Magnetic skyrmions under confinement

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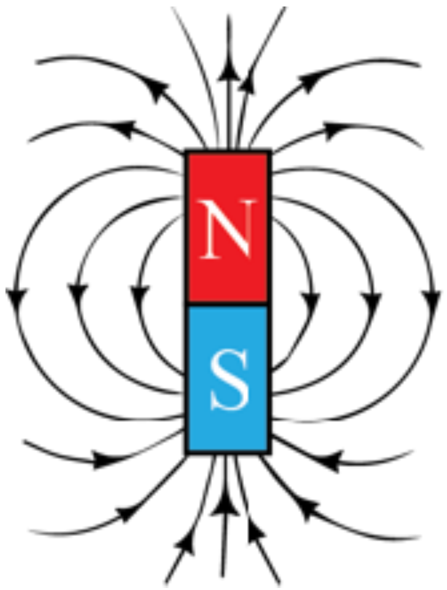
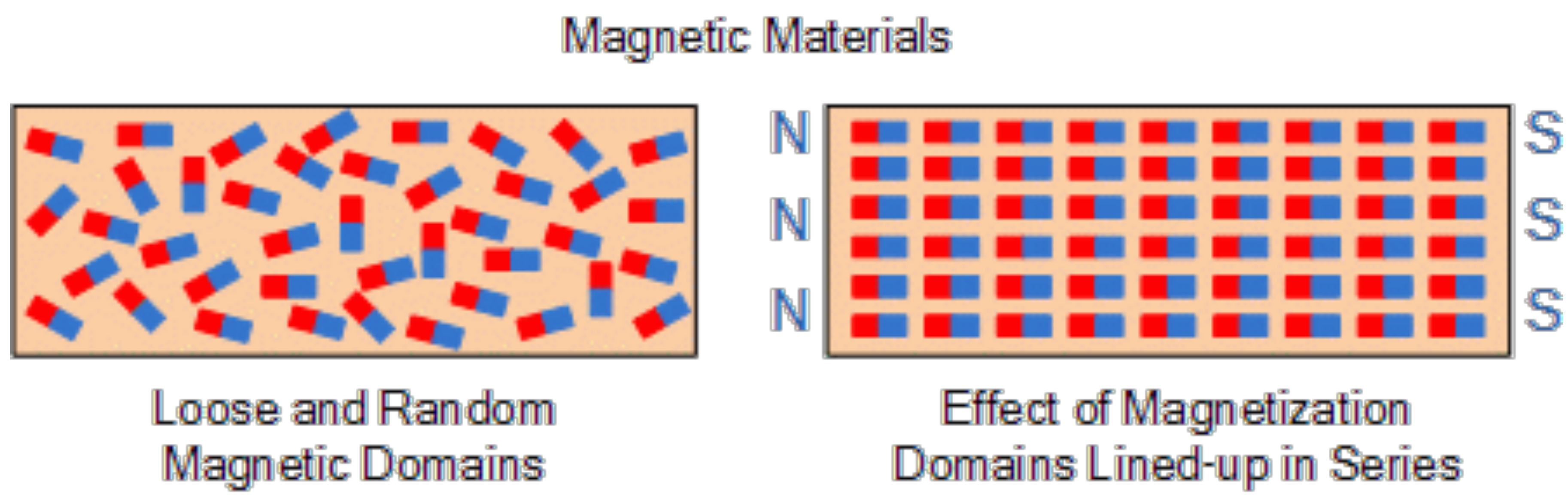
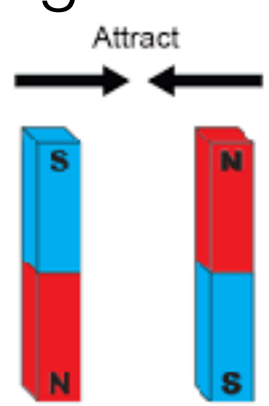
Magnetism and magnets





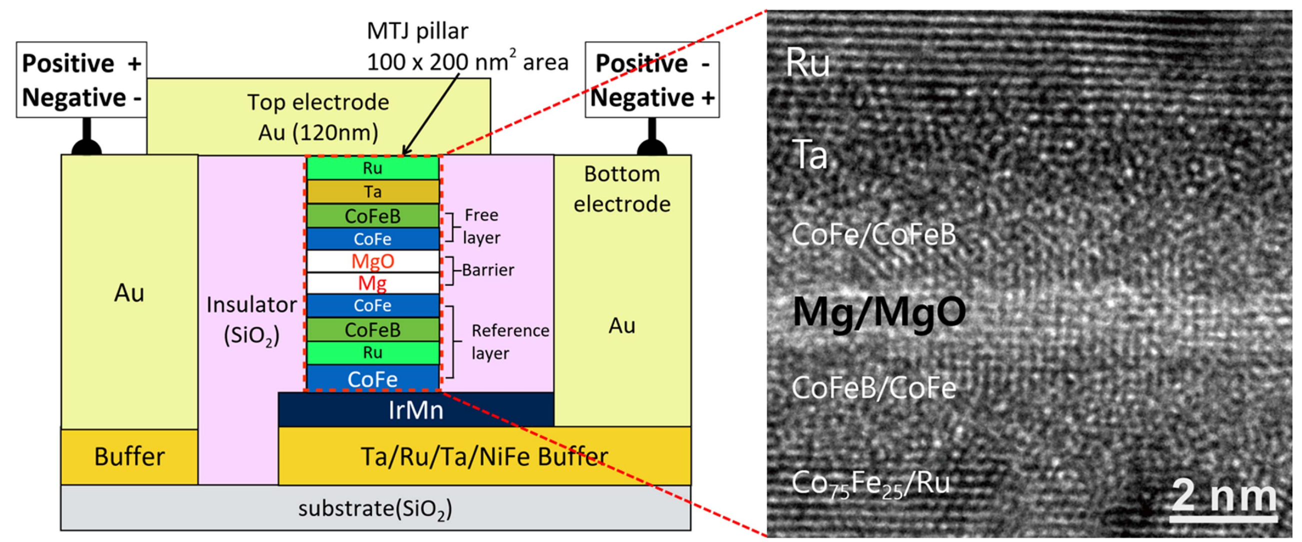
- spins act as tiny magnetic dipoles
- quantum-mechanical interaction between spins: _*exchange*_
- in transition metals below the *critical temperature*, exchange results in local spin alignment into the _*ferromagnetic state*_
- magnetic field mediates long-range attraction/repulsion between magnets



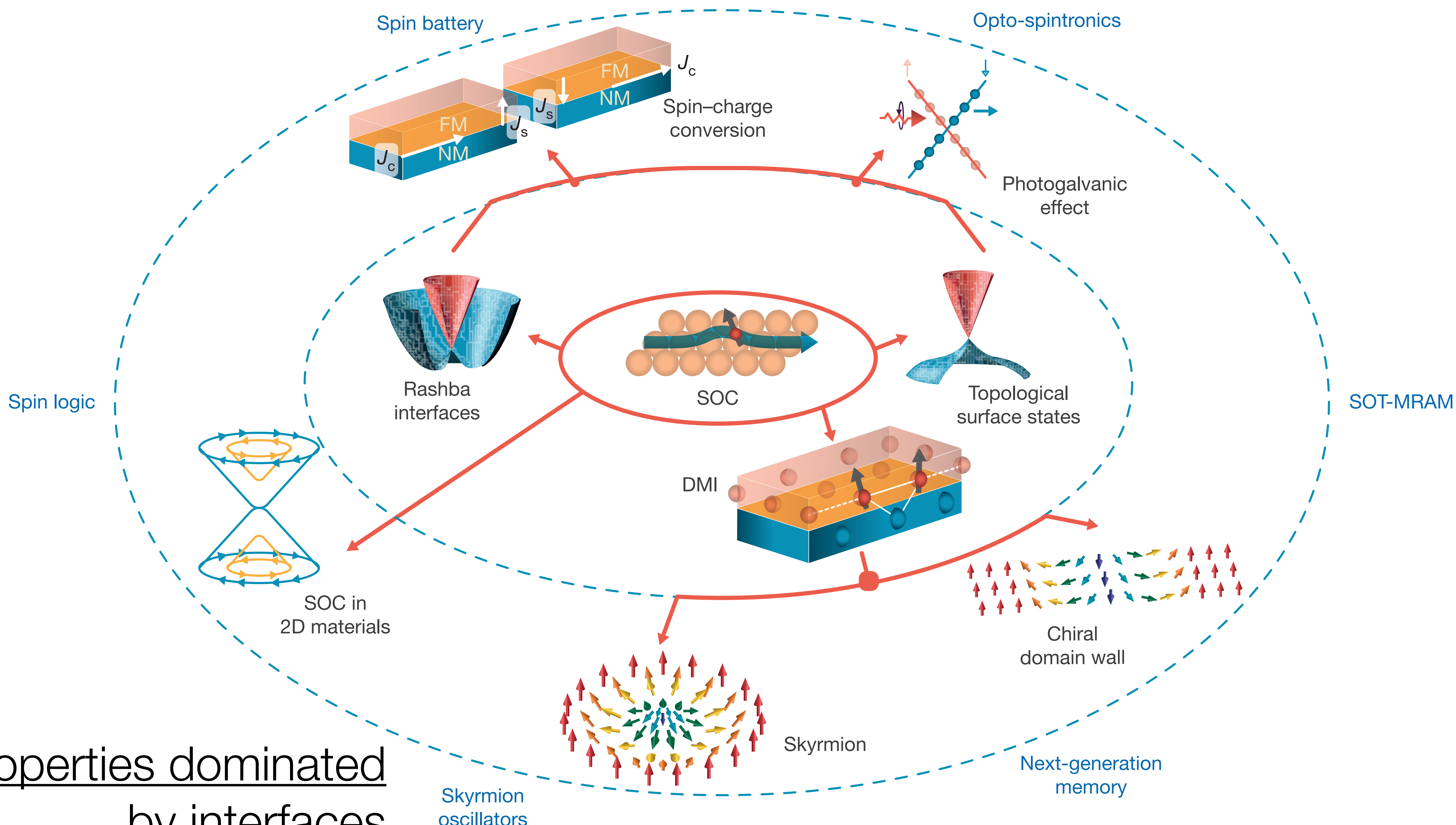
images borrowed from:
Tom Whyntie. (2016), zenodo.com
mammothmemory.net

