Bootstrapping Liouville Theory

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Conformal Field Theory

(Euclidean) QFT

- Random fields $\Psi(x)$, $x \in M$, M manifold, e.g. $\mathbb{R}^d$
- **Correlation functions** $\langle \prod_{i=1}^N \Psi(x_i) \rangle$

Gaussian fields: correlations determined by **two point functions**

Conformal field theory

- $\langle \prod_{i=1}^N \Psi(x_i) \rangle$ determined recursively by **two and three point functions** by **conformal bootstrap**.
- In $d = 2$ Belavin, Polyakov and Zamoldchicov (1984) used bootstrap to classify CFT's and find explicit predictions for the correlation functions in several cases
- In $d > 2$ bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model) by Rychkov and others.
- This talk: prove bootstrap for **Liouville theory**.

Conformal invariance

Scaling fields $V_\Delta(x)$, $x \in \mathbb{R}^d$, $\Delta \in \mathbb{R}$

Correlation functions invariant under rotations and translations of $\mathbb{R}^d$ and under scaling

$$\langle \prod_i V_{\Delta_i}(\lambda x_i) \rangle = \prod_i \lambda^{-2\Delta_i} \langle \prod_i V_{\Delta_i}(x_i) \rangle \quad (*)$$

Δ_i scaling dimension or **conformal weight**.

Conformal invariance: $(*)$ extends to conformal maps $x \to \Lambda(x)$,

In $d = 2$: $\mathbb{R}^2 \simeq \mathbb{C}$

$$\Lambda(z) = \frac{az + b}{cz + c} \qquad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

and $\lambda^{-2\Delta_i} \to |\Lambda'(z)|^{-2\Delta_i}$.

Natural setup is the **Riemann sphere**: $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Structure Constants

Use conformal map to fix three points to $\{0, 1, \infty\}$.

3-point functions are determined up to constants

$$\langle \prod_{k=1}^{3} V_{\Delta_k}(z_k) \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C(\Delta_1, \Delta_2, \Delta_3)$$

with $\Delta_{12} = \Delta_3 - \Delta_1 - \Delta_2$ etc.

$$C(\Delta_1, \Delta_2, \Delta_3) = \langle V_{\Delta_1}(0) V_{\Delta_2}(1) V_{\Delta_3}(\infty) \rangle$$

are called the **structure constants** of the CFT.

Bootstrap hypothesis

Operator Product Expansion Axiom:

$$\langle V_{\Delta_1}(x_1) V_{\Delta_2}(x_2) V_{\Delta_3}(x_3) \ldots \rangle = \sum_{\Delta \in \mathcal{S}} C^{\Delta}_{\Delta_1 \Delta_2}(x_1, x_2, \partial_{x_2}) \langle V_{\Delta}(x_2) V_{\Delta_3}(x_3) \ldots \rangle$$

- $C^{\Delta}_{\Delta_1 \Delta_2}$ are **determined** by and **linear** in the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- All correlations are determined by $C(\Delta_1, \Delta_2, \Delta_3)$

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

CFT on Riemann surfaces

CFT extends naturally to Riemann surfaces viewed as a surface Σ with **Riemannian metric** g

Diffeomorphism covariance axiom: For $\psi \in Diff(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(\psi(x_i)) \rangle_{\Sigma,g} = \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,\psi^* g}$$

Weyl covariance axiom: For $\sigma \in C^\infty(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,e^\sigma g} = e^{\frac{c}{96\pi} \int_\Sigma (|d\sigma|^2 + 2R_g \sigma) dv_g} \prod_i e^{-\Delta_i \sigma(x_i)} \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g}$$

c **central charge** of the CFT, R_g scalar curvature, v_g volume

Hence correlations defined on **moduli space** of Riemann surfaces

$$g \sim e^\sigma \psi^* g \quad \psi \in Diff(\Sigma), \;\; \sigma \in C^\infty(\Sigma)$$

