Purely linear response of the quantum Hall current to space-adiabatic perturbations

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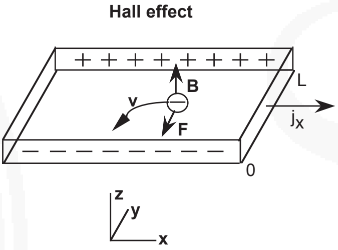
based on: G. Marcelli, D. Monaco, LMP 2022

Quantum Hall Effect

Quantized transverse (Hall) current in

- Quantum Hall systems [von Klitzing *et al.* 1980, Thouless *et al.* 1982]
- Haldane model [Haldane 1988], Chern insulators [Chang *et al.* 2015, Bestwick *et al.* 2015]



Kubo formula for the Hall conductivity (linear response)

$$j_y = \sigma_{xy} E_x + \mathcal{O}(E_x{}^2)$$

with

$$\boxed{\sigma_{xy} = \nu \frac{e^2}{h}, \quad \nu = \text{Chern number} = \text{TKNN invariant} \; \in \mathbb{Z}}$$

Exactness of Kubo formula

Theorem

Actually

$$j_y = \sigma_{xy} E_x + \mathcal{O}(E_x{}^{\infty}).$$

Comments

- This result is due to [Klein, Seiler 1990] (see also [Bachmann *et al.* 2021])
- Their proof is based on Laughlin's magnetic-flux insertion argument [Laughlin 1981] made rigorous by the use of time-adiabatic perturbation theory [Avron, Simon, ...]
- It applies to many-body, disordered electron gases at zero temperature under a spectral gap assumption

- Our proof is based on space-adiabatic perturbation theory [Nenciu, Teufel, ...]: more physical?
- It applies to non-interacting, periodic electrons at zero temperature under a spectral gap assumption
- It computes ν by means of equilibrium quantities

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- Our proof is based on space-adiabatic perturbation theory [Nenciu, Teufel, . . .]: more physical?
- It applies to non-interacting, periodic electrons at zero temperature under a spectral gap assumption
- It computes ν by means of equilibrium quantities

Review: Klein–Seiler's argument

On $\Lambda = [0, L_1] \times [0, L_2]$ consider (a fermionic many-body version of)

$$\widetilde{H}(\phi_1, \phi_2) := \frac{1}{2}\left(-\mathrm{i}\nabla - \frac{B}{2}\,\mathbf{e_3} \times \mathbf{x} - \phi_1\,\frac{\mathbf{e_1}}{L_1} - \phi_2\,\frac{\mathbf{e_2}}{L_2}\right)^2 + W(\mathbf{x})$$

- $\phi_i\,\mathbf{e}_i/L_i$ "threads a magnetic flux ϕ_i trough the loop in i-th direction"
- $\widetilde{H}(\phi_1, \phi_2)$ is 2π-periodic in ϕ_1, ϕ_2 up to gauge transformations: for $G(\phi_1, \phi_2) := \mathrm{e}^{\mathrm{i}(\phi_1 X_1/L_1 + \phi_2 X_2/L_2)}$

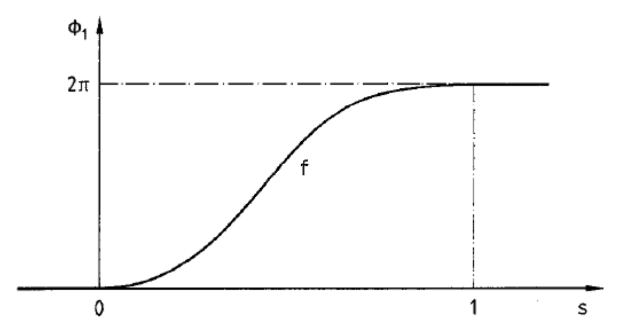
$$\begin{aligned}\widehat{H}(\phi_1, \phi_2) &:= G(\phi_1, \phi_2)\,\widetilde{H}(\phi_1, \phi_2)\,G(\phi_1, \phi_2)^* \\ &= \widehat{H}(\phi_1 + 2\pi, \phi_2) = \widehat{H}(\phi_1, \phi_2 + 2\pi)\end{aligned}$$

- Assume: $\widetilde{H}(\phi_1, \phi_2)$ has an isolated spectral island, with spectral projection $\widetilde{P}(\phi_1, \phi_2)$