Magnetic materials for spintronics

atomically thin multilayers with strong spin-orbit coupling (SOC):



C.-M. Choi et al., *Semicond. Sci. Technol.* 32, 105007 (2017)



from A. Soumyanarayanan et al., *Nature* **539**, 509-517 (2016)

properties dominated by interfaces

IN SUPREMÆ DIGNITATIS 1343

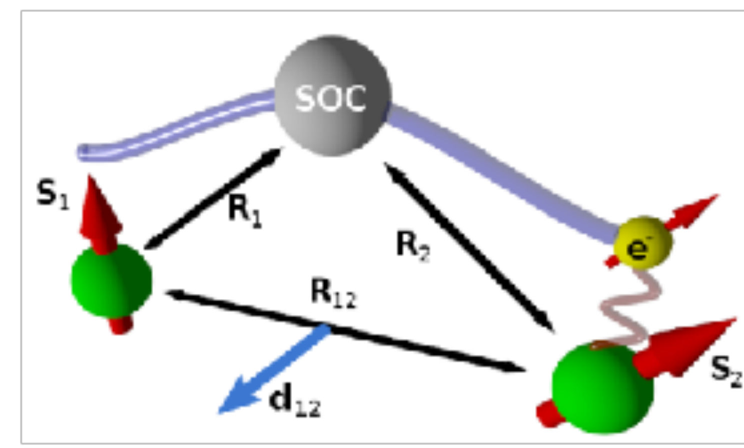
Dzyaloshinskii-Moriya interaction (DMI)

- In 1957, while trying to explain weak ferromagnetism in α-Fe_2O_3, Dzyaloshinskii, used symmetry arguments to introduce an antisymmetric exchange term that favors canted spin arrangements I. Dzyaloshinskii, Sov. Phys. JETP **5,** 1259-1272 (1957)

$$E = \boldsymbol{D}_{ij} \cdot (\boldsymbol{S}_i \times \boldsymbol{S}_j)$$

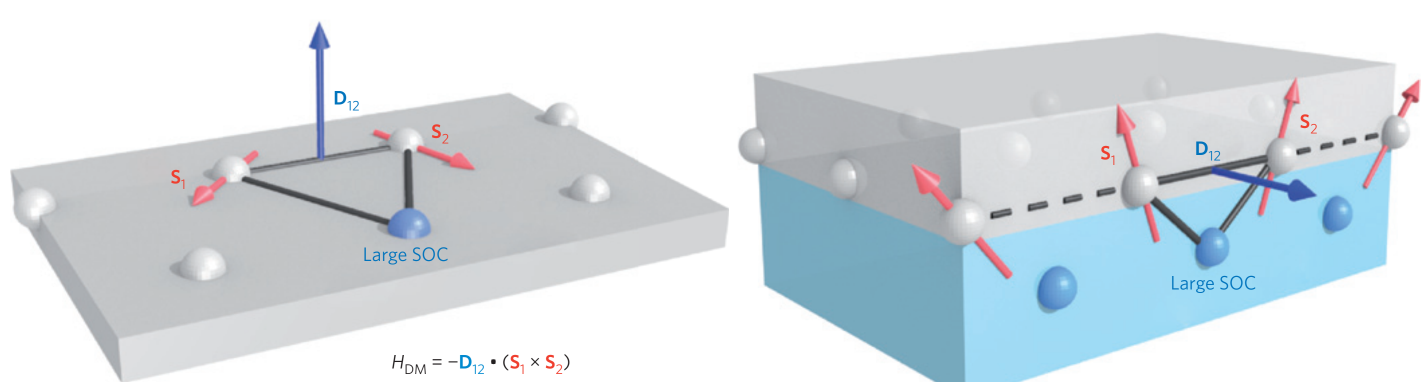
- In 1960, Moriya developed a theory of anisotropic super-exchange interaction, including spin-orbit coupling in Anderson theory of super-exchange

T. Moriya, Phys. Rev. B **120**, 91-98 (1960)

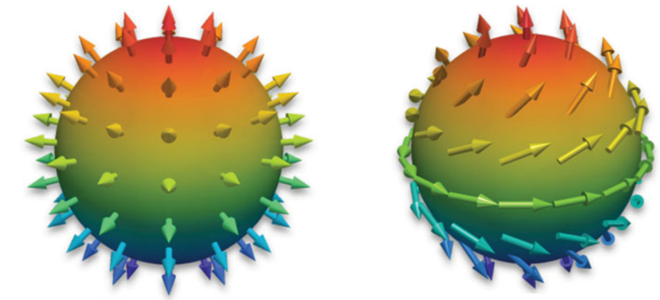


- The same effect was shown to be induced by the symmetry breaking at an interface

A. Fert and P. M. Levy, Phys. Rev. Lett. **44**, 1538-1541 (1980)
A. Crépieux and C. Lacroix, J. Magn. Magn. Mater. **182**, 341-349 (1998)



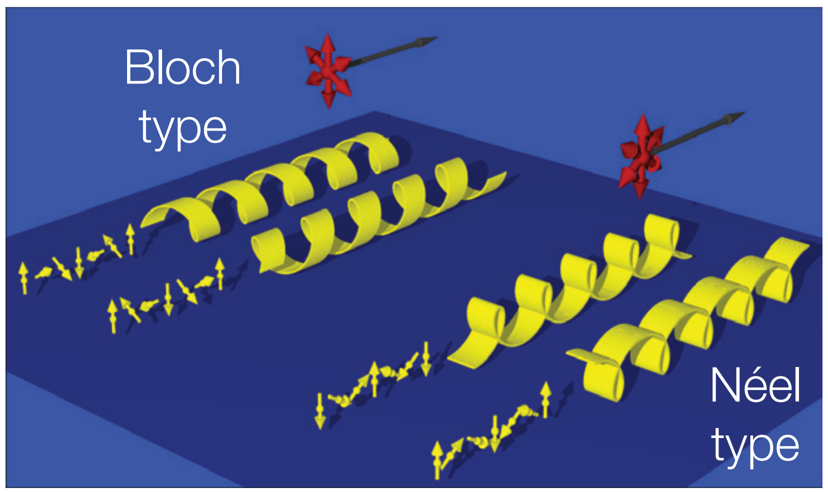
A. Fert, V. Cross and J. Sampaio, Nature Nanotechnol. **8**, 152-156 (2013)



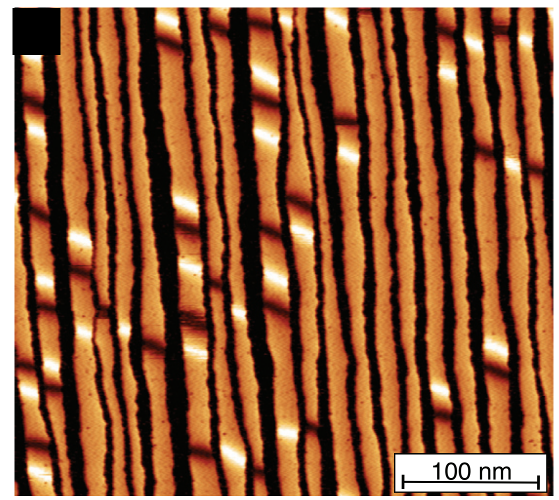
Topological spin textures

T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

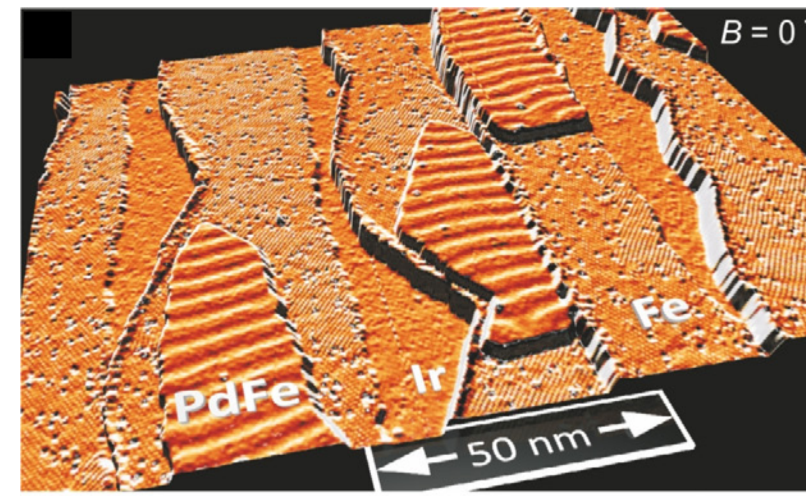
spin spirals and chiral domain walls from **Dzyaloshinskii-Moriya interaction** (DMI):