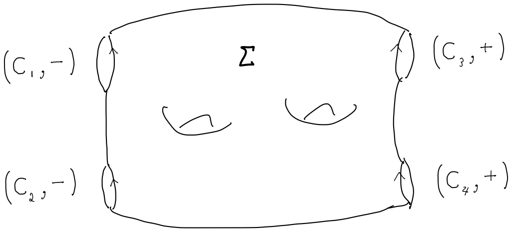
Bootstrap following G. Segal

- Σ closed oriented Riemann surface with $n \geq 0$ marked points $z_1, \ldots, z_n$ and boundary

$$\partial\Sigma = \cup_i C_i$$

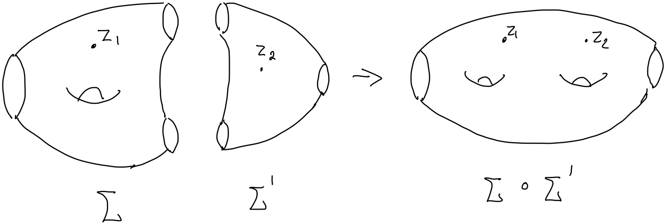
together with analytic parametrisations $\zeta_i : \mathbb{T} \to C_i$.

- Set $\sigma_i = \pm 1$ depending on whether orientation of $\zeta_i(\mathbb{T})$ agrees with that of Σ or not. Call them "in" and "out" boundaries.



Gluing surfaces

Glue "out" circles to "in" circles $(\Sigma, \Sigma') \to \Sigma \circ \Sigma'$



Segal's CFT functor

CFT consists of a **Hilbert space** $\mathcal{H}$ and an assignement

$$\Sigma \to \mathcal{A}_\Sigma$$

where

- $\mathcal{A}_\Sigma : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes n}$ is a **Hilbert-Schmidt operator**
- Σ has m in-circles and n out-circles

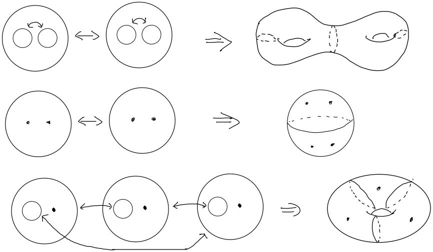
Gluing Axiom

$$\mathcal{A}_{\Sigma \circ \Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$

Building blocks

Build Σ by gluing simple topological building blocks $\mathcal{B}$:

- Pairs of pants $\mathcal{P} \sim \hat{\mathbb{C}} \setminus 3$ disks
- Annuli with one marked point $\hat{\mathbb{C}} \setminus \{2 \text{ disks}, \ 1 \text{ point}\}$
- Disks with two marked points $\hat{\mathbb{C}} \setminus \{1 \text{ disk}, \ 2 \text{ points}\}$



Bootstrap

Upshot:

Correlation function on Σ is given by composing operators $\mathcal{A}_{\mathcal{B}_a}$

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(x_i) \rangle_{\Sigma} = \prod_{a} \mathcal{A}_{\mathcal{B}_a}$$

Show:

- Operators $\mathcal{A}_{\mathcal{B}_a}$ are determined by structure constants
- $\mathcal{A}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}'}$ can be factorised by representation theory

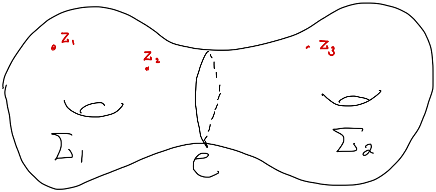
Path integrals

Motivation for axioms: let the QFT be given formally as a path integral, e.g. for a scalar field ϕ

$$\langle\prod_{i=1}^{n} V_{\Delta_i}(z_i)\rangle = \int_{\phi:\Sigma\to\mathbb{R}} \prod_{i=1}^{n} V_{\Delta_i}(\phi(z_i))e^{-S_\Sigma(\phi)}D\phi$$

with local action functional $S_\Sigma(\phi)$

Let $\Sigma = \Sigma_1 \circ \Sigma_2$, $\partial\Sigma_i = \mathcal{C}$ so that $S_\Sigma = S_{\Sigma_1} + S_{\Sigma_2}$.



