Asymptotic expansions for spin $O(N)$ models

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Based on work with A. Giuliani (on arXiv in a few days).

Spin $O(N)$ models

Definition

Probability measure on $\Omega_L = (\mathbb{S}^{N-1})^{\Lambda_L}$, $\Lambda_L = \{0, 1, \cdots, L-1\}^d$ the d-dimensional torus. Will suppose $N \geq 2$.

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Parameters : inverse temperature $\beta = 1/T \geq 0$, magnetic field $h \geq 0$.

Given by :

$$\mu_{L;\beta,h}(f) = \frac{1}{Z_{L;\beta,h}} \int_{\Omega_L} \prod_i d\nu_N(S_i) f(S) e^{\beta \sum_{i \sim j} S_i \cdot S_j + h \sum_i S_i^N}$$

ν_N the uniform measure on $\mathbb{S}^{N-1}$.

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Phase transition (symmetry breaking/long range order/spontaneous magnetization) in dimensions $d \geq 3$. *But an $O(N-1)$ symmetry survives.*

Infinite volume measures

Denote

$$\mu_{\beta,h} = \lim_{L\to\infty} \mu_{L;\beta,h}, \quad \mu_\beta = \lim_{h\to 0} \mu_{\beta,h}$$

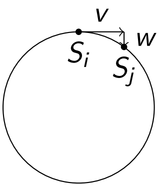
Limits have to be taken along suitable subsequences to have some form of ergodicity.

The spin wave picture

The Boltzmann weight is proportional to

$$\exp\Big(-\frac{\beta}{2}\sum_{i\sim j}|S_i - S_j|^2\Big).$$

Locally,



One has $|S_i - S_j|^2 = |v|^2 + |w|^2 \simeq |v|^2 + O(|v|^4)$ for small $|v|$. $\to$ locally a Gaussian in the *tangent space* (of dimension $N-1$).

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At small T one expects small gradients. $\rightarrow$ locally a Gaussian free field with spin dimension $N - 1$.

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One way to make sense of this : expand correlation functions in Taylor series around $T=0$, hope it is valid and check the value of the coefficients.

Formal low temperature expansion : XY example

When $N = 2$, ν_2 is the image of the uniform measure on $(-\pi, \pi]$ by $S_i(\theta) = (\sin(\theta_i), \cos(\theta_i))$. In particular, $S_i \cdot S_j = \cos(\theta_i - \theta_j)$.
Expanding the cos yields

$$\beta \sum_{i \sim j} S_i \cdot S_j = \beta \sum_{i \sim j} \sum_{k \geq 0} \frac{(-1)^k}{(2k)!} (\theta_i - \theta_j)^{2k}$$
$$= \beta \sum_{i \sim j} 1 - \frac{1}{2} \sum_{i \sim j} (\phi_i - \phi_j)^2 +$$
$$+ \sum_{k \geq 2} \frac{(-1)^k}{(2k)!} T^{k-1} \sum_{i \sim j} (\phi_i - \phi_j)^{2k}$$

where $\phi = \sqrt{\beta}\theta$.

Formal low temperature expansion : XY example

Looking at the last line

$$\beta \sum_{i \sim j} 1 - \frac{1}{2} \sum_{i \sim j} (\phi_i - \phi_j)^2 + \sum_{k \geq 2} \frac{(-1)^k}{(2k)!} T^{k-1} \sum_{i \sim j} (\phi_i - \phi_j)^{2k}$$

Constant - Gaussian + Perturbation

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Setting $W_T = \sum_{k\geq 2} \frac{(-1)^k}{(2k)!} T^{k-1} \sum_{i\sim j} (\phi_i - \phi_j)^{2k}$, (and ignoring the fact that $\phi_i \in (-\sqrt{\beta}\pi, \sqrt{\beta}\pi])$, one can perform the Gaussian integral to obtain a (formal) power series in T.

