part of main result

Theorem 1

Under Assumption S, for any $\mathcal{O}_X \in \mathcal{A}_X^{\mathcal{N}}$

$$\mathrm{Tr}_{\mathcal{F}_L} \mathcal{O}_X \rho(t) = \langle \mathcal{O}_X \rangle_{\beta,\mu,L} + \sum_{n \geqslant 1} \frac{(-\varepsilon)^n}{n!} I_n + R_{\beta,\mu,L}(\mathcal{O}_X, \varepsilon, \eta, L), \ \textit{where}$$

$$I_n := \int_{[0,\beta]^n} \mathrm{d}\underline{s} \left[\prod_{j=1}^{n} g_{\beta,\eta}(t - i s_j) \right] \langle \mathbf{T} \gamma_{s_1}(\mathcal{P}); \gamma_{s_2}(\mathcal{P}); \cdots; \gamma_{s_n}(\mathcal{P}); \mathcal{O}_X \rangle_{\beta,\mu,L}$$

and

$$|R_{\beta,\mu,L}(\mathcal{O}_X, \varepsilon, \eta, L)| \leqslant K(X, \|\mathcal{O}_X\|) \frac{|\varepsilon|}{\eta^{d+2} \beta}$$

uniformly in L.

Reminder: $\gamma_{.}$ is the *Euclidean* evolution for $\mathcal{H} = \mathcal{H}(-\infty)$, $\rho(t)$ is evolved by

$$\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon\, g(\eta t) \mathcal{P}$$

Comments on the result

- Most interesting with absolute convergence of the series (especially for $L \to \infty$), I'll come back to this
- Error terms diverge for $\eta \to 0^+$ with β, ε fixed, so this applies to the adiabatic regime only holds at zero temperature
- At zero temperature the imaginary-time expansion gives an exact expression for this class of time evolutions
- There are special cases

$$g(\eta t) = \sum_{n=1}^{\infty} \tilde{h}_n \exp\left(\frac{2\pi n}{\beta} t\right)$$

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Assumption D

For each $\beta > 0$ *and each* $\mathcal{O} \in \mathcal{A}^{\mathcal{N}}$, *there exist* $\mathfrak{c} = \mathfrak{c}(\beta, \mathcal{P})$ *and* $C = C(\mathcal{O})$*:*

$$\int_{[0,\beta]^n} \mathrm{d}\underline{t}\,(1 + |\underline{t}|_\beta) \sum_{X_1,\ldots \subset \Lambda_L} |\langle \mathbf{T}\gamma_{t_1}(\Psi_{X_1}); \cdots; \gamma_{t_n}(\Psi_{X_n}); \mathcal{O}\rangle| \leqslant C\,\mathfrak{c}^n\, n!$$

where $|\underline{t}|_\beta = \sum_{j=1}^n \min_{m\in\mathbb{Z}} |t_j - m\,\beta|$.

Theorem 2

Under assumptions S and D, $\exists \varepsilon_0 = \varepsilon_0(\mathfrak{c})$ *such that for* $|\varepsilon| < \varepsilon_0$,

1. *The series in Theorem 1 is absolutely convergent*
2. *With* $\langle\cdot\rangle_t$ *denoting the Gibbs state of* $\mathcal{H}(\eta t)$,

$$|\mathrm{Tr}_{\mathcal{F}_L}\, \mathcal{O}_X\, \rho(t) - \langle \mathcal{O}_X\rangle_t| \leqslant K\left(|\varepsilon|\eta + \frac{|\varepsilon|}{\eta^{d+2}\,\beta}\right)$$

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$$|\mathrm{Tr}_{\mathcal{F}_L}\, \mathcal{O}_X\, \rho(t) - \langle \mathcal{O}_X\rangle_t| \leqslant K\left(|\varepsilon|\eta + \frac{|\varepsilon|}{\eta^{d+2}\,\beta}\right)$$

- Also implies a convergent perturbative expansion for the instantaneous Gibbs state (see next slide)
- If $\mathcal{H}$ and $\mathcal{P}$ are both quadratic in $a^{\pm}$, the bound follows from Wick's rule and the relation to the one-particle correlations. In particular if 1-particle H is gapped (uniformly in L) and μ is in the gap, $\mathfrak{c}$ is *independent of* β.
- If $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$ with $\mathcal{H}_0$ as above and $\mathcal{V}$, $\mathcal{P}$ finite range this extends to λ not too large via cluster expansion (Battle-Brydges-Federbush-Kennedy formula) this is the main reason for considering Fermions rather than lattice spin systems.
- Related to spectral properties (e.g. de Roeck-Salmhofer proof of stability of gap) but not as strong as (uniform) gap