2ML Fe on W(110)

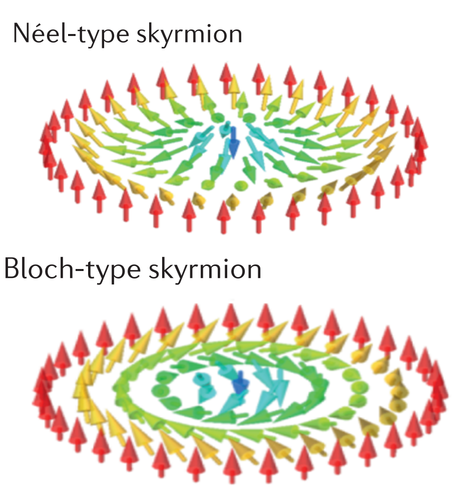


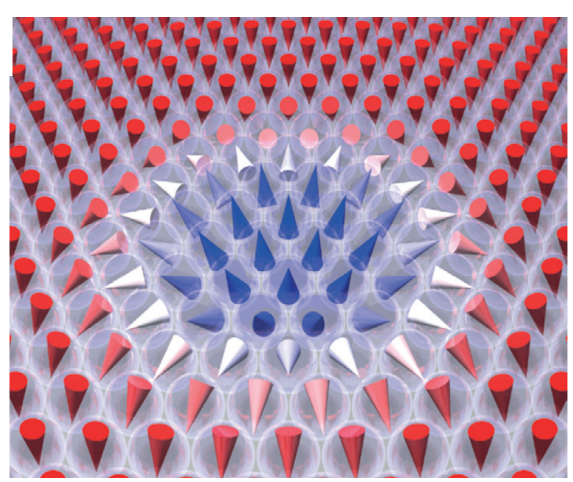
Pd/Fe bilayer on Ir(111)

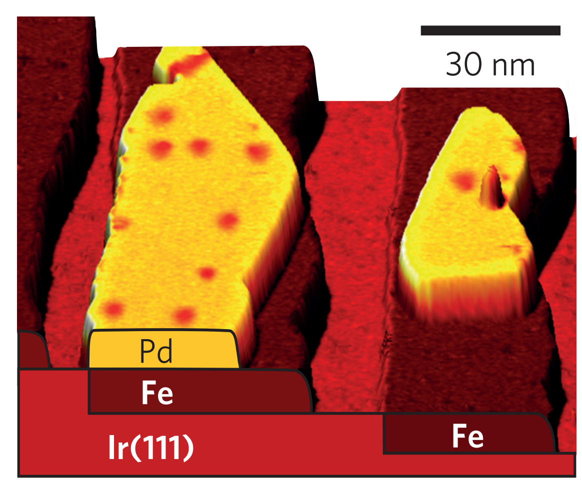


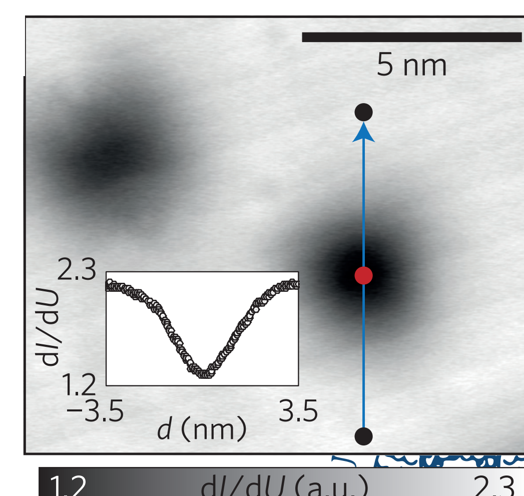
K. von Bergmann et al., J. Phys.: Condens. Matter **26**, 394002 (2014)

magnetic skyrmions:



