Universality in Condensed Matter and Statistical Mechanics

Universality of the magnetization ripple: A singular SPDE-perspective

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Randomness in ferromagnets on mesoscopic level

Thermal noise due to positive temperature

statistical mechanics:

space-time white noise

Example: thermal switching in small elements

Quenched noise due to material inhomogeneities

reduced model:

spatial white noise

Example: magnetization ripple in films (Harte '68, Hoffmann '68)

$\rightsquigarrow$ singular **S**tochastic **P**artial **D**ifferential **E**quations

"singular" = noise too rough for naive treatment of nonlinearity

A model for the ripple in thin-film ferromagnets

Magnetization m is of unit length $|m|^2 = 1$,
in-plane (m_1, m_2), constant in thickness direction $m(x_1, x_2)$.

Short range attraction via penalization of $\nabla m \in L^2$,
long-range repulsion via penalization of $\nabla \cdot m \in \dot{H}^{-\frac{1}{2}}$.

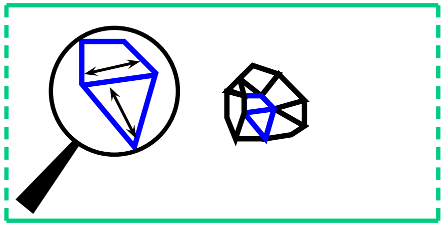
Crystalline anisotropy: m couples to lattice,
$m \approx (1, 0)$ effectively experiences random field $(0, \xi)$;
geometric approximation $m \approx (1 - \frac{1}{2}u^2, u)$,
anisotropic rescaling of x_1, x_2:

$$\int dx_1 dx_2 (\partial_1 u)^2 + (|\partial_1|^{-\frac{1}{2}}(-\partial_1 \tfrac{1}{2}u^2 + \partial_2 u))^2 - 2\xi u$$

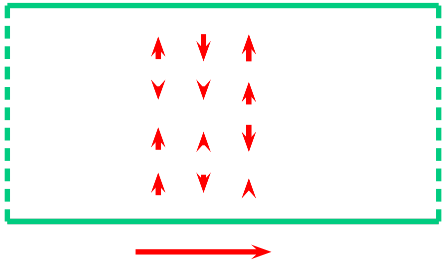
Impose (artificial) periodicity $\mathbb{R}^2 \rightsquigarrow [0, 1)^2$