Review: Klein–Seiler's argument

$H_\tau(t,\Phi) := \widetilde{H}(\phi_1 = f(t/\tau), \phi_2 = \Phi)$

τ : time-adiabatic parameter
$\varepsilon = \tau^{-1} \propto V \ll 1$: Hall voltage
$s = t/\tau$



$P(t,\Phi) := \widetilde{P}(\phi_1 = f(t/\tau), \phi_2 = \Phi)$ spectral projection

Review: Klein–Seiler's argument

- Physical evolution:

$$\mathrm{i}\,\partial_s U_\tau(s,\Phi) = \tau\, H_\tau(s,\Phi)\, U_\tau(s,\Phi), \qquad U_\tau(0,\Phi) = \mathbf{1}$$

- Physical state:

$$P_\tau(s,\Phi) = U_\tau(s,\Phi)\, P(0,\Phi)\, U_\tau(s,\Phi)^* \qquad \Big(\neq P(s,\Phi) \Big)$$

- Adiabatic evolution:

$$\mathrm{i}\,\partial_s U_\tau^{(\mathrm{a})}(s,\Phi) = \tau\, H_\tau^{(\mathrm{a})}(s,\Phi)\, U_\tau^{(\mathrm{a})}(s,\Phi), \qquad U_\tau^{(\mathrm{a})}(0,\Phi) = \mathbf{1}$$

where

$$H_\tau^{(\mathrm{a})}(s,\Phi) := H_\tau(s,\Phi) + \frac{\mathrm{i}}{\tau} \left[\partial_s P(s,\Phi), P(s,\Phi) \right]$$

Review: Klein–Seiler's argument

Theorem (Adiabatic theorem)

- *Since* $H_\tau(s,\Phi)$ *is constant near* $s = 1$

$$P(0,\Phi)\, U_\tau^{(a)}(1,\Phi)^*\, U_\tau(1,\Phi)\, P(0,\Phi) = \mathcal{O}(\tau^{-\infty})$$

- *Intertwining property of the adiabatic evolution:*

$$P(s,\Phi)\, U_\tau^{(a)}(s,\Phi) = U_\tau^{(a)}(s,\Phi)\, P(0,\Phi)$$

Review: Klein–Seiler's argument

Hall current

$$I_\tau(s,\Phi) := \operatorname{Tr} P_\tau(s,\Phi)\, \partial_\Phi H_\tau(s,\Phi)$$

Average transported charge

$$\begin{aligned}
\langle Q\rangle &:= \int_0^{2\pi} \frac{d\Phi}{2\pi}\left(\tau \int_0^1 ds\, I(s,\Phi)\right) \\
&= \frac{i}{2\pi}\int_0^{2\pi} d\Phi \int_0^1 ds\partial_s \operatorname{Tr} P(0,\Phi)\, U_\tau(s,\Phi)^*\, \partial_\Phi U_\tau(s,\Phi) \\
&= \frac{i}{2\pi}\int_{\partial[0,1]\times[0,2\pi]} \operatorname{Tr} P(0,\Phi)\, U_\tau^{(a)}(s,\Phi)^*\, dU_\tau^{(a)}(s,\Phi) + \mathcal{O}(\tau^{-\infty}) \\
&= \underbrace{\frac{1}{2\pi}}_{=e^2/h} \underbrace{\left(i\int_{\mathbb{T}^2} d\phi_1\, d\phi_2 \operatorname{Tr} \widehat{P}\big[\partial_{\phi_1}\widehat{P}, \partial_{\phi_2}\widehat{P}\big]\right)}_{\text{Chern number } \in\mathbb{Z}} + \mathcal{O}(\tau^{-\infty})
\end{aligned}$$

where last equality employs Chern–Simons formula:

$$\operatorname{Tr} P_U\, dP_U \wedge dP_U = \operatorname{Tr} P\, dP \wedge dP + d\big(\operatorname{Tr} P\, U^{-1}\, dU\big)$$