Path integrals

Let for $\varphi : \mathcal{C} \to \mathbb{R}$

$$\mathcal{A}_{\Sigma_j}(\varphi) = \int_{\phi|_{\Sigma_j = \varphi}} \prod_{i: z_i \in \Sigma_j} V_{\Delta_i}(z_i) e^{-S_{\Sigma_j}(\phi)} D\phi \quad j = 1, 2$$

Then formally get

$$\langle \prod_{i=1}^{n} V_{\Delta_i}(z_i) \rangle = \int_{\varphi : \mathcal{C} \to \mathbb{R}} \mathcal{A}_{\Sigma_1}(\varphi) \mathcal{A}_{\Sigma_2}(\varphi) D\varphi$$

View $\mathcal{A}_\Sigma$ as an integral kernel $\mathcal{A}_\Sigma(\varphi_{in}, \varphi_{out})$ **"Amplitude"**

Next :

- Probabilistic construction of $\mathcal{A}_\Sigma$ for **Liouville CFT**
- Prove gluing $\mathcal{A}_{\Sigma \circ \Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$
- Use this to prove bootstrap and compute correlations.

Liouville Theory

Action functional

$$S_\Sigma(\phi) = \int_\Sigma (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + QR_g\phi + \mu e^{\gamma\phi})dv_g$$

- $\gamma \in (0, 2]$
- $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, $\mu > 0$.
- R_g scalar curvature of the Riemannian metric g

Classical theory: for $Q = \frac{\gamma}{2}$ the **minimizer** ϕ_0 gives rise to constant negative curvature metric $e^{\gamma\phi_0}|dz|^2$ (Picard, Poincare)

Quantum theory $\langle\cdot\rangle$:

- Noncritical string theory (Polyakov 1981)
- 2d gravity: Knizhnik, Polyakov, Zamolodchikov (1988)
- 4d SuSy Yang-Mills: Alday, Gaiotto, Tachikawa (2010)

Probablistic Liouville Theory

We define

$$\langle F \rangle_{\Sigma,g} := Z_g \int_{\mathbb{R}} \mathbb{E}\big(F(\phi) e^{-\int_\Sigma (QR_g\phi + \mu :e^{\gamma\phi}:) dv_g}\big) dc$$

$\phi = c + X$ with X free field:

$$\mathbb{E}X(z)X(z') = -\Delta_g^{-1}(z,z')$$

Liouville theory is **superrenormalisable**: normal ordering

$$: e^{\gamma\phi} := \lim_{\epsilon\to 0} e^{\gamma c} e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E}X_\epsilon(z)^2}$$

suffices for renormalisation but it is **nonperturbative**:

$$c \to c + t \implies \mu \to e^t c$$

No small coupling limit!

Existence

Theorem (David, K, Rhodes, Vargas, CMP 2016) *Let*

$$V_\alpha(z) =: e^{\alpha\phi_g(z)} :$$

The correlation functions $\langle \prod_i V_{\alpha_i}(z_i)\rangle_{\Sigma,g}$ *exist and are nontrivial* ***iff***

$$(1) \quad \sum_{i=1}^{n} \alpha_i + \chi(\Sigma) Q > 0, \quad \text{and} \quad (2) \;\; \alpha_i < Q \;\; \forall i$$

V_α *are* ***primary fields*** *with scaling dimension* $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$

- (1): convergence of c-integral using $\int_\Sigma R_g dv_g = -\chi(\Sigma)$.
- (2): regularity of Gaussian Multiplicative Chaos measure

$$\int |z|^{-\gamma\alpha} : e^{\gamma X(z)} : d^2 z < \infty \;\; a.s. \;\; iff \;\; \alpha < Q$$

(1): Gauss-Bonnet: $\int_\Sigma R_g dv_g = -\chi(\Sigma)$.

$$\langle\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle = \mathbb{E}\left[\prod_i e^{\alpha_i X(z_i)} \int_{\mathbb{R}} e^{(\sum_i \alpha_i + \chi(\Sigma))c - \mu e^{\gamma c}\int :e^{\gamma X(z)}:d^2z} dc\right]$$