Model validated by experiment

small grains,
random easy axis

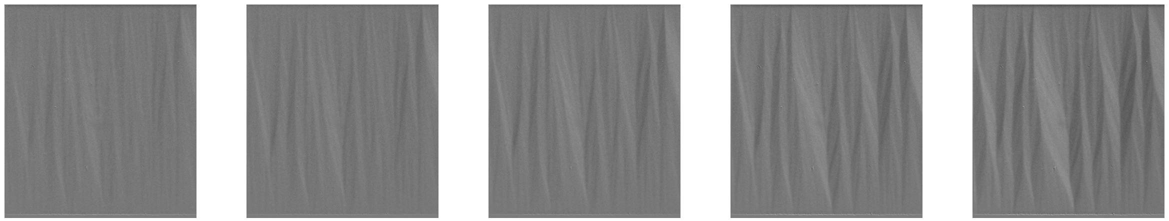


random external
transversal field



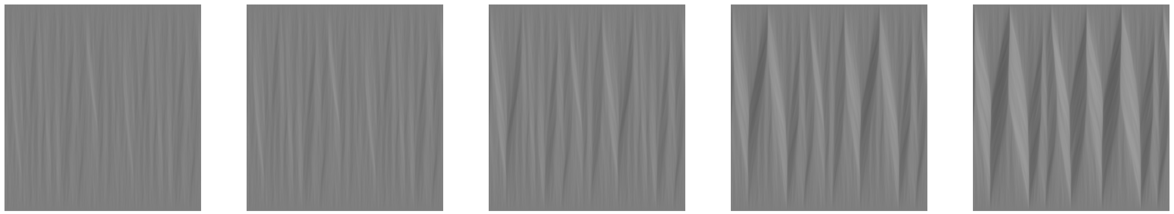
Experiments

u via gray scale



Simulation

of reduced model



Steiner, Schäfer, Wieczoreck, McCord, O., Phys. Rev. B '12

Three issues with the model

As $\xi \to$ white noise ($\Longleftrightarrow$ grain size $\ll$ ripple scales),

Problem 1: $\min E \to -\infty$.
Fix 1: $E_{ren} := E - E(v)$ where v solves linearized problem.

Problem 2: coercivity of E_{ren} unclear.
Fix 2: super-quadratic coercivity from Burger's nonlinearity.

Problem 3: term in E_{ren} features a singular product.
Fix 3: stochastic construction à la rough paths.

Goal "universality": Define random variational problem
that is the suitable limit under
any reasonable approximation of white noise.

Problem 1: $\min E \downarrow -\infty$

Consider linearized problem

$$E_{lin}(v) := \int_{[0,1)^2} dx\Big((\partial_1 v)^2 + (|\partial_1|^{-\frac{1}{2}}\partial_2 v)^2 - 2\xi v\Big).$$

Claim: As $\xi \to$ white noise, $\mathbb{E}\min E_{lin} \downarrow -\infty$.

In terms of Fourier series $\mathcal{F}$ and wave number $k \in 2\pi\mathbb{Z}^2$, $k_1 \neq 0$:

$$(k_1^2 + |k_1|^{-1}k_2^2)\mathcal{F}v(k) = \mathcal{F}\xi(k).$$

Energy of minimizer: $-\sum_k (k_1^2 + |k_1|^{-1}k_2^2)^{-1}|\mathcal{F}\xi(k)|^2$.

Its expected value $\mathbb{E}$ in case of white noise:

$$-\sum_k (k_1^2 + |k_1|^{-1}k_2^2)^{-1} = -\infty.$$

Fix 1: renormalize energy

Recall: As $\xi \to$ white noise, $\mathbb{E} \min E_{lin} \downarrow -\infty$.

Let v be the minimizer of E_{lin}.

Consider renormalized $E_{ren}(w) := E(v+w) - E(v)$.

E_{ren} is of multi-linear form:

$B(w,w) +$ linear

$+ \ C(w,w,w) + 3C(v,w,w) + 3C(v,v,w)$

$+ \ Q(v+w, v+w, v+w, v+w) - Q(v,v,v,v)$,

– coercivity unclear.

Problem 2: coercivity?

Fix 2: conservative term in Burgers is coercive

Have more specific structure:

$$E(u) = B(Tu + \Gamma(u,u), Tu + \Gamma(u,u)) + \text{linear},$$

and super-quadratic coercivity:

$$B(Tw + \Gamma(w,w), Tw + \Gamma(w,w)) \gtrsim \Big(B(Tw, Tw)\Big)^{\frac{3}{2}-}.$$

Assumes form

$$\int (\partial_1 u)^2 + (|\partial_1|^{-\frac{1}{2}}(\partial_2 u - \partial_1 \tfrac{1}{2} u^2))^2 \gtrsim \Big(\int (\partial_1 u)^2 + (|\partial_1|^{-\frac{1}{2}} \partial_2 u)^2\Big)^{\frac{3}{2}-},$$

relies on if $\partial_2 u - \partial_1 \frac{1}{2} u^2 = \rho$ then not only (à la Horwarth-Karman-Monin)

$$\frac{d}{dx_2} \int dx_1 \tfrac{1}{2}(u(\cdot{+}h)\text{-}u)^2 - \frac{d}{dh} \int dx_1 \tfrac{1}{6}(u(\cdot{+}h)\text{-}u)^3 = \int dx_1 (u(\cdot{+}h)\text{-}u)\rho$$

but also with coercive cubic term

$$\frac{d}{dx_2} \int dx_1 \tfrac{1}{2}(u(\cdot{+}h)\text{-}u)_+^2 - \frac{d}{dh} \int dx_1 \tfrac{1}{6}(u(\cdot{+}h)\text{-}u)_+^3 = \int dx_1 (u(\cdot{+}h)\text{-}u)_+\rho$$