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&= \frac{\mathrm{i}}{2\pi} \int_{\partial[0,1]\times[0,2\pi]} \operatorname{Tr} P(0,\Phi)\, U_\tau^{(\mathrm{a})}(s,\Phi)^*\, \mathrm{d}U_\tau^{(\mathrm{a})}(s,\Phi) + \mathcal{O}(\tau^{-\infty}) \\
&= \underbrace{\frac{1}{2\pi}}_{=e^2/h} \underbrace{\left(\mathrm{i} \int_{\mathbb{T}^2} \mathrm{d}\phi_1\, \mathrm{d}\phi_2 \operatorname{Tr} \widehat{P}\left[\partial_{\phi_1}\widehat{P}, \partial_{\phi_2}\widehat{P}\right] \right)}_{\text{Chern number } \in\mathbb{Z}} + \mathcal{O}(\tau^{-\infty})
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Our argument

The model

- H_0 : unperturbed 2D (magnetic-)periodic gapped Hamiltonian (e.g. Bloch–Landau operator, Haldane Hamiltonian, ...)

  Π_0 : equilibrium spectral projection below the gap
- H_ε : perturbed Hamiltonian, the electric field is modelled by a slowly-varying linear potential (space-adiabatic perturbation)

$$H_\varepsilon = H_0 - \varepsilon X_1$$

These two have very different spectral properties: typically $\sigma(H_\varepsilon) = \mathbb{R}$

What we need

- Π_ε : a non-equilibrium state
- J_2 : a transverse Hall current operator
- $\mathcal{T}(\cdot)$: a trace functional to compute expectations of extensive observables in extended states (in view of periodicity)

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Exactness of Kubo formula (reprise)

Theorem ([Marcelli, M., LMP 2022])

$$\mathcal{T}(J_2 \Pi_\varepsilon) = \varepsilon \, \sigma_{12} + \mathcal{O}(\varepsilon^\infty)$$

with

$$\sigma_{12} = -\mathrm{i}\, \mathcal{T}\Big(\Pi_0 \Big[[\Pi_0, X_1], [\Pi_0, X_2] \Big] \Pi_0 \Big) \in \frac{1}{2\pi}\mathbb{Z}$$

Exactness of Kubo formula (reprise)

Theorem ([Marcelli, M., LMP 2022])

For all $n \in \mathbb{N}$ *there is* $\Pi_\varepsilon^{(n)}$ *such that*

$$\mathcal{T}(J_2 \Pi_\varepsilon^{(n)}) = \varepsilon \, \sigma_{12} + \mathcal{O}(\varepsilon^{n+1})$$

with

$$\sigma_{12} = -\mathrm{i}\,\mathcal{T}\Big(\Pi_0 \Big[[\Pi_0, X_1], [\Pi_0, X_2] \Big] \Pi_0 \Big) \in \frac{1}{2\pi}\mathbb{Z}$$

Hall current operator

Hall current operator

$$J_2 := \mathrm{i}\,[H_0, X_2] = \mathrm{i}\,[H_\varepsilon, X_2]$$

Remark: Bloch–Floquet transform

Periodic operators can be fibered in the Bloch–Floquet representation:

$$\mathcal{U}\, H_0\, \mathcal{U}^{-1} = \int_{\mathbb{R}^2/(2\pi\mathbb{Z}^2)}^{\oplus} \mathrm{d}k\, H_0(k) \text{ with } \boxed{(\mathcal{U}\psi)(k,x) := \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{-\mathrm{i}k\cdot(x-n)}\,(T_n\psi)(x)}$$

Assumption

Since

$$\mathrm{i}\,[H_0, X_j](k) = \partial_{k_j} H_0(k)$$

we assume that

$$k \mapsto H_0(k) \text{ is } C^\infty\text{-smooth (in norm-resolvent topology)}$$

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Trace per unit volume

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$$\mathcal{T}(A) := \lim_{L\to\infty} \frac{1}{L^2} \operatorname{Tr} \chi_{[-L/2,L/2]^2} A \chi_{[-L/2,L/2]^2}$$

(provided A is trace-class on compact sets)

For periodic operators A

$$\begin{aligned} \mathcal{T}(A) &= \operatorname{Tr} \chi_{\mathcal{C}} A \chi_{\mathcal{C}} && \left[\mathcal{C} = [-1/2,1/2]^2 : \text{ unit cell}\right] \\ &= \int_{\mathcal{C}} \mathrm{d}x \, A(x;x) && \left[A(x;y) : \text{ integral kernel of } A\right] \\ &= \int_{\mathbb{T}^2} \mathrm{d}k \, \operatorname{Tr}_{L^2(\mathcal{C})} A(k) && \left[A(k) : \text{ Bloch–Floquet fibers of } A\right] \end{aligned}$$