The c-integral converges **if** $\sum_i \alpha_i > 2Q$:

$$\langle\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle = \frac{\Gamma(s)}{\mu^s\gamma}\mathbb{E}\left[\prod_i e^{\alpha_i X(z_i)}\left(\int :e^{\gamma X(z)}: d^2z\right)^{-s}\right], \quad s := \frac{\sum_i \alpha_i - 2Q}{\gamma}$$

(2): Shift in X-integral (Girsanov theorem):

$$X(z) \to X(z) - \sum_i \Delta_g^{-1}(z, z_i) \sim X(z) - \sum_i \alpha_i \log|z - z_i|$$

gives

$$\langle\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle \sim \mathbb{E}\left(\int \prod_i \frac{1}{|z-z_i|^{\gamma\alpha_i}} :e^{\gamma X(z)}: d^2z\right)^{-s}$$

Now $\frac{1}{|z-z_i|^{\gamma\alpha_i}}$ is $:e^{\gamma X(z)}: d^2z$-integrable (almost surely) **if and only if**

$$\alpha_i < Q \quad \text{i.e.} \quad \gamma\alpha_i < 2 + \frac{\gamma^2}{2}.$$

Structure constants

For the structure constants we take $\Sigma = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81)

Let α_i satisfy the Seiberg bounds. Then

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty)\rangle_{\hat{\mathbb{C}}} = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

$C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ is an explicit formula conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov in 1995 involving 16 Barnes G-functions (!).

Proof involves GMC analysis to derive recursive equations determining $C(\alpha_1, \alpha_2, \alpha_3)$

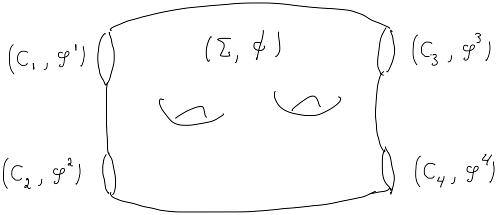
Amplitudes

Let $\partial\Sigma = \cup_{a=1}^{n} \mathcal{C}_a$. For $\phi : \Sigma \to \mathbb{R}$ set

$$\phi|_{\mathcal{C}_a} = \varphi^a, \quad \boldsymbol{\varphi} := (\varphi^1, \ldots, \varphi^n)$$

How to make sense of

$$\mathcal{A}_\Sigma(\boldsymbol{\varphi}) = \int_{\phi|_{\partial\Sigma} = \boldsymbol{\varphi}} \prod_i V_{\alpha_i}(z_i) e^{-S_\Sigma(\phi)} D\phi \ \ ?$$



Free field

Let $\phi = \phi_0 + \zeta$ with

- ϕ_0 harmonic extension of φ: $\Delta_g \phi_0 = 0, \quad \phi_0|_{\partial\Sigma} = \varphi$
- $\zeta|_{\partial\Sigma} = 0$

Then the free field action factorises

$$S^0(\phi) := \int_\Sigma g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \, dv_g = S^0(\phi_0) + S^0(\zeta)$$

$S^0(\phi_0)$ reduces to a boundary term

$$S^0(\phi_0) = \int_{\partial\Sigma} \phi_0 \partial^\perp \phi_0 = (\varphi, D_\Sigma \varphi)$$

where D_Σ is the **Dirichlet-Neumann** operator acting on the boundary fields

$$\phi|_{\partial\Sigma} = \varphi = (\varphi^1, \varphi^2, \ldots \varphi^n)$$

Liouville amplitudes

Let $\varphi^a(\theta) = \sum_{k\in\mathbb{Z}} \varphi^a_k e^{ik\theta}$. Then

$$S^0(\phi_0) = \frac{1}{4}\sum_{a=1}^{n}\sum_{k\in\mathbb{Z}} |k||\hat{\varphi}^a_k|^2 + (\varphi, \tilde{D}_\Sigma \varphi)$$

$\tilde{D}_\Sigma$ is **smoothing**: $(\varphi, \tilde{D}_\Sigma\varphi)$ defined on $\varphi^a \in H^{-s}(\mathbb{T})\ \forall s > 0$.

