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Spectral gaps and spectral gap stability for quantum lattice systems

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Three main questions

1. Existence of a gap for specific Hamiltonians.
2. Stability of the gap under ‘gentle’ perturbations (‘universality’).
3. Classification of equivalence classes gapped phases, for example, those defined by gapped curves (Chen-Gu-Wen 2011).

Outline

1. AKLT chain
2. gapped ground state phases
3. stability of the spectral gap
4. AKLT model on the (decorated) honeycomb lattice
5. $O(n)$ spin chains

Example 1: The AKLT chain

The AKLT chain (Affleck-Kennedy-Lieb-Tasaki 1987-88) is the spin-1 chain with nearest neighbor interaction given by the projection onto the spin-2 states:

$$H_{[a,b]} = \sum_{x \in [a,b]} P^{(2)}_{x,x+1}, \quad P^{(2)}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2.$$

$[a,b] \in \mathbb{Z}$, $H_{[a,b]}$ is the Hamiltonian, acts on $\bigotimes_{x\in[a,b]} \mathbb{C}^3$, self-adjoint.

Ground state space is 4-dimensional and given by $\ker H_{[a,b]}$, for all $b > a \in \mathbb{Z}$. AKLT proved that the infinite chain has a unqiue ground state with a spectral gap and exponential decay of correlations (Haldane's Conjecure).

- $\lim_n \langle \psi_n, A\psi_n \rangle = \omega(A)$, independent of the sequence of unit vectors $\psi_n \in \ker H_{[a_n,b_n]}$, $a_n \to -\infty$, $b_n \to \infty$.
- There exists $\gamma > 0$ such that $\mathrm{spec} \ker H_{[a,b]} \subset \{0\} \cup [\gamma, \infty)$, for all $b > a \in \mathbb{Z}$.
- $|\omega(A_x B_y)| \leq 4\|A_x\|\|B_y\| \frac{1}{3}^{|x-y|}$.

The exact ground state is a Matrix Product State (MPS) (Fannes-N-Werner 1989-1992).

AKLT settled Q1 (existence of the gap).

Q2 (stability) was first addressed by Yarotsky (2004), who proved that translation-invariant, finite-range perturbations of the AKLT chain do not close the gap for sufficiently small coupling constants.

$$H(s) = \sum_x P^{(2)}_{x,x+1} + s \sum_{X \subset \mathbb{Z}} \Phi(X).$$

$\Phi(X) = \Phi(X)^*$ acts non-trivially only on spins at $x \in X \subset \mathbb{Z}$. Finite range R: $\Phi(X) = 0$ if $\operatorname{diam} X > R$.

Other proofs and generalizations of stability for the AKLT chain by Michalakis-Zwolak 2013, Szehr-Wolf 2015, Moon-N 2018, Sims-N-Young 2021,

and for other models by Bravyi-Hastings-Michalakis 2010-11, Sims-N-Young 2018, De Roeck-Salmhofer 2019, Hastings 2019, Fröhlich-Pizzo 2018-2020, Del-Vecchio-Fröhlich-Pizzo-Rossi 2020-2022.

Q3 (classification of phases)

One can construct a C^1-curve of projections $P(s)$ such that $P(1) = P^{(2)}$ and the model with nn interaction $P(0)$ has a unique product ground state (for the infinite chain) and prove a uniform positive lower bound for the gap for $s \in [0, 1]$ (Bachmann-N 2014).

This implies that the AKLT chain belongs to the same phase as the model with a unique product ground state (the trivial phase).

In contrast, if we one restricts to interpolations $P(s)$ that respect spin rotation symmetry about 1 axis and an additional $\mathbb{Z}_2$ symmetry, an index argument shows that any curve connecting the AKLT model with a model in the trivial phase, must pass through a phase transition where the gap closes (Tasaki 2018, Ogata 2019-20).

This implies that the AKLT chain belongs to a SPT phase distinct from the trivial phase.