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NEASS

Theorem

Let $H_\varepsilon = H_0 - \varepsilon X_1$, as before. There exist a sequence

$$\mathcal{S}_\varepsilon^{(n)}, \quad n \in \mathbb{N},$$

of bounded, periodic operators with smooth Bloch–Floquet fibers such that the non-equilibrium almost stationary state (NEASS)

$$\Pi_\varepsilon^{(n)} := \exp\left(\mathrm{i}\,\varepsilon\,\mathcal{S}_\varepsilon^{(n)}\right) \Pi_0 \exp\left(-\mathrm{i}\,\varepsilon\,\mathcal{S}_\varepsilon^{(n)}\right)$$

satisfies

$$\boxed{\left[H_\varepsilon, \Pi_\varepsilon^{(n)}\right] = \varepsilon^{n+1}\left[R_\varepsilon^{(n)}, \Pi_\varepsilon^{(n)}\right] = \mathcal{O}\left(\varepsilon^{n+1}\right).}$$

Comments

- Older approaches to space-adiabatic perturbation theory: [Nenciu, Teufel, Panati, ...] $\rightsquigarrow \Pi_{\varepsilon}^{(n)}$ as an asymptotic power series in ε
- Here: $\Pi_{\varepsilon}^{(n)}$ is unitarily conjugated to equilibrium state Π_0 (useful!)...
  ... but $\mathcal{S}_{\varepsilon}^{(n)}$ is still constructed order by order in ε
- This type of NEASS was constructed previously
  - in [M., Teufel 2019] for time-dependent lattice Hamiltonian in finite volume with uniform gap (estimates uniform in system size)
  - in [Teufel 2020] for space-adiabatically perturbed lattice Hamiltonian in finite volume (estimates uniform in system size)
  - in [Marcelli, Panati, Teufel 2021] in our context, up to first order

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Inverse Liouvillian

The proof relies on the following

Lemma

Decompose $\mathcal{H} = \operatorname{Ran}\Pi_0 \oplus (\operatorname{Ran}\Pi_0)^{\perp}$*, and correspondigly operators as*

$$A = A^{\mathrm{D}} + A^{\mathrm{OD}}$$

with $\quad A^{\mathrm{D}} := \Pi_0 A \Pi_0 + \Pi_0^{\perp} A \Pi_0^{\perp}, \quad A^{\mathrm{OD}} := \Pi_0 A \Pi_0^{\perp} + \Pi_0^{\perp} A \Pi_0$.

Define the Liouvillian

$$\mathcal{L}_{H_0}(A) := -\mathrm{i}\,[H_0, A].$$

The Liouvillian is invertible on OD operators:

$$\mathcal{L}_{H_0}(A) = B = B^{\mathrm{OD}}$$

$$\implies \quad A = A^{\mathrm{OD}} = \mathcal{L}_{H_0}^{-1}(B) := \frac{1}{2\pi}\oint \mathrm{d}z\,(H_0 - z\mathbf{1})^{-1}\,[\Pi_0, A]\,(H_0 - z\mathbf{1})^{-1}.$$

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For example

$$\boxed{\mathcal{S}_{\varepsilon}^{(1)} = -\mathcal{L}_{H_0}^{-1}\left(X_1^{\mathrm{OD}}\right) = -\mathcal{L}_{H_0}^{-1}\left([[X_1, \Pi_0], \Pi_0]\right).}$$

Exactness of Kubo formula (proof)

Theorem ([Marcelli, M., LMP 2022])

$$\mathcal{T}(J_2 \Pi_\varepsilon^{(n)}) = \varepsilon \sigma_{12} + \mathcal{O}(\varepsilon^{n+1})$$

with

$$\sigma_{12} = -\mathrm{i}\,\mathcal{T}\Big(\Pi_0\big[[\Pi_0, X_1], [\Pi_0, X_2]\big]\Pi_0\Big) \in \frac{1}{2\pi}\mathbb{Z}$$

Denote $\Pi_\varepsilon^{(n)} \equiv \Pi$, $\exp\big(\mathrm{i}\,\varepsilon\,\mathcal{S}_\varepsilon^{(n)}\big) \equiv U$ in what follows.