Goldman&Josien&O. '15, Golse&Perthame '11

Problem 3: (borderline) singular product

$E_{ren}(w)$ contains the term $4\int w\, v\, R_1\partial_2 v$,

where $R_1 = \mathrm{sign} k_1$ is Hilbert transform in x_1.

Recall linear (elliptic) operator $(-\partial_1^2) \quad + |\partial_1|^{-1}(-\partial_2^2)$,

Carnot-Carathéodory distance $|x_1 - y_1| + |x_2 - y_2|^{\frac{2}{3}}$

and effective dimension $1 \quad + \quad \frac{3}{2} = \frac{5}{2}$.

Hölder exp. of white noise ξ given by $-\frac{1}{2} \times \frac{5}{2}- = -\frac{5}{4}-$,

Hölder exponent of v given by $2 - \frac{5}{4}- = \frac{3}{4}-$,

Hölder exponent of $R_1\partial_2 v$ given by $-\frac{3}{2} + \frac{3}{4}- = -\frac{3}{4}-$.

Product $F = v\, R_1\partial_2 v$ is singular – no canonical meaning.

No problem in absence of R_1: $v\partial_2 v = \partial_2\frac{1}{2}v^2$

Fix 3: stochastic construction of singular product

Recall: $E_{ren}(w)$ contains $\int w\,F$ with singular product $F = vR_1\partial_2 v$.

Suppose law of ξ is invariant under shift & reflection, and satisfies Spectral Gap (SG) inequality.

Then $\exists!$ $C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$-valued random variable (v, F)

with $\big((-\partial_1^2) + |\partial_1|^{-1}(-\partial_2^2)\big)v = \xi$ distributionally

and $F = \lim_{\epsilon\downarrow 0} v\,R_1\partial_2 v_\epsilon$ in $C^{-\frac{3}{4}-}$, both almost surely.

For weakly convergent laws of ξ with uniform SG, the law of (v, F) converges weakly.

(v, F) analogous to (Brownian motion, $\frac{d}{dt}$iterated integral)

in Lyons' rough paths, very simple version of Hairer's model

Fix 3: Γ-topology

Extend definition of $E_{ren}(w) = E(v+w) - E(v)$
by replacing $\int w\, v\, R_1 \partial_2 v$ with $\int w\, F$.

This leads to the map $(v,F) \mapsto \{w \mapsto E_{ren}(w)\}$.

Then $C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-} \ni (v,F) \mapsto \{w \mapsto E_{ren}(w)\}$ is
continuous w. r. t. Γ-convergence, based on $L^2 \ni w$ topology.
Moreover, for $(v,F) \in C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$,
sub-level sets $\{E_{ren}(w) \le M\} \subset L^2$ are compact.

topology inspired by Dal Maso & Modica '86
on stochastic homogenization of variational problems

analogies with Gubinelli & Barashkov '22
variational approach to Euclidean QFT (Φ^4_2)

Goal achieved: universality of the ripple

$(v, F) \mapsto \{w \mapsto E_{ren}(w)\}$ is continuous
w. r. t. Hölder and Γ topology. $\{E_{ren}(w) \leq M\}$ compact.

(law of ξ) $\mapsto$ (law of (v, F)) is continuous
under weak convergence. Need uniform SG constant.

$\Longrightarrow$ (law of ξ) $\mapsto$ (law of E_{ren}) is continuous
under weak convergence.

Any approximation of white noise with uniform SG
has same limit
as a variational problem that admits minimizers.

Next step: $[0, 1)^2 \rightsquigarrow \mathbb{R}^2$, gain scaling