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- Last point follows by comparison with cumulant expansion for the instantaneous Gibbs state

$$\langle \mathcal{O} \rangle_{\eta t} = \langle \mathcal{O} \rangle_{\beta,\mu,L} + \sum_{n=1}^{\infty} \frac{(-\varepsilon g(\eta t))^n}{n!} \int_{[0,\beta]^n} \mathrm{d}\underline{s} \, \langle \mathbf{T} \gamma_{s_1}(\mathcal{P}); \gamma_{s_2}(\mathcal{P}); \cdots; \gamma_{s_n}(\mathcal{P}); \mathcal{O} \rangle_{\beta,\mu,L}$$

  which is the series in the technical theorem with $g_{\beta,\eta}(t - i\, s_j) \to g(\eta t)$; hence additional error term of order $|\varepsilon|\eta$

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- When the decay property holds uniformly for $\beta \to \infty$, we recover a standard adiabatic theorem in the zero temperature limit

Structure of the proof of Theorem 1

I. Changing the switching function $g(\eta\, t)$ to $g_{\beta,\eta}(t)$ a sum of "Matsubara" exponentials gives a correction $R_{\beta,\mu,L}(\mathcal{O}_X, \varepsilon, \eta, L)$, which given Assumption S can be estimated based on a Lieb-Robinson bound

II. Taking advantage of the imaginary-time periodicity and analyticity of $g_{\beta,\eta}$ we can deform the integrals in the resulting Duhamel expansion into imaginary-time integrals of (non-connected) correlation functions (Wick rotation); using the oscillation of the (now imaginary) exponentials in $g_{\beta,\eta}$ the non-connected parts of the correlations cancel in the integration

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Approximating the real-time dynamics 1

Lemma 1

Let $\tilde{\mathcal{H}}$, $\tilde{\rho}$ denote the versions with $g_{\beta,\eta}$ replacing g; then

$$\left|\mathrm{Tr}_{\mathcal{F}_L}\, \mathcal{O}\, [\rho(t) - \tilde{\rho}(t)]\right| \leqslant \frac{C_{\mathcal{O}} |\varepsilon|}{\eta^{d+2}\beta}$$

Sketch of the proof; for simplicity $g(\eta\, t) = \exp(\eta\, t)$, $g_{\beta,\eta}(t) = \exp(\eta_\beta\, t)$ with $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}_+$ a little bigger than η, $|\eta - \eta_\beta| \leqslant \frac{2\pi}{\beta}$.

Letting $\mathcal{U}$, $\tilde{\mathcal{U}}$ denote the unitary evolutions associated with $\mathcal{H}, \tilde{\mathcal{H}}$,

$$\begin{aligned}
&\left|\mathrm{Tr}_{\mathcal{F}_L}\, \mathcal{O}\, [\rho(t) - \tilde{\rho}(t)]\right| \\
&= \left| \lim_{T\to\infty} \mathrm{Tr}_{\mathcal{F}_L} \big[\mathcal{U}(-T;t)\, \mathcal{O}\, \mathcal{U}(t;-T) - \tilde{\mathcal{U}}(-T;t)\, \mathcal{O}\, \tilde{\mathcal{U}}(t;-T)\,\big]\, \rho_{\beta,\mu,L} \right| \\
&\leqslant \limsup_{T\to\infty} \left\| \mathcal{O} - \mathcal{U}(t;-T)\, \tilde{\mathcal{U}}(-T;t)\, \mathcal{O}\, \tilde{\mathcal{U}}(t;-T)\, \mathcal{U}(-T,t) \right\|
\end{aligned}$$

- Let $\zeta_{\beta,\eta}(t) = g(\eta\, t) - g_{\beta,\eta}(t)$, so that $\mathcal{H}(\eta\, t) = \tilde{\mathcal{H}}(\eta, t) + \zeta_{\beta,\eta}(t)$; then

$$\begin{aligned}&\left\| \mathcal{O} - \mathcal{U}(t; -T)\, \tilde{\mathcal{U}}(-T; t)\, \mathcal{O}\, \tilde{\mathcal{U}}(t; -T)\, \mathcal{U}(-T, t) \right\| \\ =& \left\| \int_{-T}^{t} \frac{\partial}{\partial s} \Big[\mathcal{U}(s; -T)\, \tilde{\mathcal{U}}(-T; s)\, \mathcal{O}\, \tilde{\mathcal{U}}(s; -T)\, \mathcal{U}(-T, s) \Big]\, \mathrm{d}s \right\| \\ \leqslant& \int_{-T}^{t} |\varepsilon|\, |\zeta_{\eta,\beta}(s)| \left\| \big[\mathcal{P}, \tilde{\mathcal{U}}(t; s)\, \mathcal{O}\, \tilde{\mathcal{U}}(s; t) \big] \right\| \mathrm{d}s\end{aligned}$$

- Using a Lieb-Robinson bound for non-autonomous dynamics [Bachman, Michalakis, Nachtergaele, Sims; Bru, Pedra] gives the estimate $\leqslant C_{\mathcal{O}} \int_{-T}^{t} |\varepsilon|\, |\zeta_{\eta,\beta}(s)|\, (1 + |t - s|^d)\, \mathrm{d}s$

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Approximating the real-time dynamics 2