$$\mathcal{T}(J_2\Pi) = \mathrm{i}\,\mathcal{T}(\Pi[H_\varepsilon, X_2]\Pi) = \mathrm{i}\,\mathcal{T}([\Pi H_\varepsilon \Pi, \Pi X_2 \Pi]) + \mathcal{O}(\varepsilon^{n+1})$$

$$= \underbrace{\mathrm{i}\,\mathcal{T}([\Pi H_0 \Pi, \Pi X_2 \Pi])}_{\text{persistent current}} + \varepsilon \underbrace{\big\{-\mathrm{i}\,\mathcal{T}([\Pi X_1 \Pi, \Pi X_2 \Pi])\big\}}_{\sigma_{12}} + \mathcal{O}(\varepsilon^{n+1})$$

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$$\begin{aligned}\mathcal{T}(J_2 \Pi) &= \mathrm{i}\,\mathcal{T}\left(\Pi\,[H_\varepsilon, X_2]\,\Pi\right) = \mathrm{i}\,\mathcal{T}\left([\Pi H_\varepsilon \Pi, \Pi X_2 \Pi]\right) + \mathcal{O}(\varepsilon^{n+1}) \\ &= \underbrace{\mathrm{i}\,\mathcal{T}\left([\Pi H_0 \Pi, \Pi X_2 \Pi]\right)}_{\text{persistent current}} + \varepsilon \underbrace{\left\{ -\mathrm{i}\,\mathcal{T}\left([\Pi X_1 \Pi, \Pi X_2 \Pi]\right)\right\}}_{\sigma_{12}} + \mathcal{O}(\varepsilon^{n+1})\end{aligned}$$

Exactness of Kubo formula (proof)

Lemma (Vanishing of the persistent current)

$$\mathcal{T}\left(\left[\Pi H_0 \Pi, \Pi X_2 \Pi\right]\right) = 0.$$

Proof.

$$\left[\Pi H_0 \Pi, \Pi X_2 \Pi\right] = \left[\Pi H_0 \Pi, X_2\right] - \left[\Pi H_0 \Pi, X_2^{\mathrm{OD},\varepsilon}\right]$$

and $\Pi H_0 \Pi$, $X_2^{\mathrm{OD},\varepsilon}$ are bounded with smooth Bloch–Floquet fibers. □

Exactness of Kubo formula (proof)

Lemma (Chern–Simons formula)

Since $\Pi = U \Pi_0 U^{-1}$

$$\mathcal{T}\left([\Pi X_1 \Pi, \Pi X_2 \Pi]\right) = \mathcal{T}\left([\Pi_0 X_1 \Pi_0, \Pi_0 X_2 \Pi_0]\right)$$
$$= \mathcal{T}\Big(\Pi_0 \Big[[\Pi_0, X_1], [\Pi_0, X_2]\Big] \Pi_0\Big).$$

$$\sigma_{12} = \frac{1}{2\pi}\left(\frac{1}{2\pi}\int_{\mathbb{T}^2} \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{Tr}\Big(\Pi_0(k)\left[\partial_{k_1}\Pi_0(k), \partial_{k_2}\Pi_0(k)\right]\Pi_0(k)\Big)\right)$$
$$= \frac{1}{2\pi}(\text{Chern number}) \in \frac{1}{2\pi}\mathbb{Z}$$

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Lemma (Chern–Simons formula)

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$$= \mathcal{T}\Big(\Pi_0 \Big[[\Pi_0, X_1], [\Pi_0, X_2] \Big] \Pi_0 \Big).$$

$$\sigma_{12} = \frac{1}{2\pi} \left(\frac{1}{2\pi} \int_{\mathbb{T}^2} \mathrm{d}k_1 \, \mathrm{d}k_2 \, \mathrm{Tr}\Big(\Pi_0(k) [\partial_{k_1} \Pi_0(k), \partial_{k_2} \Pi_0(k)] \, \Pi_0(k) \Big) \right)$$
$$= \frac{1}{2\pi} (\text{Chern number}) \in \frac{1}{2\pi} \mathbb{Z}$$

What next?

- Inclusion of ergodic disorder $\rightsquigarrow$ noncommutative Chern number [Bellissard, Van Elst, Schulz-Baldes 1994; Bouclet, Germinet, Klein, Schenker 2005]
- Inclusion of interactions (on a lattice) [Teufel 2020; Henheik, Teufel 2021]
- Spin transport [Marcelli, Panati, Teufel 2021]