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$$\beta \sum_{i \sim j} 1 - \frac{1}{2} \sum_{i \sim j} (\phi_i - \phi_j)^2 + \sum_{k \geq 2} \frac{(-1)^k}{(2k)!} T^{k-1} \sum_{i \sim j} (\phi_i - \phi_j)^{2k}$$

Constant - Gaussian + Perturbation

Problem : the series is not convergent as the Gaussian propagator (covariances) decay too slowly :

$$G_{ij} \asymp |i - j|^{2-d}.$$

Asymptotic expansions : existing results

Gawedzki and Kupiainen, 1980 : asymptotic expansion of free energy/correlation in scalar $T(\nabla\phi)^4$ model via block-spin renormalization.

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Garban and Sepúlveda, 2021 : in $d = 2$ Villain model, bounds on the (super-polynomial) correction to Gaussian behaviour.

Expansions via Infrared bounds : the BFLLS procedure ($d \geq 3$)

Preparation 1 : Gaussian Integration by Parts

Let A be an $n \times n$ covariance matrix and E_A the associated Gaussian expectation. Then, for any $m > 0$,

$$E_A(\varphi_1 F(\varphi)) = \sum_{k=1}^{n} A^m_{1k} E_A(\partial_k F(\varphi)) + m^2 \sum_{k=1}^{n} A^m_{1k} E_A(\varphi_k F(\varphi)),$$

where $A^m = (A^{-1} + m^2)^{-1}$.

Preparation 2 : Infrared bound

Theorem (Fröhlich, Simon, and Spencer, 1976)

For any $h, \beta \geq 0$, and any f with finite support,

$$\mu_{\beta,h}\left(e^{\sum_x (S_x - \mu_{\beta,h}(S_x))\cdot f(x)}\right) \leq e^{\frac{1}{2\beta}(f,Gf)}. \tag{1}$$

Moreover,

$$\mu_{\beta,h}(S_0^N)^2 \geq 1 - NTG_{00}. \tag{2}$$

G the Green function of the Laplacian on $\mathbb{Z}^d$.

Preparation 2 : Infrared bound

From this, one deduces :

Theorem

There exists $C < \infty$ independent of $T < 1$ such that

$$\mu_\beta\left(e^{\sqrt{\beta}|S_0^k|}\right) \leq C, \quad \mu_\beta\left(e^{\sqrt{\beta}(1-S_0^N)}\right) \leq C.$$

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Which, after some manipulations, gives (in the case of XY) that $\phi = \sqrt{\beta}\theta$ has moments of all order *bounded uniformly in* $T > 0$.

The procedure

Suppose we want to compute $\mu_\beta\big(\cos(\theta_0 - \theta_{\mathrm{e}_1})\big)$ to order 2. One first expands

$$\mu_\beta\big(\cos(\theta_0 - \theta_{\mathrm{e}_1})\big) = \sum_{k\geq 0} \frac{(-1)^k}{(2k)!} T^k \mu_\beta\big((\phi_0 - \phi_{\mathrm{e}_1})^{2k}\big)$$

$$= 1 - \frac{T}{2}\mu_\beta\big((\phi_0 - \phi_{\mathrm{e}_1})^2\big) + \frac{T^2}{24}\mu_\beta\big((\phi_0 - \phi_{\mathrm{e}_1})^4\big) + O(T^3),$$

by the previous uniform bound on moments.

The procedure

Then, (formally and forgetting $\phi \in (-\sqrt{\beta}\pi, \sqrt{\beta}\pi]$)

$$\mu_\beta(f) = \frac{E_G(fe^{W_T})}{E_G(e^{W_T})}$$

(recall $W_T = \sum_{k\geq 2} \frac{(-1)^k}{(2k)!} T^{k-1} \sum_{i\sim j}(\phi_i - \phi_j)^{2k}$). So, using (regularized) Gaussian Integration by Parts,

$$\begin{aligned} \mu_\beta((\phi_0 - \phi_{e_1})^2) &= E_{G^m}((\varphi_0 - \varphi_{e_1})^2) \\ &+ \sum_x E_{G^m}(\varphi_x(\varphi_0 - \varphi_{e_1}))\mu_\beta((\phi_0 - \phi_{e_1})\partial_x W_T) \\ &+ m^2 \sum_x E_{G^m}(\varphi_x(\varphi_0 - \varphi_{e_1}))\mu_\beta((\phi_0 - \phi_{e_1})\phi_x) \end{aligned}$$

for any $m > 0$. Massive propagator gives convergence.