Gapped ground state phases

Quantum spins systems on a finite-dimensional lattice Γ, e.g. $\mathbb{Z}^\nu$. For finite $\Lambda \subset \Gamma$,

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}_x^{d_x}, \quad \mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda), \quad \mathcal{A} = \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}^{\|\cdot\|}.$$

Local Hamiltonians:

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X).$$

Dynamics:

$$\tau_t^{(\Lambda)}(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad \tau_t(A) = \lim_{\Lambda\uparrow\Gamma} \tau_t^{(\Lambda)}(A).$$

Similarly, infinite-volume dynamics also exists for time-dependent Hamiltonians with short-range interactions $\Phi(X,t)$.

Gapped phase

Def: two interactions, Φ_0 and Φ_1, with a (unique) gapped ground state belong to the same gapped phase if there exists a (piecewise) differentiable interpolation $[0,1] \ni s \mapsto \Phi_s$, that is uniformly gapped (Chen-Gu-Wen 2011).

Symmetry Protected / Enhanced gapped phase

Def: Given a symmetry G, defined as 'gapped phase' above, but with G-symmetric Φ_s, for all $s \in [0,1]$ (Pollman-Turner-Berg-Oshikawa, 2010).

There are other definitions in the literature which, under suitable conditions, are equivalent to those above.

See *From Lieb-Robinson bounds to automorphic equivalence*, in Rupert L. Frank, Ari Laptev, Mathieu Lewin, and Robert Seiringer (eds), The Physics and Mathematics of Elliott Lieb, vol. 2, pp. 79–92, European Mathematical Society Press, 2022, arXiv:2205.10460.

Stability of the gap with(out) symmetry breaking

Consider perturbations of frustration-free models with Hamiltonians

$$H_\Lambda(s) = \sum_{x\in\Lambda} h_x + s \sum_{x\in\Lambda, n\geq 0} \Phi(b_x(n))$$

where $h_x \in \mathcal{A}_{b_x(R)}$, $\sup_x \|h_x\| < \infty$; $b_x(r)$ is the ball of radius r, centered at x. $\Phi(b_x(n)) \in \mathcal{A}_{b_x(n)}$, self-adjoint.

Unperturbed model is frustration-free: $h_x \geq 0$ and $\ker H_\Lambda(0) \neq \{0\}$.

Suppose the unperturbed model has N pure ground states: $\omega_1, \ldots, \omega_N$, related by a finite group of symmetries, G, of the Hamiltonian and:
C1: There are $C > 0, q \geq 0$ such that $\mathrm{gap}(H_{b_x(n)}(0)) \geq Cn^{-q}$ (non-zero edge modes do not vanish faster than a power law).
C2: The ω_i are gapped: there exists $\gamma_0 > 0$, s.t., for all A with $\omega_i(A) = 0$,

$$\lim_m \omega_i(A^*[H_{b_0(m)}, A]) \geq \gamma_0 \omega_i(A^* A).$$

C3: $\|\Phi(b_x(n))\| \leq \|\Phi\| e^{-an^\theta}$, for some $a > 0, \theta > 0$.
C4: the $\Phi(b_x(n))$ are G-symmetric.

C5 (LTQO): there are projections $P_1^{(m)}, \ldots, P_N^{(m)} \in \mathcal{A}_{b_x(m)}$, and a function G_0, for which

$$\sum_{n\geq 1} n^{q+3\nu/2}\sqrt{G_0(n)} < \infty.$$

and for all $A \in \mathcal{A}_{b_x(k)}$, $m \geq k \geq 0$, $1 \leq i, j \leq N$,

$$\|P_i^{(m)} A P_j^{(m)} - \delta_{ij}\omega_i(A) P_i^{(m)}\| \leq \|A\|(k+1)^\nu G_0(m-k).$$

For frustration-free spin chains with N pure ground states of Matrix Product form (MPS) related by a spontaneously broken discrete symmetry, C5 always holds with an exponential G_0.