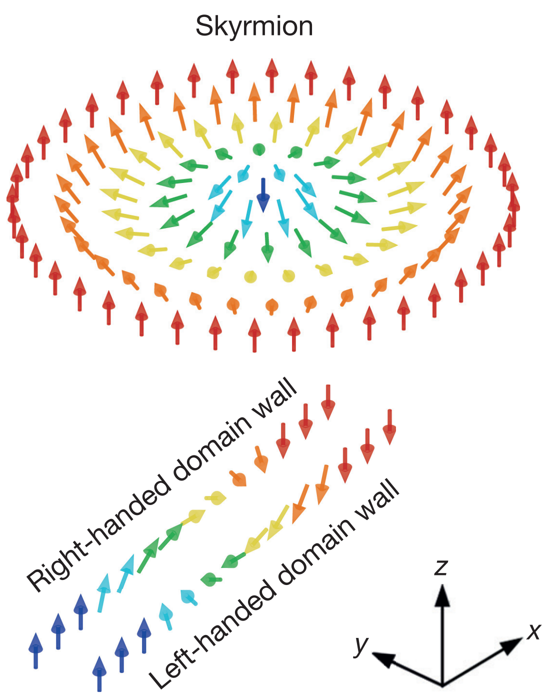


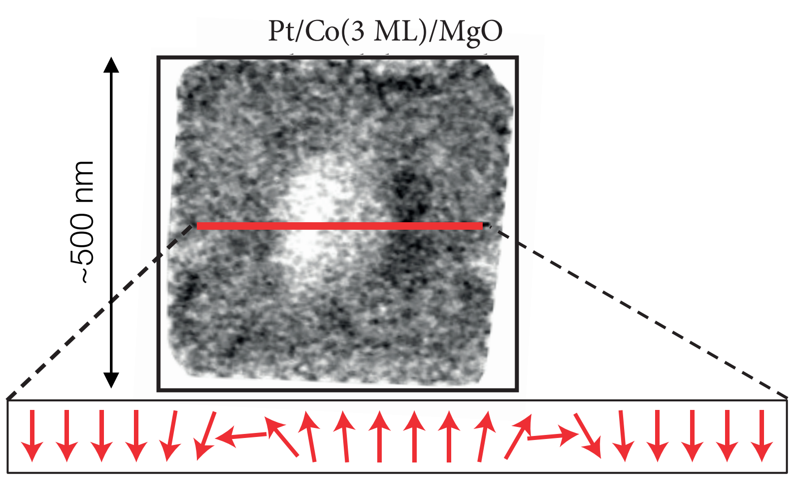


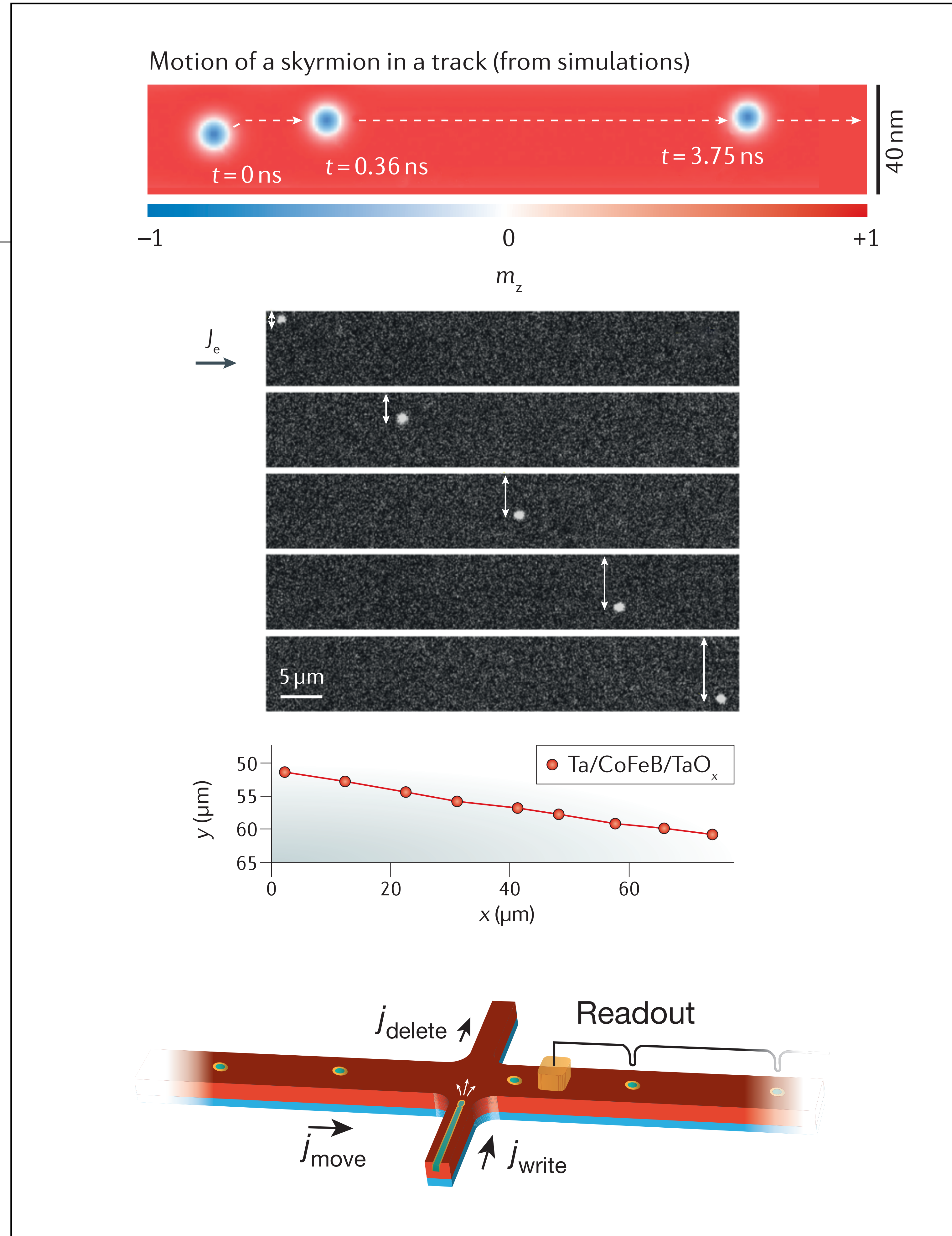


C. Hanneken et al., *Nature Nanotechnol.* **10**, 1039–1042 (2015)

Skyrmions on the track

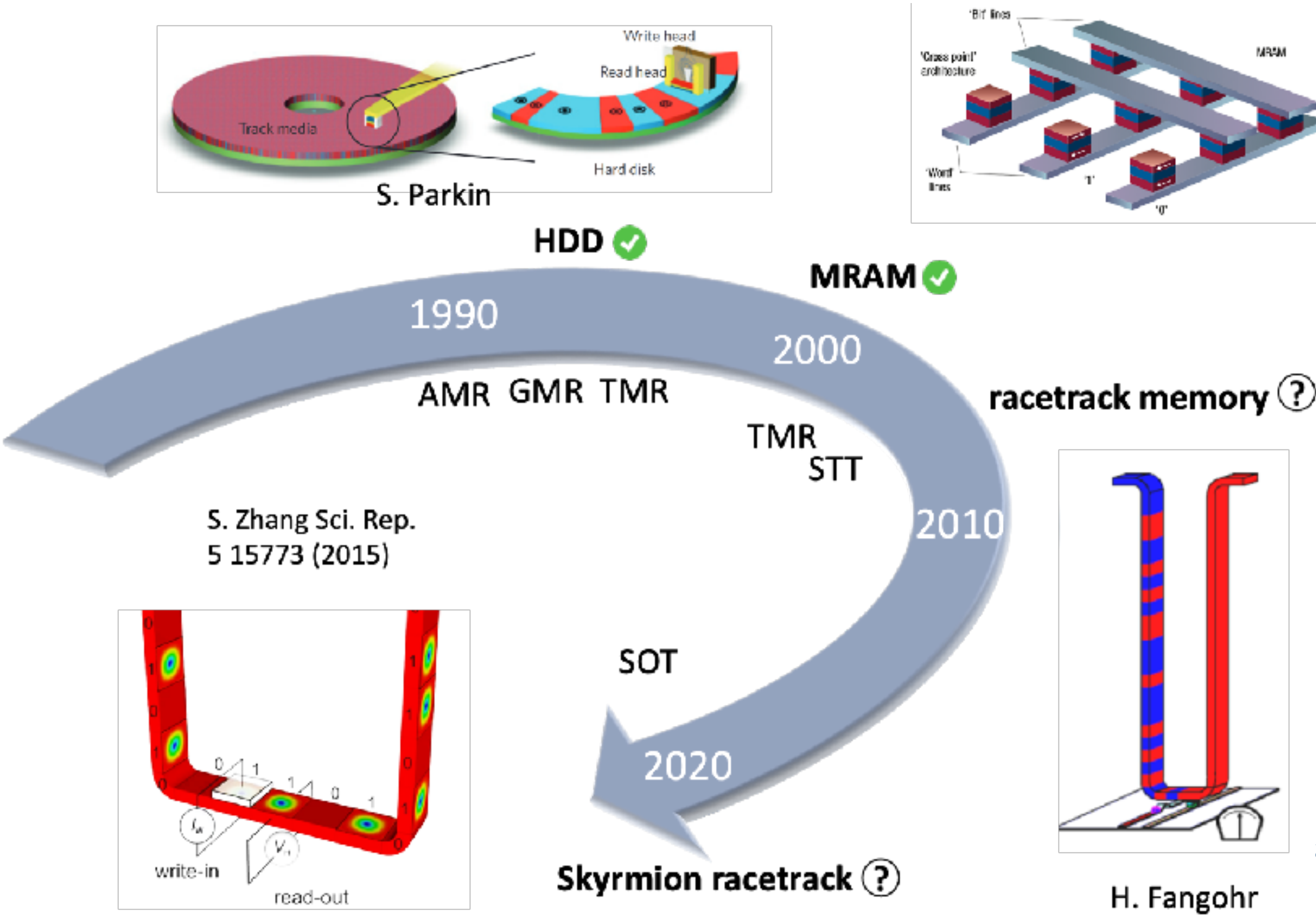




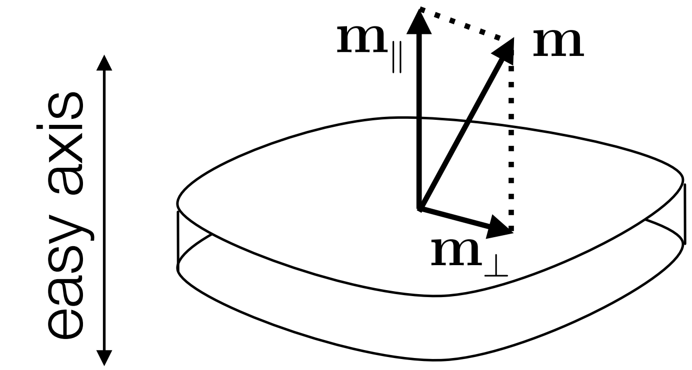


O. Boulle et al., *Nature Nanotechnol.* **11**, 449-455 (2016)
A. Soumyanarayanan et al., *Nature* **539**, 509-517 (2016)
A. Fert et al., *Nature Mat.* **2**, 17031 (2017)

Magnetic memory applications



Micromagnetics of ultrathin films



atomically thin extended ferromagnetic films with PMA and DMI

state variable $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2$ — *normalized magnetization per unit area*

micromagnetic energy: $\mathbf{m} = (\mathbf{m}_\perp, m_\parallel)$ $\quad m_\parallel \to -1 \text{ as } |\mathbf{r}| \to \infty$

$$E(\mathbf{m}) = E_{\mathrm{ex}}(\mathbf{m}) + E_{\mathrm{a}}(\mathbf{m}) + E_{\mathrm{Z}}(\mathbf{m}) + E_{\mathrm{DMI}}(\mathbf{m}) + E_{\mathrm{s}}(\mathbf{m})$$

where:

$$E_{\mathrm{ex}}(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r, \quad E_{\mathrm{a}}(\mathbf{m}) = Q \int_{\mathbb{R}^2} |\mathbf{m}_\perp|^2 d^2 r, \quad E_{\mathrm{Z}}(\mathbf{m}) = -2h \int_{\mathbb{R}^2} (1 + m_\parallel) d^2 r,$$

$$E_{\mathrm{DMI}}(\mathbf{m}) = \kappa \int_{\mathbb{R}^2} \left(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel \right) d^2 r,$$

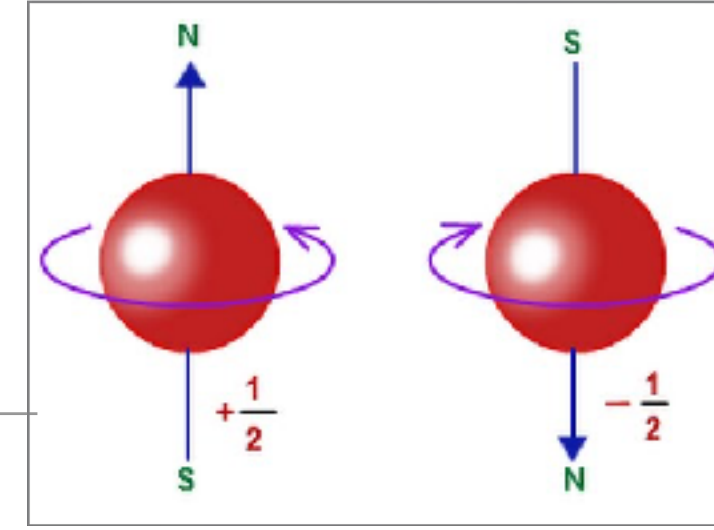
dimensionless parameters:

$$Q = \frac{K_{\mathrm{u}}}{K_{\mathrm{d}}}, \qquad \kappa = \frac{D}{\sqrt{A K_{\mathrm{d}}}}, \qquad h = \frac{H}{M_{\mathrm{s}}},$$

unit of length:

$$\ell = \sqrt{A / K_{\mathrm{d}}}, \qquad K_{\mathrm{d}} = \tfrac{1}{2} \mu_0 M_{\mathrm{s}}^2$$