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The procedure

Studying W_T,

$$\partial_x W_T = \sum_{k\geq 1} \frac{(-1)^{k+1}}{(2k+1)!} T^k \sum_{\xi} (\phi_x - \phi_{x+\xi})^{2k+1}.$$

So,

$$\sum_x E_{G^m}\big(\varphi_x(\varphi_0 - \varphi_{\mathrm{e}_1})\big)\mu_\beta((\phi_0 - \phi_{\mathrm{e}_1})\partial_x W_T) =$$

$$= \sum_{k\geq 1} a_k T^k \sum_x E_{G^m}\big(\varphi_x(\varphi_0 - \varphi_{\mathrm{e}_1})\big) \sum_\xi \mu_\beta\big((\phi_0 - \phi_{\mathrm{e}_1}))(\phi_x - \phi_{x+\xi})^{2k+1}\big)$$

$$= \sum_{k\geq 1} a_k T^k \sum_x \sum_{\mathrm{e}} E_{G^m}\big((\varphi_x - \varphi_{x+\mathrm{e}})(\varphi_0 - \varphi_{\mathrm{e}_1})\big) \times$$

$$\times \mu_\beta\big((\phi_0 - \phi_{\mathrm{e}_1})(\phi_x - \phi_{x+\mathrm{e}})^{2k+1}\big).$$

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Other key estimates are

$$\sum_x E_{G^m}(\varphi_0\varphi_x) = m^{-2}, \quad \sum_x \left|E_{G^m}\big((\varphi_0 - \varphi_{\mathrm{e}})\varphi_x\big)\right| \leq cm^{-1},$$
$$\sum_x \left|E_{G^m}\big((\varphi_0 - \varphi_{\mathrm{e}})(\varphi_x - \varphi_{x+\mathrm{e}'})\big)\right| \leq c|\log m|.$$

Removing the mass

Using these give

$$\begin{aligned}\mu_\beta((\phi_0-\phi_{\mathrm{e}_1})^2) &= E_{G^m}\big((\varphi_0-\varphi_{\mathrm{e}_1})^2\big)\\ &+\sum_{k\geq 1} a_k T^k \sum_x \sum_{\mathrm{e}} E_{G^m}\big(\nabla_x^{\mathrm{e}}\varphi\nabla_0^{\mathrm{e}_1}\varphi\big)\mu_\beta\big(\nabla_0^{\mathrm{e}_1}\phi(\nabla_x^{\mathrm{e}}\phi)^{2k+1}\big)\\ &+ m^2\sum_x E_{G^m}\big(\varphi_x(\varphi_0-\varphi_{\mathrm{e}_1})\big)\mu_\beta((\phi_0-\phi_{\mathrm{e}_1})\phi_x)\\ &+\mathrm{error}(0).\end{aligned}$$

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&+ m^2 \sum_x E_{G^m}\big(\varphi_x(\varphi_0 - \varphi_{\mathrm{e}_1})\big)\mu_\beta((\phi_0 - \phi_{\mathrm{e}_1})\phi_x) \\
&+ \mathrm{error}(m).
\end{aligned}
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$$+ \mathrm{error}\big(m + m^2 m^{-1}\big).$$

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$$+ \mathrm{error}\big(m + m^2 m^{-1} + T^{n+1} |\log(m)|\big).$$

Choosing $m = e^{-(\log \beta)^2}$ does the job !

Limitations

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A fix : one can treat non-gradient correlations by using quantitative bounds on the decay of correlations : e.g.

$$|\mu_\beta(\cos(\theta_0))^2 - \mu_\beta(\cos(\theta_0 - \theta_x))| \leq \frac{c \log |x|}{\beta |x|}$$

using a bound deduced from Reflection-Positivity and the Infrared.