Definition. The Liouville amplitude with primary fields V_{α_i} at z_i

$$\mathcal{A}_\Sigma(\varphi) = Z\ e^{-(\varphi,\tilde{D}_\Sigma\varphi)}\,\mathbb{E}\left(\prod V_{\alpha_i}(z_i) e^{-\int_\Sigma QR_g\phi dv_g - \mu\int_\Sigma :e^{\gamma\phi}:dv_g}\right)$$

with

- $\phi = \phi_0 + \zeta$
- ϕ_0 harmonic extension of φ
- $\mathbb{E}$ is over the Dirichlet GFF ζ on Σ.

Gluing

Let μ be measure on distributions $\varphi = \sum_{k\in\mathbb{Z}} \varphi_k e^{ik\theta} \in H^s(\mathbb{T}), \quad s < 0$

$$d\mu(\varphi) = d\varphi_0 \prod_{k>0} e^{-\frac{1}{2}|k|\,|\hat{\varphi}_k|^2} \frac{d^2\varphi_k}{\pi|k|}$$

and define the **Liouville Hilbert space**

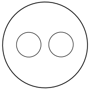
$$\mathcal{H} = L^2(H^s(\mathbb{T}), d\mu).$$

Proposition (GKRV'22). $\mathcal{A}_\Sigma$ are Hilbert-Schmidt operators and

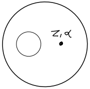
$$\mathcal{A}_{\Sigma\circ\Sigma'} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}$$

Examples

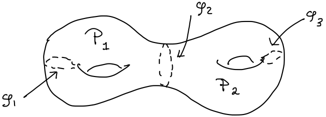
Building blocks: $\mathcal{P}$



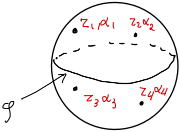
$(\mathbb{C}, z, \alpha)$



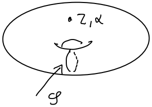
$(\mathbb{D}, \underset{\sim}{z}, \underset{\sim}{\alpha})$



$$\langle 1 \rangle_{\Sigma} = \int A_{\mathcal{P}_1}(\varphi_1, \varphi_1, \varphi_2) A_{\mathcal{P}_2}(\varphi_2, \varphi_3, \varphi_3) \prod_{i=1}^{3} d\mu(\varphi_i)$$

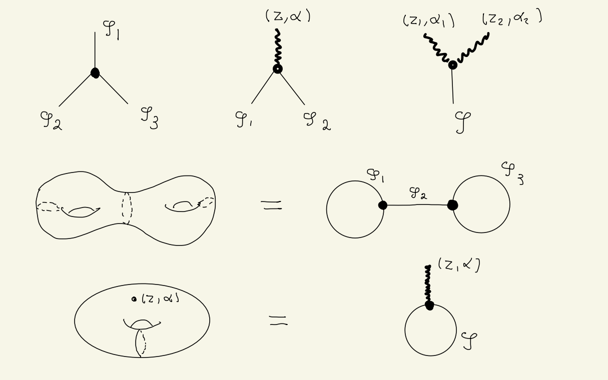


$$\left\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle_{\hat{\mathbb{C}}} = \int A_{\mathbb{D}, z_1 z_2 \alpha_1 \alpha_2}(\varphi) A_{\mathbb{D}, z_3 z_4 \alpha_3 \alpha_4}(\varphi) \, d\mu(\varphi)$$



$$\langle V_\alpha(z) \rangle_{\mathbb{T}^2} = \int A_{\mathbb{C}, z, \alpha}(\varphi, \varphi) \, d\mu(\varphi)$$

ϕ^3 graphs



Express the scalar product $\int F(\varphi)G(\varphi)d\mu(\varphi)$ using eigenfunctions of the **Hamiltonian** of LCFT:

$$H = H_0 + \mu \int_0^{2\pi} : e^{\gamma\varphi(\theta)} : d\theta$$

where H_0 is the Hamiltonian of the free field.