Stray field energy



electron spins are magnetic dipoles

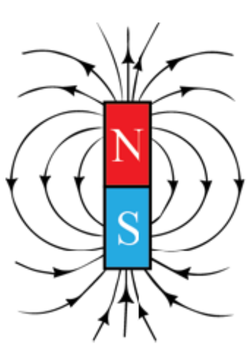
in a thin film the stray field is due to the *bulk* and *surface* magnetic charges:

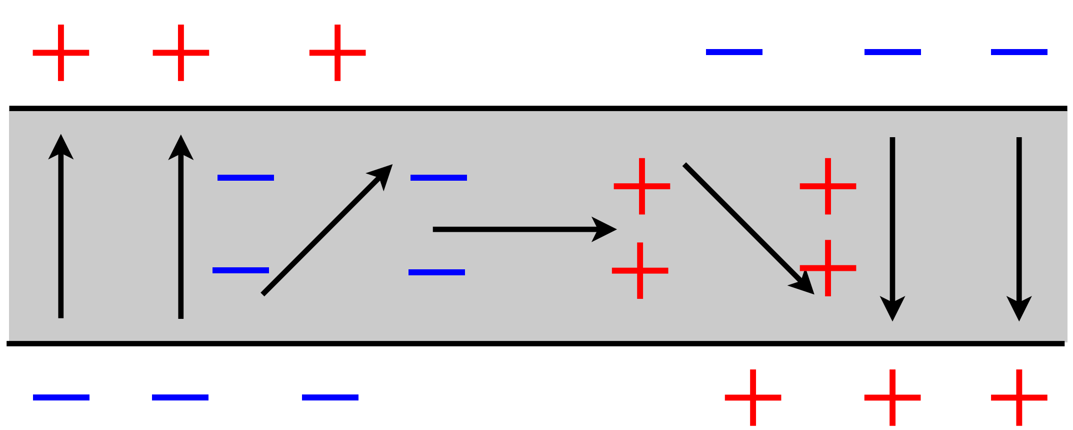
$$E_s(\mathbf{m}) \simeq -\int_{\mathbb{R}^2} |\mathbf{m}_\perp|^2 d^2r + \frac{\delta}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot\mathbf{m}_\perp(\mathbf{r})\cdot\nabla\cdot\mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r d^2r' - \frac{\delta}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

Dietze and Thomas, 1961; Garcia-Cervera, 1999; De Simone et al., 2000; M, 2019; Knüpfer, M and Nolte, 2019

here $\delta \ll 1$ is the *effective* film thickness

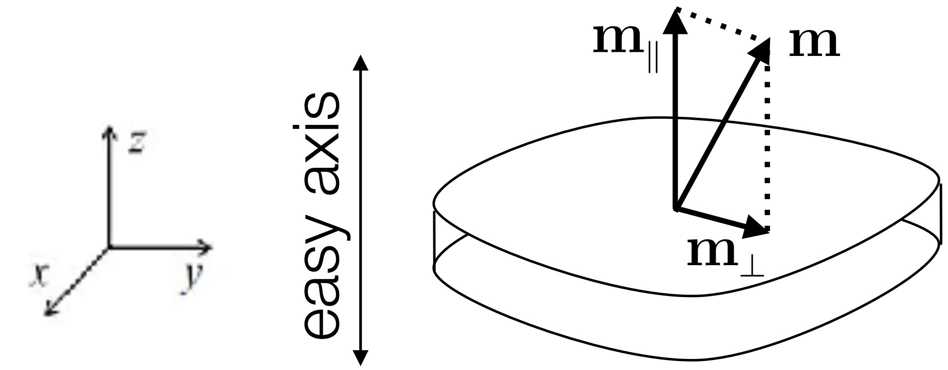
$$|\mathbf{m}_\perp|^2 = 1 - m_\parallel^2$$





$$\rho = -\nabla\cdot\mathbf{m}$$

The minimal model



use the local approximation for the stray field

Winter, 1961; Gioia and James, 1997

two-dimensional micromagnetic energy:

Bogdanov and Yablonskii, 1989
Rohart and Thiaville, 2013
Bernand-Mantel, M and Simon, 2020

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}_{\perp} \cdot \nabla m_{\parallel} + (Q-1)|\mathbf{m}_{\perp}|^2\right) d^2 r \qquad Q > 1$$

for $|\kappa| < \sqrt{Q-1}$ the ground state is $\mathbf{m} = \pm \hat{\mathbf{z}}$. Indeed, for $\mathbf{m} \neq \pm \hat{\mathbf{z}}$

$$E(\mathbf{m}) \geq \|\nabla \mathbf{m}\|_2^2 - 2|\kappa| \cdot \|\mathbf{m}_{\perp}\|_2 \|\nabla m_{\parallel}\|_2 + (Q-1)\|\mathbf{m}_{\perp}\|_2^2 > E(\pm \hat{\mathbf{z}})$$

specify a non-trivial *topological degree*: $m_{\parallel} \to -1$ as $|\mathbf{r}| \to \infty$

$$\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathbf{m} \cdot (\partial_1 \mathbf{m} \times \partial_2 \mathbf{m})\, d^2 r \in \mathbb{Z}$$

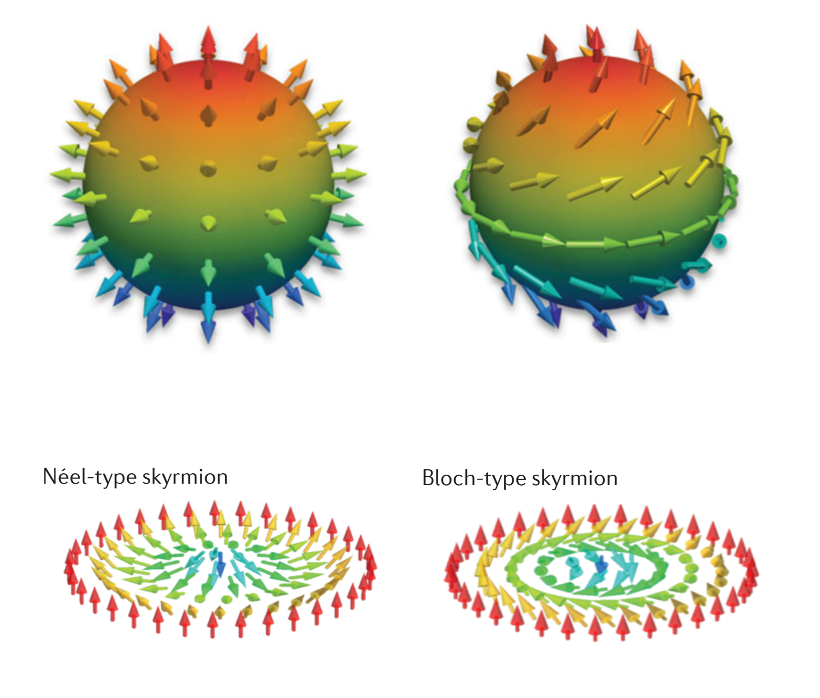
Brezis and Coron, 1983

sharp topological lower bound:

$$\int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r \geq 8\pi \, |\mathcal{N}(\mathbf{m})|$$

$$|\nabla m|^2 \pm 2m \cdot (\partial_1 m \times \partial_2 m) = |\partial_1 m \mp m \times \partial_2 m|^2$$

Magnetic skyrmions



T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

maps $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2$ with non-trivial topology

example: harmonic maps

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r$$

all minimizers with prescribed degree are known — Belavin and Polyakov, 1975

after the stereographic projection, they are rational functions of $z = x + iy$ or their complex conjugates

specifically, all degree 1 minimizing maps belong to:

Eells and Sampson, 1964
Lemaire, 1978
Wood, 1974
Brezis, Coron, 1985

$$\mathcal{B} := \left\{ S\Phi(\rho^{-1}(\bullet - x)) : S \in \mathrm{SO}(3),\ \rho > 0,\ x \in \mathbb{R}^2 \right\}$$

i.e., *dilations, rotations and translations* of:

$$\Phi(x) := \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right)$$

Belavin-Polyakov (BP) profiles

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Compact magnetic skyrmions

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel + (Q-1)|\mathbf{m}_\perp|^2 \right) d^2 r$$

consider the topologically non-trivial admissible class

$$\mathcal{A} := \left\{ \mathbf{m} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \mathcal{N}(\mathbf{m}) = 1, \ \mathbf{m} + \hat{\mathbf{z}} \in L^2(\mathbb{R}^2; \mathbb{R}^3) \right\}$$

note that the last condition simply selects the limit at infinity, since

$$\min \left\{ \int_{\mathbb{R}^2} |m_\parallel + 1|^2 \, d^2 r, \int_{\mathbb{R}^2} |m_\parallel - 1|^2 \, d^2 r \right\} \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 \, d^2 r \int_{\mathbb{R}^2} |\mathbf{m}_\perp|^2 \, d^2 r$$

we have the following *non-optimal* existence result: Bernand-Mantel, M and Simon, 2020

Theorem 1. *Let* $Q > 1$ *and let* $\kappa \in \mathbb{R}$ *be such that* $0 < |\kappa| < \frac{1}{\sqrt{2}}\sqrt{Q-1}$. *Then there exists* $\mathbf{m} \in \mathcal{A}$ *such that*

$$E(\mathbf{m}) = \inf_{\widetilde{\mathbf{m}} \in \mathcal{A}} E(\widetilde{\mathbf{m}}).$$

adapting arguments of
Melcher, 2014
Döring and Melcher, 2017
see also Greco, 2019

<u>Note</u>: no minimizers if $\kappa = 0$ (Derrick-Pohozaev) or $|\kappa| > \frac{4}{\pi}\sqrt{Q-1}$ (*E* unbounded below: stripes)

Compact magnetic skyrmions (cont.)