Expansions via Infrared bounds :
$N > 2$ case

Preparation 3 : (u, θ)-coordinates

Generalization of cylindrical coordinates :

$$(S^1, \cdots, S^N) = \left(u^1, \cdots, u^{N-2}, \sqrt{1-|u|^2}\sin\theta, \sqrt{1-|u|^2}\cos\theta\right),$$

$u \in \mathbb{R}^{N-2}, |u| \leq 1$, $\theta \in (-\pi, \pi]$. S uniform on $\mathbb{S}^{N-1}$ equivalent to θ uniform on $(-\pi, \pi]$ and u uniform on $\{v \in \mathbb{R}^{N-2} : \ |v| \leq 1\}$.

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Rescale them as :

$$\tilde{u} = \sqrt{\beta} u, \quad \phi = \sqrt{\beta}\theta.$$

In these coordinates,

$$\beta \sum_{i \sim j} S_i \cdot S_j =$$

$$= \sum_{i \sim j} \tilde{u}_i \cdot \tilde{u}_j + \beta \sqrt{1 - T\tilde{u}_i^2}\sqrt{1 - T\tilde{u}_j^2}\cos(\sqrt{T}\phi_i - \sqrt{T}\phi_j).$$

Preparation 4 : A priori decay of correlations

As in the XY case, by Infrared and Reflection Positivity,

$$\mu_\beta(S_i^k S_j^k) \leq \frac{c \log|i-j|}{\beta|i-j|}, \quad \mu_\beta(S_i^N; S_j^N) \leq \frac{c \log|i-j|}{\beta|i-j|},$$

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Still, one can get

$$\mu_\beta(\phi_i \phi_j) \leq \frac{c \log|i-j|}{|i-j|} + \mathrm{error},$$

with error as small as we want as a function of T *but independent of* $|i-j|$. The same type of bounds hold for general correlations.

New procedure : example

Change of variable

Try to expand the magnetization $\mu_\beta(S_0^N)$ to first order. First,

$$\mu_\beta(S_0^N) = \mu_\beta\big(\sqrt{1 - T|\tilde{u}_0|^2}\cos(\sqrt{T}\phi_0)\big)$$

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&= 1 - \frac{T}{2}\mu_\beta(|\tilde{u}_0|^2) - \frac{T}{2}\mu_\beta(\phi_0^2) + O(T^2) \\
&= 1 - \frac{T(N-2)}{2}\mu_\beta((\tilde{u}_0^1)^2) - \frac{T}{2}\mu_\beta(\phi_0^2) + O(T^2).
\end{aligned}$$

Change of variable

Moreover, using the remaining $O(N-1)$ symmetry,

$$\mu_\beta((\tilde{u}_0^1)^2) = \mu_\beta((\sqrt{\beta} S_0^1)^2)$$

.

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So,

$$\mu_\beta(S_0^N) = 1 - \frac{T(N-1)}{2}\mu_\beta(\phi_0^2) + O(T^2).$$

Integration by part formula

For F function of ϕ, $\mathcal{F}$ function of $\tilde{u}$, and $m > 0$

$$\mu_\beta(\phi_0 F\mathcal{F}) = \sum_y G^m_{0y}\mu_\beta(\partial_y F\mathcal{F}) + m^2\sum_x G^m_{0x}\mu_\beta(\phi_x F\mathcal{F}) +$$

$$+\sum_x\sum_{\mathrm{e}} E_{G^m}\big(\varphi_0\nabla^{\mathrm{e}}_y\varphi\big)\mu_\beta\big(F\mathcal{F}[\nabla^{\mathrm{e}}_x\phi - \rho_x\rho_{x+\mathrm{e}}\sqrt{\beta}\sin(\sqrt{T}\nabla^{\mathrm{e}}_x\phi)]\big)$$

where $\rho_x = \sqrt{1-|u_x|^2}$, and the constraint $\phi \in (-\sqrt{\beta}\pi, \sqrt{\beta}\pi]$ has been ignored.

Controlling the error terms

Back to the example, choosing m appropriately (i.e. : suitable power of T)

$$\mu_\beta(\phi_0^2) = G^m_{00} + m^2 \sum_x G^m_{0x}\mu_\beta(\phi_x\phi_0) +$$

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Iterate.

Ongoing

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Treat gradient observables in $d = 2$ (a bit less soon).

Thank You !