Theorem (Stability of the bulk gap, N-Sims-Young, AHP 2022, arXiv:2102.07209)

If conditions C1-C5 are satisfied, then, for all $\gamma \in (0, \gamma_0)$, there is a constant $\beta > 0$, such that the perturbed model with

$$|s| \leq s_0 := \frac{\gamma_0 - \gamma}{\beta \gamma_0}$$

has N pure ground states $\omega_i^{(s)}$ related by the symmetry G and each with a $\mathrm{gap} > \gamma$. The simplex of ground states spanned by $\{\omega_i^{(s)}\}$ is automorphically equivalent to the unperturbed one:

$$\omega_i^{(s)} \circ \alpha_s(A) = \omega_i(A), \quad A \in \mathcal{A}, 1 \leq i \leq N,$$

for a differentiable curve of quasi-local automorphisms $\alpha_s(A), |s| < s_0$, which commute with the symmetry.

In particular, model stays in same gapped phase for $s < |s_0|$ and the automorphisms $\alpha_s(A)$ can be used to show that certain qualitative characteristics of the phase, such as topological indices, are constant along the curve (Ogata 2020-22). This holds as long as the gap stays open (Bachmann-Michalakis-N-Sims).

A few words about the proof:

- In outline, the proof follows the Bravyi-Hastings-Michalakis (BHM) strategy (BHM 2010, BH 2011, M-Zwolak 2013).
- Uses Hastings' quasi-adiabatic dynamics to transform the Hamiltonian into a form where the perturbation satisfies a relative form bound (Michalakis-Zwolak 2013). The technical challenge was to make this work for the unbounded H^{GNS}.
- The proof avoids the need for a uniform gap estimate for finite systems by perturbing the GNS Hamiltonian.
- Local topological quantum order (LTQO) condition is essential.

Example 3: AKLT model on honeycomb lattice

Affleck, Kennedy, Lieb, and Tasaki (1987-88) introduced a class of nearest neighbor Hamiltonians on regular lattices, later generalized by Kirillov and Korepin (1989) to general graphs G. For each $x \in G$, $\mathcal{H}_x = \mathbb{C}^{d_x}$, with $d_x =$ degree of x +1. The d_x- dimensional irrep of $SU(2)$ acts on $\mathcal{H}_x$ $(d_x = 2j_x + 1)$.
Let $z(e)$ denote the sum of the degrees of the vertices of the an edge e in G. Then

$$H_G^{\mathrm{AKLT}} = \sum_{\text{edges } e \text{ in } G} P_e^{(z(e)/2)},$$

where $P_e^{(j)}$ denoted the orthogonal projection on the states on the edge e of total spin j. Recall

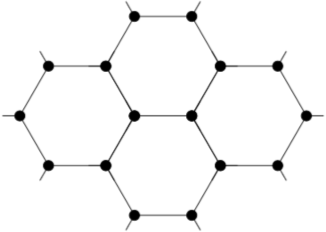
$$V_{j_1} \otimes V_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_j.$$

AKLT model on hexagonal (honeycomb) lattice

At each vertex sits a spin of magnitude $S = 3/2$ ($\mathcal{H}_x = \mathbb{C}^4$).

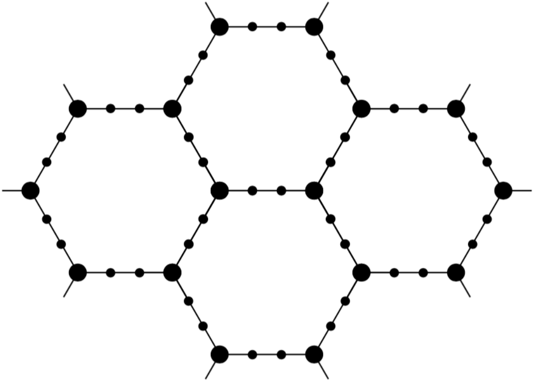
Hamiltonian:

$$H^{AKLT} = \sum_{\text{edges } \{x,y\}} h^{AKLT}_{x,y}.$$



The AKLT on n-decorated honeycomb.