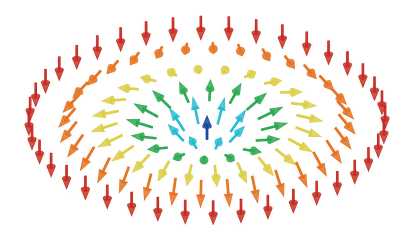
$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel + (Q-1)|\mathbf{m}_\perp|^2 \right) d^2 r$$

conformal limit: $0 < |\kappa| \ll 1\,, \;\; Q > 1$

Bernand-Mantel, M and Simon, 2021

Theorem 2. *Let $Q > 1$, $0 < \kappa \ll 1$ and let $\mathbf{m}_\kappa$ be a minimizer of $E(\mathbf{m}) = E_{\kappa,Q}(\mathbf{m})$ over all $\mathbf{m} \in \mathcal{A}$. Then there exist $\mathbf{r}_\kappa \in \mathbb{R}^2$ and $\rho_\kappa > 0$ such that $\mathbf{m}_\kappa - \Phi(\rho_\kappa^{-1}(\bullet - \mathbf{r}_\kappa)) \to 0$ in $\mathring{H}^1(\mathbb{R}^2; \mathbb{R}^3)$ and $\rho_\kappa \kappa^{-1} \log \kappa^{-1} \to \frac{1}{2}(Q-1)^{-1}$ as $\kappa \to 0$.*

as the $\mathring{H}^1(\mathbb{R}^2)$-norm is translation and dilation invariant, this means that a rescaled and translated minimizing profile $\mathbf{m}_\kappa$ converges to the canonical Belavin-Polyakov profile — *the Néel skyrmion*



relies crucially on the rigidity estimate for degree 1 almost harmonic maps:

$$c \min_{\phi \in \mathcal{B}} \int_{\mathbb{R}^2} |\nabla(\mathbf{m} - \phi)|^2 d^2 r \le \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r - 8\pi$$

for some universal c > 0 and all $\mathbf{m} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) \;:\; \mathcal{N}(\mathbf{m}) = \pm 1$

Bernand-Mantel, M and Simon, 2021; see also Gustafson, Kang and Tsai, 2007; Hirsch and Zemas, 2022; Topping, 2022

Bounded domains

restrict to $\Omega \subset \mathbb{R}^2$ — bounded simply connected domain with C^2 boundary

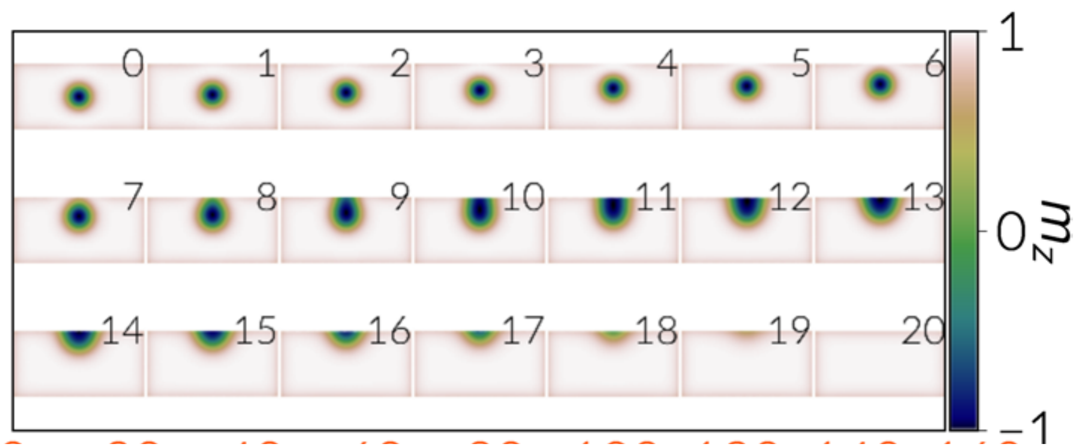
which energy? Thin film limit of 3D micromagnetics: Gioia and James, 1997

$$E(\mathbf{m}) = \int_\Omega \left(|\nabla \mathbf{m}|^2 + \kappa(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) + (Q-1)|\mathbf{m}_\perp|^2\right) d^2r$$

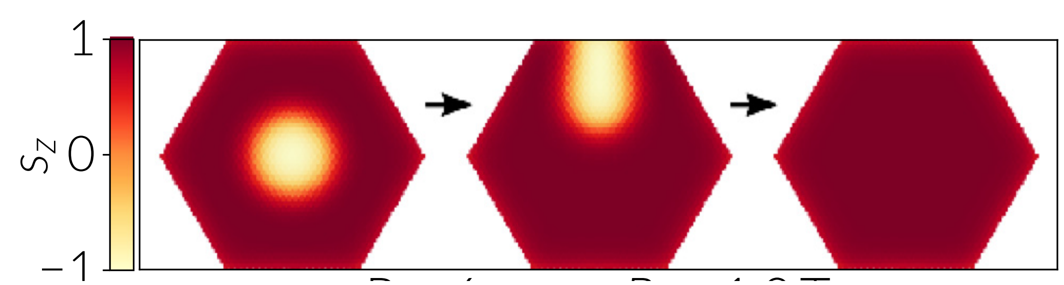
radial critical points when $\Omega = B_R$, but are they minimizers (even local)? Rohart and Thiaville, 2013

the degree is no longer defined => *no topological protection*

escape through the boundary:



Cortes-Ortuño et al., 2017



Cortes-Ortuño et al., 2019

Restoring the topological protection

supplement the energy with Dirichlet b.c.: $\mathbf{m}|_{\partial\Omega} = -\hat{\mathbf{z}}$

defines a non-trivial admissible class

$$\mathcal{A} = \left\{ \mathbf{m} \in H^1(\Omega; \mathbb{S}^2),\ \mathbf{m} = -\hat{\mathbf{z}} \text{ on } \partial\Omega,\ \mathcal{N}(\mathbf{m}) = 1 \right\}$$

where

$$\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi} \int_\Omega \mathbf{m} \cdot (\partial_1 \mathbf{m} \times \partial_2 \mathbf{m})\, d^2 r$$

family of energies:

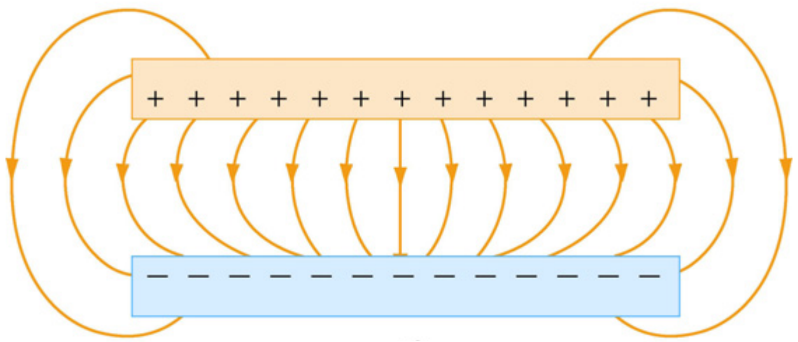
simplify even further by setting $Q = 1$

$$\mathcal{E}_\kappa(\mathbf{m}) = \int_\Omega \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel \right) d^2 r$$

extend the magnetization by $\mathbf{m} = -\hat{\mathbf{z}}$ outside Ω

conformal limit: $0 < |\kappa| \ll 1$ wlog assume $\kappa > 0$

Micromagnetics of the film edge



two-dimensional energy accounting for the dipolar interactions in $\mathbb{R}^2$:

$$
\begin{aligned}
E(\mathbf{m}) = & \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 + \kappa(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) \right\} d^2 r \\
& + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' \\
& - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r'
\end{aligned}
$$

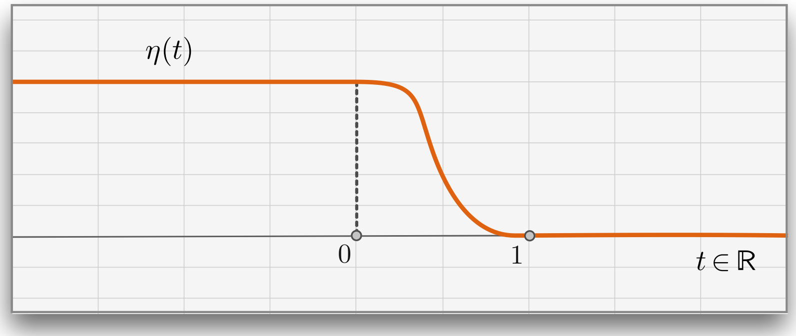
restricting the integrals to Ω would neglect the *fringe fields*

solution: extend $\mathbf{m}$ by zero outside Ω ?