Spectrum of Liouville theory

Theorem (GKRV, Acta Math. to appear) H has spectral resolution

$$L^2(d\mu(\varphi)) = \int_{\mathbb{R}_+}^{\oplus} \mathcal{H}_P \otimes \tilde{\mathcal{H}}_P \, dP$$

$\mathcal{H}_P$ and $\tilde{\mathcal{H}}_P$ are Verma modules of two commuting **Virasoro algebras** with central charge $c = 1 + 6Q^2$ of highest weights $(\Delta_{Q+iP}, \Delta_{Q+iP})$.

Complete set of generalised eigenfunctions $\Psi_{P,\nu,\tilde{\nu}}$

- $P \in \mathbb{R}_+$ and ν, $\tilde{\nu}$ are Young diagrams
- $\Psi_{P,0,0}$ is amplitude of $(\mathbb{D}, \alpha)$, $\alpha = Q + iP$
- **CFT spectrum** of LCFT is $\{\Delta_{Q+iP}\}_{P\in\mathbb{R}}$

Proof. For the free field $\mu = 0$ this is well known. Deformation to $\mu \neq 0$ is obtained by **scattering theory**.

Ward identities

Need to evaluate amplitudes of building blocks at eigenstates:

Proposition (GKRV'22). Let $\mathcal{B}$ be a pair of pants. Then

$$A_{\mathcal{B}}(\otimes_{j=1}^{3}\Psi_{P_j,\nu_i,\tilde{\nu}_i}) = D(\boldsymbol{\nu},\boldsymbol{P})D(\tilde{\boldsymbol{\nu}},\boldsymbol{P})A_{\mathcal{B}}(\otimes_{j=1}^{3}\Psi_{P_j,0,0}).$$

where $D(\boldsymbol{\nu},\boldsymbol{P})$ is **explicit**, representation theoretic and

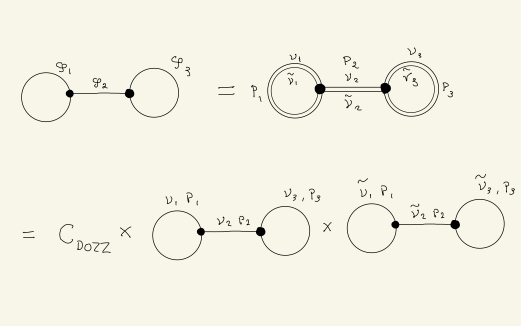
$$A_{\mathcal{B}}(\otimes_{j=1}^{3}\Psi_{P_j,0,0}) = C_{DOZZ}(Q+iP_1,Q+iP_2,Q+iP_3)$$

the LCFT structure constant given by the DOZZ formula.

Similar factorisation holds for other building blocks with some $Q+iP_i$ replaced by α_i of vertex insertions.

Proof is based on **probabilistic Ward identities**.

Integrability of Liouville theory



Theorem (GKRV (2022). Let Σ have genus g. Then

$$\langle \prod_{i=1}^{m} V_{\alpha_i}(z_i) \rangle_\Sigma = \int_{\mathbb{R}_+^{3g+m-3}} |\mathcal{F}(\mathbf{z}, \mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

- Conformal block $\mathcal{F}(\mathbf{z}, \mathbf{P})$ is purely representation theoretic and **holomorphic in the moduli** of the surface $(\Sigma, z_1, \dots, z_m)$
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha, \alpha', \alpha'')$ with $\alpha, \alpha', \alpha'' \in \{\alpha_i, Q \pm iP_j\}$

Work in progress

For $\gamma \in \mathbb{R}$, $c = 1 + 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2 \geq 25$. How about other c?

For $\gamma = i\beta$ with $\beta \in \mathbb{R}$, $c \in (-\infty, 1]$. This also has a probabilistic formulation. Can one recover the minimal models from imaginary Liouville?

Connection to the AdS_3 σ-model

$$S = k \int (\bar{\partial}\phi\partial\phi + e^{2\phi}\bar{\partial}v\partial\bar{v})d^2z$$

and analytic Langlands correspondence

Integrable perturbations of LCFT: e.g. $e^{\gamma\phi} \to \cosh(\gamma\phi)$

Thank you!