E.g.: 2-decorated hexagonal lattice:



Theorem (Abdul-Rahman-Lemm-Lucia-N-Young 2020)

For all $n \geq 3$, the spectral gap above the ground state of the AKLT model on an n-decorated hexagonal lattice is bounded below by some $\gamma > 0$, independent of lattice size.

Comments and further results

- Q1 (spectral gap) for the AKLT model on the (undecorated) hexagonal lattice: two variations of the arguments above have been used in combination with numerical computation to obtain estmates (Lemm-Sandvik-Wang 2020, Pomata-Wei 2020).
- For decoration number $n \geq 5$, Stability (Q2) was recently proved by Lucia-Moon-Young arXiv:arXiv:2209.01141, by proving LTQO, but Q2 remains open (for now) for the undecorated hexagonal lattice.
- The method generalizes to AKLT models in many other 'decorated' lattices; relies on calculations for finite-dimensional objects. For example, the proof of a spectral gap has been extended to lattices of degree d with sufficiently large decoration number $n(d)$ by Lucia-Young arXiv:2212.11872.

Example 4: $O(n)$ spin chains

Equivalent form of the AKLT interaction by a local unitary change of basis:

$$P^{(2)}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \sim \frac{1}{2}(\mathbb{1} + T_{x,x+1} - 2Q_{x,x+1}),$$

where $T_{x,x+1}$ is the swap operator and $Q_{x,x+1}$ is the projection onto $\frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_0 \otimes e_0 + e_{-1} \otimes e_{-1})$.

T and Q generalize to n-dimensional spins and arbitrary coupling constants as follows

$$uT + vQ, \quad u, v \in \mathbb{R}$$

where Q is the projection to

$$\psi = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle.$$

Both T and Q commute with the natural action of $O(n)$ on the spins in this basis. It is the general $O(n)$ invariant nearest neighbor interaction for $n \geq 2$, which was studied by Tu & Zhang, 2008.

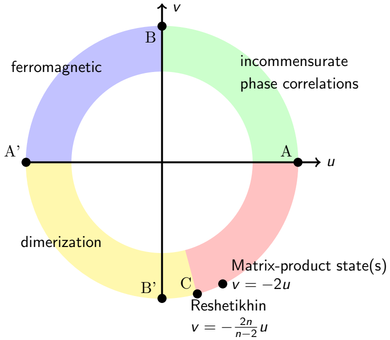


Figure: Ground state phase diagram for the chain with nearest-neighbor interactions $uT + vQ$ for $n \geq 3$.

- $v = -2nu/(n-2)$, $n \geq 3$, Bethe ansatz point (Reshetikhin, 1983)
- $v = -2u$: frustration free point, equivalent to $\perp$ projection onto symmetric vectors $\ominus$ one. Unique g.s. if n odd; two 2-periodic g.s. for even n; spectral gap in all cases and stable phase (N-Sims-Young, 2022).
- $u = 0, v = -1$. Equivalent to the $SU(n) - P^{(0)}$ models aka Temperley-Lieb chain; Affleck, 1990, Nepomechie-Pimenta 2016). Dimerized for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020). Proof of open region for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021). Proof of 2 distinct 2-periodic phases for even n (N-Ragone, in prep).

Dimerization and spectral gap for large n

(Björnberg-Mühlbacher-N-Ueltschi, CMP2021)

Model: chain of n-dimensional spins with $O(n)$-invariant nearest neighbor interaction $h = uT + vQ$, $u, v \in \mathbb{R}$, T is the swap operator and Q projects onto $\psi = n^{-1/2} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle$.