No! - the dipolar energy is generically *undefined* (e.g., for $\mathbf{m} = -\hat{\mathbf{z}}$)

=> regularize the edge with a smooth cutoff: $|\mathbf{m}| = \eta_\delta$

$$\eta_\delta(\mathbf{r}) = \eta(\delta^{-1} \text{dist}(\mathbf{r}, \Omega)) \qquad \delta \ll 1$$



Chaves-O'Flynn and M, 2013; M, Osipov and Vanden-Eijnden, 2015; Lund, M and Slastikov, 2018; Morini et al., 2023+; Di Fratta, M and Slastikov, 2023+

Micromagnetics of the film edge (cont.)

two-dimensional energy with a regularized edge:

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \eta_\delta^2 \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 + \kappa(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) \right\} d^2 r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot (\eta_\delta \mathbf{m}_\perp)(\mathbf{r}) \nabla \cdot (\eta_\delta \mathbf{m}_\perp)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'$$
$$- \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\eta_\delta(\mathbf{r}) m_\parallel(\mathbf{r}) - \eta_\delta(\mathbf{r}') m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r'$$

make the size of Ω depend on $\delta = \delta_\varepsilon$, with a new parameter $\varepsilon \ll 1$, rescale:

$$\Omega^{\delta_\varepsilon} = \varepsilon^{-1} \delta_\varepsilon \, \Omega, \quad Q_\varepsilon = 1 + \frac{\varepsilon |\ln \varepsilon|}{2\pi \gamma_\varepsilon} \alpha, \quad \kappa_\varepsilon = \left(\frac{\varepsilon |\ln \varepsilon|}{2\pi \gamma_\varepsilon} \right)^{1/2} \lambda, \quad \delta_\varepsilon = \left(\frac{2\pi \varepsilon \gamma_\varepsilon}{|\ln \varepsilon|} \right)^{1/2}$$

then after an integration by parts $\qquad \gamma_\varepsilon > 0$

Micromagnetics of the film edge (cont.)

$$\eta_\varepsilon(x) = \eta\left(\frac{\mathrm{dist}(x,\Omega)}{\varepsilon}\right)$$

rescaled two-dimensional energy with a regularized edge: $\mathbf{m} \in H^1_{\mathrm{loc}}(\Omega + B_\varepsilon; \mathbb{S}^2)$

$$\begin{aligned} E_\varepsilon(\mathbf{m}) = & \int_{\mathbb{R}^2} \eta_\varepsilon^2 \left\{ |\nabla \mathbf{m}|^2 + \alpha |m_\perp|^2 + \lambda (m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) \right\} d^2 r \\ & + \frac{\gamma_\varepsilon}{2|\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot (\eta_\varepsilon \mathbf{m}_\perp)(\mathbf{r}) \nabla \cdot (\eta_\varepsilon \mathbf{m}_\perp)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r d^2 r' \\ & - \frac{\gamma_\varepsilon}{2|\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla (\eta_\varepsilon m_\parallel)(\mathbf{r}) \cdot \nabla (\eta_\varepsilon m_\parallel)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r d^2 r' \end{aligned}$$

in the limit $\varepsilon \to 0$ with $\gamma_\varepsilon = \gamma$ we get $\Gamma - \lim E_\varepsilon = E_0$, where

$$\begin{aligned} E_0(\mathbf{m}) := & \int_\Omega \left(|\nabla \mathbf{m}|^2 + \alpha |\mathbf{m}_\perp|^2 \right) d^2 r + \lambda \int_\Omega \left(m_\parallel \nabla \cdot \mathbf{m}_\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel \right) d^2 r \\ & + \gamma \int_{\partial\Omega} \left((\mathbf{m}_\perp \cdot \mathbf{n})^2 - m_\parallel^2 \right) d\mathcal{H}^1(\mathbf{r}) \end{aligned}$$

compare with Kohn and Slastikov, 2006

if $1 \ll \gamma_\varepsilon \ll |\ln \varepsilon|$ => Dirichlet b.c. (after renormalization)

if $\gamma_\varepsilon \ll 1$ => free b.c.

Di Fratta, M and Slastikov, 2023+

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Skyrmions under confinement

$$\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi}\int_\Omega \mathbf{m}\cdot(\partial_1\mathbf{m}\times\partial_2\mathbf{m})\, d^2r$$

minimize

$$\mathcal{E}_\kappa(\mathbf{m}) = \int_\Omega \left(|\nabla\mathbf{m}|^2 - 2\kappa\mathbf{m}_\perp\cdot\nabla m_\parallel\right) d^2r$$

in the class

$$\mathcal{A} = \left\{\mathbf{m}\in H^1(\Omega;\mathbb{S}^2),\ \mathbf{m} = -\hat{\mathbf{z}} \text{ on } \partial\Omega,\ \mathcal{N}(\mathbf{m}) = 1\right\}$$

Theorem 3. *There exists $\kappa_0 > 0$ depending only on Ω such that for all $0 < \kappa < \kappa_0$ there exists a minimizer of $\mathcal{E}_\kappa$ over $\mathcal{A}$.*

note that there is *no minimizer* for $\varkappa = 0$ => limit as $\varkappa \longrightarrow 0$ is singular!

expecting the minimizer to concentrate on a shrinking BP profile

formally optimizing among BP, expect $\inf_{\mathcal{A}} \mathcal{E}_\kappa - 8\pi = O(\kappa^2)$, $\rho = O(\kappa)$

=> study the $\Gamma - \lim_{\kappa\to 0} \frac{\mathcal{E}_\kappa - 8\pi}{\kappa^2}$ in $\mathcal{A}_\kappa = \{m\in\mathcal{A} \ :\ \mathcal{E}_\kappa(m) - 8\pi < 0\}$

Monteil, M, Simon and Slastikov, 2022+

Topology for the Γ-limit

Definition 4. *Let*

$$\widetilde{\mathcal{A}}_0 := \{R_0 \in SO(3) : R_0 \hat{\mathbf{z}} = \hat{\mathbf{z}}\} \times (0, \infty) \times \Omega.$$

We then say that a sequence $m_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *BP-converges to* $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}}_0$ *as* $\kappa_n \to 0$ *if and only if the following holds: There exist* $R_n \in SO(3)$, $\rho_n > 0$, $a_n \in \Omega$ *such that for* $\phi_n := R_n \Phi(\rho_n^{-1}(\bullet - a_n)) \in \mathcal{B}$ *we have*

$$\limsup_{n\to\infty} \kappa_n^{-2} \int_{\mathbb{R}^2} |\nabla(\mathbf{m}_{\kappa_n} - \phi_n)|^2 d^2r < \infty,$$

$$R_0 = \lim_{n\to\infty} R_n, \quad r_0 = \lim_{n\to\infty} \frac{\rho_n}{\kappa_n}, \quad a_0 = \lim_{n\to\infty} a_n.$$

expecting the above definition to be satisfied by the minimizers of the rigidity estimate

$$c \min_{\phi\in\mathcal{B}} \int_{\mathbb{R}^2} |\nabla(\mathbf{m} - \phi)|^2 d^2r \le \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2r - 8\pi$$

$$\mathcal{E}_{\kappa,\lambda}(\mathbf{m}) := \int_\Omega \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}_\perp \cdot \nabla m_\parallel + \frac{\lambda}{|\log \kappa|} |\mathbf{m}_\perp|^2 \right) d^2 r$$

Limit energy

Definition 5. *For* $(R_0, r_0, a_0) \in \widetilde{\mathcal{A}}_0$ *let*

$$\mathcal{E}_0(R_0, r_0, a_0) := r_0^2 T(a_0) - 2r_0 \int_{\mathbb{R}^2} (R_0 \Phi)_\perp \cdot \nabla \Phi_\parallel \, \mathrm{d}x,$$

where the Dirichlet contribution of the tail correction is

$$T(a_0) := \inf_{u \in \mathring{H}^1(\mathbb{R}^2;\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x : u(x) = 2\frac{x - a_0}{|x - a_0|^2} \ \text{in } \mathbb{R}^2 \setminus \Omega \right\}.$$

We furthermore define a restricted admissible set

$$\mathcal{A}_0 := \left\{ (R_0, r_0, a_0) \in \widetilde{\mathcal{A}}_0 : \mathcal{E}_0(R_0, r_0, a_0) < 0 \right\}.$$

anisotropy can be added as a continuous perturbation:

$$\mathcal{E}_0(R_0, r_0, a_0) := r_0^2 (T(a_0) + 8\pi\lambda) - 2r_0 \int_{\mathbb{R}^2} (R_0 \Phi)_\perp \cdot \nabla \Phi_\parallel \, \mathrm{d}x$$