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&\left\| \mathcal{O} - \mathcal{U}(t; -T)\, \tilde{\mathcal{U}}(-T; t)\, \mathcal{O} \tilde{\mathcal{U}}(t; -T)\, \mathcal{U}(-T, t) \right\| \\
&= \left\| \int_{-T}^{t} \frac{\partial}{\partial s} \Big[\mathcal{U}(s; -T)\, \tilde{\mathcal{U}}(-T; s)\, \mathcal{O} \tilde{\mathcal{U}}(s; -T)\, \mathcal{U}(-T, s) \Big]\, \mathrm{d}s \right\| \\
&\leqslant \int_{-T}^{t} |\varepsilon|\, |\zeta_{\eta,\beta}(s)| \left\| \big[\mathcal{P}, \tilde{\mathcal{U}}(t; s)\, \mathcal{O} \tilde{\mathcal{U}}(s; t) \big] \right\| \mathrm{d}s
\end{aligned}$$

- Using a Lieb-Robinson bound for non-autonomous dynamics [Bachman, Michalakis, Nachtergaele, Sims; Bru, Pedra] gives the estimate $\leqslant C_{\mathcal{O}} \displaystyle\int_{-T}^{t} |\varepsilon|\, |\zeta_{\eta,\beta}(s)|\, (1 + |t - s|^{d})\, \mathrm{d}s$

- The result follows from this and $|\zeta_{\eta,\beta}(s)| = |e^{\eta s} - e^{\eta \beta s}| \leqslant \dfrac{2\pi}{\beta} |s| e^{\eta s}$

Wick rotation: overview

We now take the Duhamel series for the dynamics of $\tilde{\mathcal{H}}(\eta, t)$ and rewrite each term as an imaginary-time integral.

Lemma 2

$$\int_{s_n \leqslant \cdots \leqslant s_1 \leqslant t} \mathrm{d}\underline{s} \left[\prod_{j=1}^{n} g_{\eta,\beta}(s_j)\right] \langle [\cdots[\tau_t(\mathcal{O}), \tau_{s_1}(\mathcal{P})]\cdots, \tau_{s_n}(\mathcal{P})]\rangle_{\beta,\mu,L}$$

$$= \frac{(-i)^n}{n!} \int_{[0,\beta]^n} \left[\prod_{j=1}^{n} g_{\eta,\beta}(t - i\, s_j)\right] \langle \mathbf{T}\gamma_{s_1}(\mathcal{P}); \cdots; \gamma_{s_n}(\mathcal{P}); \mathcal{O}\rangle_{\beta,\mu,L}$$

This is based on a step in the proof of stability of KMS states in [Bratteli-Robinson vol. 2], which has a time independent permutation and so uses a clustering assumption for integrability in time.

Wick rotation: lowest order

- Using KMS and periodicity of $g_{\beta,\eta}$ and expanding the commutator,

$$\begin{aligned}&\int_{-\infty}^{t} g_{\beta,\eta}(s)\langle[\tau_s(\mathcal{P}),\tau_t(\mathcal{O})]\rangle_{\beta,\mu,L}\mathrm{d}s\\ &=\int_{-\infty}^{t}[g_{\beta,\eta}(s)\langle\tau_s(\mathcal{P})\tau_t(\mathcal{O})\rangle - g_{\beta,\eta}(s-i\beta)\langle\tau_{s-i\beta}(\mathcal{P})\tau_t(\mathcal{O})\rangle]\mathrm{d}s\\ &=i\int_0^{\beta} g_{\beta,\eta}(t-is)\langle\tau_{t-is}(\mathcal{P})\tau_t(\mathcal{O})\rangle_{\beta,\mu,L}\mathrm{d}s\end{aligned}$$

- then using stationarity of the Gibbs state

$$\int_0^{\beta} g_{\beta,\eta}(t-i\,s)\,\langle\tau_{t-is}(\mathcal{P})\tau_t(\mathcal{O})\rangle\,\mathrm{d}s = \int_0^{\beta} g_{\beta,\eta}(t-i\,s)\,\langle\gamma_s(\mathcal{P})\mathcal{O}\rangle\mathrm{d}s$$

- and we can replace the correlation with the connected one using

$$\int_0^{\beta} g_{\beta,\eta}(t-i\,s)\,\langle\gamma_s(\mathcal{P})\rangle\langle\mathcal{O}\rangle\,\mathrm{d}s = \langle\mathcal{P}\rangle\,\langle\mathcal{O}\rangle\int_0^{\beta} g_{\beta,\eta}(t-i\,s)\,\mathrm{d}s = 0.$$

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&=i\int_0^{\beta} g_{\beta,\eta}(t-is)\langle\tau_{t-is}(\mathcal{P})\tau_t(\mathcal{O})\rangle_{\beta,\mu,L}\mathrm{d}s
\end{aligned}$$

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Conclusion

- We show a sort of adiabatic behaviour (closeness of non-autonomously evolved system to instantaneous Gibbs state) for low temperature, $T \lesssim \eta^{d+2}$ with time scale η^{-1}
- Decay of correlations and convergence of cumulant expansion doesn't quite require a gap, are there cases where we can extend our result?
- Replace with more sophisticated expansions?
- Exactness of linear response (where appropriate)?

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