Finite chains of 2ℓ spins, with Hamiltonian: $H_\ell = \sum_{x=-\ell+1}^{\ell-1} h_{x,x+1}$.
Consider ground states as limits of Gibbs states:

$$\langle A \rangle_{\ell,\beta,u} = \frac{\mathrm{Tr} A e^{-\beta H_\ell}}{\mathrm{Tr} e^{-\beta H_\ell}}.$$

Basic observables: generators of $SO(n)$:

$$L^{\alpha,\alpha'} = |\alpha\rangle\langle\alpha'| - |\alpha'\rangle\langle\alpha|, 1 \leq \alpha < \alpha' \leq n.$$

Theorem (Dimerization)

There exist constants $n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$, $v = -1$, and $|u| < u_0$, we have that for all $1 \leq \alpha < \alpha' \leq n$,

$$\lim_{\beta\to\infty} \left[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \right] > c \quad \text{for } \ell \text{ odd;}$$

$$\lim_{\beta\to\infty} \left[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \right] < -c \quad \text{for } \ell \text{ even.}$$

Let $E_0^{(\ell)} < E_1^{(\ell)} < \dots$ be the eigenvalues of $H_{[-\ell+1,\ell]}$, and define the ground state gap $\Delta^{(\ell)}$ by

$$\Delta^{(\ell)} = E_1^{(\ell)} - E_0^{(\ell)}.$$

The gap is obviously positive but is there is a positive lower bound independent of ℓ?

Theorem (Spectral gap)

There exist constants $n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$, $v = -1$, and $|u| < u_0$, we have

(a) $E_0^{(\ell)}$ *is non-degenerate.*

(b) $\Delta^{(\ell)} \geq c$ *for all ℓ.*

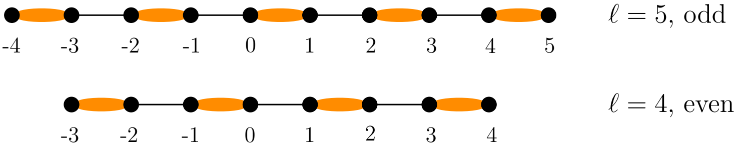
The frustration-free point

(N-Ragone, in prep)

n-dimensional spins with nn interaction $\frac{1}{2}T - Q$.

- odd $n \geq 3$: similar to $n = 3$, the AKLT model with a unique ground state, obviously different phase than the point $v = -1, u = 0$, where dimerization leads to 2 ground pure states.
- even $n \geq 4$: two pure ground with broken translation invariance, gapped, and stable, rather like in the case $v = -1, u = 0$. Question: same gapped phase or different phases?.

$v = -1, u = 0$ has dimerization: monogamy of entanglement sets up a competition between pairings, leading to spontaneous breaking of translation symmetry.



For the $O(n)$ chains maximally entangled pairs dominate for large n.

For even n, nn interaction $\frac{1}{2}T - Q$, the two ground states are not dimerized, actually related by a local unitary: any $O \in O(n)$, with $\det O = -1$. Symmetry-breaking is Ising-like, with unbroken $SO(n)$ symmetry.

ω_1, ω_2, 2-periodic,

$$\omega_2(A) = \omega_1(\sigma(A)), \quad A \in \mathcal{A},$$

where σ is the translation by one unit on the chain. And also

$$\omega_2(A) = \omega_1((\otimes O^*)A(\otimes O)), \quad A \in \mathcal{A}.$$

Unexpected property:

$$\omega_1(A) = \omega_2(A), \quad A \in \mathcal{A}_{[1,n/2-1]},$$

Since the ground states of the $-Q$ model have the full $O(n)$ symmetry, it represents a distinct phase.

Comments and Outlook

- There are many more examples of models for which proofs a spectral gap exist, but not as many as we would like! Most are frustration free models, though not all.
- It would be interesting to prove restricted forms of stability when there is no general stability. Has been done for systems with discrete symmetry breaking (N-Sims-Young 2022).
- The classification of gapped phase has also seen a lot of progress in the past few years (Ogata 2020-22, Kapustin et al 2020-22, others).