Theorem 6. *The* Γ*-limit as* $\kappa \to 0$ *of the functionals* $\frac{\mathcal{E}_\kappa - 8\pi}{\kappa^2}$ *restricted to* $\mathcal{A}_\kappa$ *with respect to the BP-convergence is given by* $\mathcal{E}_0$ *restricted to* $\mathcal{A}_0$ *in the sense that we have the following:*

(i) *For every sequence of* $\kappa_n \to 0$ *and* $\mathbf{m}_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *with* $\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(\mathbf{m}_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0$ *there exists a subsequence (not relabeled) and* $(R_0, r_0, a_0) \in \mathcal{A}_0$ *such that* $\mathbf{m}_{\kappa_n}$ *BP-converges to* (R_0, r_0, a_0)*.*

(ii) *Let* $\kappa_n \to 0$*, let* $\mathbf{m}_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *BP-converge to* $(R_0, r_0, a_0) \in \mathcal{A}_0$ *and let*

$$\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(\mathbf{m}_{\kappa_n}) - 8\pi}{\kappa_n^2} < 0.$$

Then we have

$$\liminf_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(\mathbf{m}_{\kappa_n}) - 8\pi}{\kappa_n^2} \geq \mathcal{E}_0(R_0, r_0, a_0).$$

(iii) *For every* $(R_0, r_0, a_0) \in \mathcal{A}_0$ *and every sequence of* $\kappa_n \to 0$ *there exist* $\mathbf{m}_{\kappa_n} \in \mathcal{A}_{\kappa_n}$ *BP-converging to* (R_0, r_0, a_0) *such that*

$$\limsup_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(\mathbf{m}_{\kappa_n}) - 8\pi}{\kappa_n^2} \leq \mathcal{E}_0(R_0, r_0, a_0).$$

Convergence of minimizers

$$T(a_0) := \inf_{u \in \mathring{H}^1(\mathbb{R}^2;\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x : u(x) = 2\frac{x-a_0}{|x-a_0|^2} \ \text{in } \mathbb{R}^2 \setminus \Omega \right\}$$

an immediate consequence of the Γ-convergence is:

Theorem 7. *Let $\kappa_n \to 0$ as $n \to \infty$ and let $\mathbf{m}_{\kappa_n}$ be minimizers of $\mathcal{E}_{\kappa_n}$ over $\mathcal{A}$. Then there exists a subsequence (not relabeled) and $a_0 \in \operatorname{argmin}_{a\in\Omega} T(a)$ such that with*

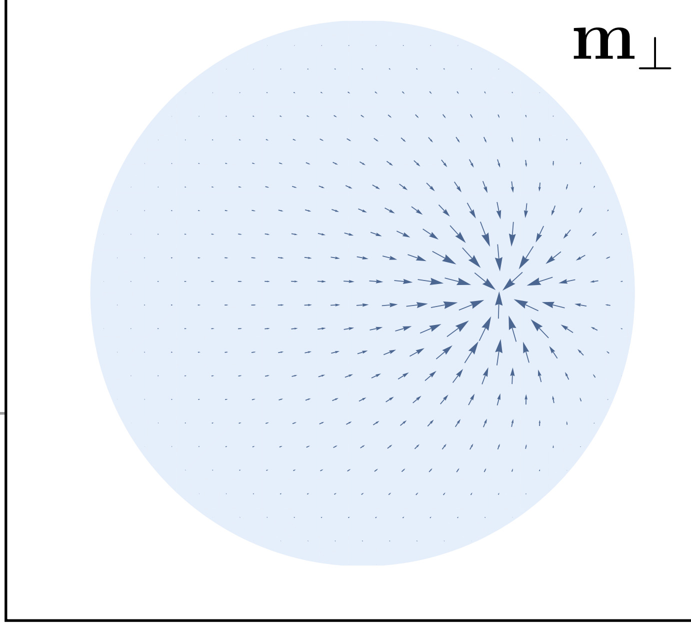
$$r_0 := \frac{4\pi}{T(a_0)} \qquad \text{and} \qquad R_0 := \mathrm{id}$$

we get for $\phi_n := \Phi\left(\frac{\bullet - a_0}{r_0\kappa_n}\right) \in \mathcal{B}$ and all $n \in \mathbb{N}$ that

$$\int_{\mathbb{R}^2} |\nabla(\mathbf{m}_{\kappa_n} - \phi_n)|^2 \, \mathrm{d}x \le C\kappa_n^2 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\mathcal{E}_{\kappa_n}(\mathbf{m}_{\kappa_n}) - 8\pi}{\kappa_n^2} = -\frac{16\pi^2}{T(a_0)}.$$

=> for $\varkappa \ll 1$ every minimizer is close to a *Néel skyrmion* of radius $\varkappa r_0$ centered at a_0

Application: disks



Proposition 8. *For $\Omega = B_\ell(0)$ and $z_0 \in \Omega$, the map achieving $T(z_0)$ is given by*

$$u_{z_0}(z) = \begin{cases} \frac{2z}{\ell^2 - \bar{z}_0 z} & \text{if } z \in B_\ell(0), \\ \frac{2}{\bar{z} - \bar{z}_0} & \text{if } z \in \mathbb{C} \setminus B_\ell(0). \end{cases}$$

Its energy is given by

$$T(z_0) = \frac{16\pi\ell^2}{(\ell^2 - |z_0|^2)^2},$$

which is minimized by $z_0 = 0$ with $T(0) = \frac{16\pi}{\ell^2}$. The rescaled skyrmion radius is $r_0 = \frac{\ell^2}{4}$ and the corresponding limiting energy is $\mathcal{E}_0\left(\mathrm{id}, \frac{\ell^2}{4}, 0\right) = -\pi\ell^2$.

skyrmion goes to the center

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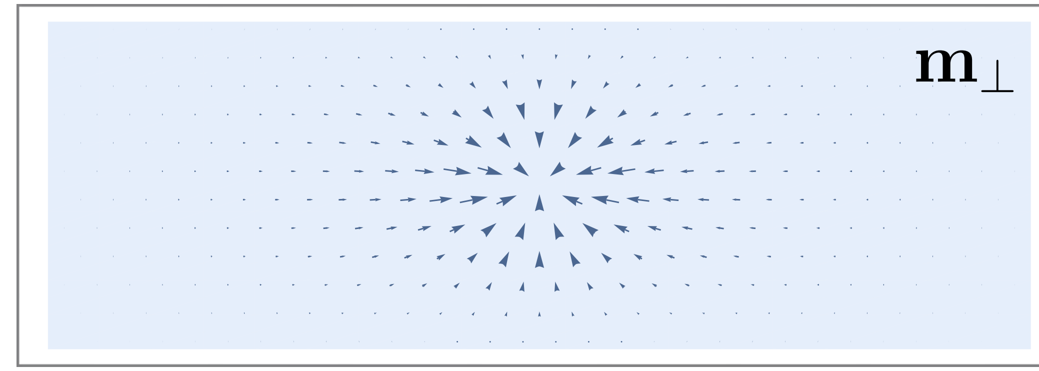
Application: strip



Proposition 9. *For $\ell > 0$, $\Omega_\ell = \mathbb{R} \times (-\ell/2, \ell/2)$, and $y_0 \in (-\ell/2, \ell/2)$, the map achieving $T(iy_0)$ is given by*

$$u_{y_0}(z) = \begin{cases} \frac{\pi}{\ell} \tanh\left(\frac{\pi}{2\ell}(z + iy_0)\right) - \frac{\pi}{\ell} \coth\left(\frac{\pi}{2\ell}(\bar{z} + iy_0)\right) + \frac{2}{\bar{z}+iy_0} & \text{if } z \in \Omega_\ell, \\ \frac{2}{\bar{z}+iy_0} & \text{if } z \in \mathbb{C} \setminus \Omega_\ell. \end{cases}$$

Its energy is given by

$$T(iy_0) = \frac{4\pi^3}{\ell^2 \cos^2\left(\frac{\pi y_0}{\ell}\right)},$$

which is minimized by $y_0 = 0$ with $T(0) = \frac{4\pi^3}{\ell^2}$. The rescaled skyrmion radius is $r_0 = \frac{\ell^2}{\pi^2}$ and the corresponding limiting energy is $\mathcal{E}_0\left(\mathrm{id}, \frac{\ell^2}{\pi^2}, 0\right) = -\frac{4\ell^2}{\pi}$.

skyrmion goes to the midline

Summary

- starting with thin film micromagnetics, obtained the minimal variational model describing a single skyrmion under confinement

- established existence of topologically non-trivial energy minimizing magnetization configurations

- characterized the behavior of degree 1 configurations in the conformal limit

- every degree 1 minimizer in the low DMI regime is close to a single Néel BP profile that is repelled from the sample boundary

- solved for the energy minimizers of the limit problem in several geometries

- **note** a close analogy with the theory of Ginzburg-Landau vortices (but only one vortex/skyrmion up to now)