Edge Transport in Interacting Quantum Hall Systems

Marcello Porta

Joint work with G. Antinucci, V. Mastropietro

Summary

- Introduction: universality of transport in condensed matter systems. Integer quantum Hall effect, bulk-edge duality.
- Interacting quantum Hall systems on the cylinder. Quantization of edge response function from a microscopic model.
- General approach: Wick rotation, RG analysis of correlations, resolution of the scaling limit, Ward identities.
- Conclusions and open problems.

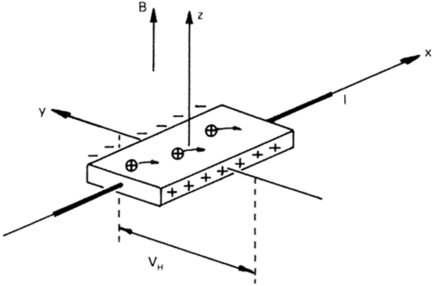
Introduction

Integer quantum Hall effect

- Bulk topological order in condensed matter systems is deeply related to the emergence of gapless edge modes.

- Example. Integer quantum Hall effect [von Klitzing *et al.* '80]

  $2d$ insulators exposed to transv. magnetic field and in-plane electric field.



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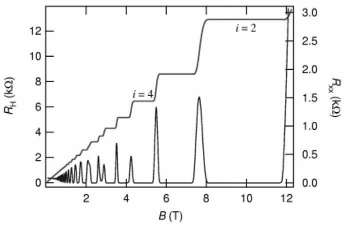
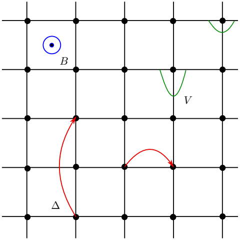
- Example. Integer quantum Hall effect [von Klitzing *et al.* '80]
  $2d$ insulators exposed to transv. magnetic field and in-plane electric field.
  Linear response: $J = \sigma E + o(E)$ with $\sigma =$ conductivity matrix:

$$\sigma = \begin{pmatrix} 0 & \frac{n}{2\pi} \\ -\frac{n}{2\pi} & 0 \end{pmatrix} , \qquad n \in \mathbb{Z}.$$



Noninteracting systems

- Noninteracting fermions on a $2d$ lattice. Hamiltonian H on $\ell^2(\mathbb{Z}^2;\mathbb{C}^M)$, $H(x;y)$ short-ranged. Example:



$\Delta =$ lattice hopping; $V =$ external potential; $B =$ magnetic field.

Here $H = -\Delta_A + V$, with

$$\Delta_A(x;y) = \Delta(x;y)e^{i\int_{x\to y} d\ell\cdot A(\ell)}\,, \qquad \int_{\partial(\text{plaquette})} d\ell \cdot A(\ell) = \text{Flux}(B)$$

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- The state of ∞-many, noninteracting fermions at $T = 0$ is described by the Fermi projector, $P_\mu = \chi(H \leq \mu)$: $\langle O \rangle_\mu := \mathrm{Tr}_{\mathfrak{h}} O P_\mu$. We have:

$$|P_\mu(x,y)| \leq C e^{-c|x-y|} \ .$$

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- Assuming the validity of linear response, the transverse conductivity of the system can be expressed via the Fermi projector. One has:

$$\sigma_{12} = \lim_{L\to\infty} \frac{i}{L^2} \mathrm{Tr}_{\mathfrak{h}} \chi\big(x \in [0,L]^2\big) P_\mu [[\hat{x}_1, P_\mu],[\hat{x}_2, P_\mu]]$$

- Remarkably, $\sigma_{12} \in \frac{1}{2\pi}\mathbb{Z}$ (Chern number/index thm) Rigorous results:

TKNN '82; Bellissard et al. '94; Avron, Seiler, Simon '94; Aizenman-Graf '98...

Edge states

- Consider now a lattice model on the cylinder Λ_L:

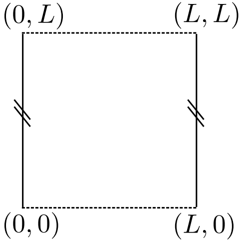

Figure: Dotted lines: Dirichlet b.c.. Identify vertical sides.

H = restriction to the cylinder of a gapped Hamiltonian on $\mathbb{Z}^2$.

- **Important:** in general, H may not have a spectral gap uniformly in L. A nonzero Hall conductivity is related to the emergence of gapless modes on the boundary [Halperin '82] (algebraic decay of correlations).

Edge states

- The gap might be closed by edge modes: $(H = \bigoplus_{k \in S^1} \hat{H}(k))$

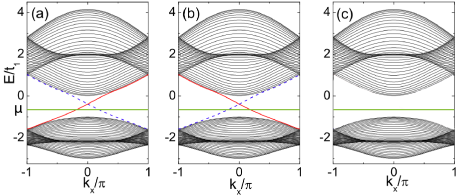

Figure: Blue/Red curves: edge modes. Gray: "bulk" spectrum.

- Red curve: eigenvalue branch, with generalized eigenstates:

$$\varphi_x(k) = e^{ikx_1}\xi_{x_2}(k) \; , \qquad \text{with } |\xi_{x_2}(k)| \leq Ce^{-c|x_2|} \; .$$

Localized in proximity of the lower edge, extended along the edge.
$\rightsquigarrow$ metallic transport along the boundary.

The bulk-edge duality

- Bulk-edge duality: relation between σ_{12} and the edge states of H.

$$\boxed{\sigma_{12} = \sum_{\omega} \frac{\mathrm{sgn}(v_\omega)}{2\pi}}$$

$=$ sum of chiralities of edge modes (also equal to edge conductance).

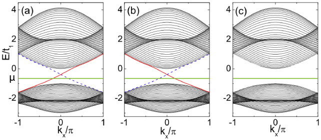

Figure: $(a): \ \sigma_{12} = \frac{1}{2\pi}, \quad (b): \sigma_{12} = -\frac{1}{2\pi}, \quad (c): \sigma_{12} = 0.$

- Argument for bulk-edge duality based on anomaly cancellation: Wen 90, Fröhlich et al. '91...

- Rigorous results for noninteracting systems: Hatsugai '93; Schulz-Baldes, Kellendonk, Richter '00; Elbau-Graf '02; Elgart, Graf, Schenker '05; Graf, P. '13; Cornean, Moscolari, Teufel '21...

Many-body systems

- Quantization of transport and bulk-edge duality from interacting many-body models?

  IQHE for interacting systems:

  - Hastings-Michalakis 2016: many-body topological approach.
  - Giuliani-Mastropietro-P. 2017: analytic QFT method. Ward ids.
  - Bachmann-Bols-de Roeck-Fraas 2018: many-body index theorem.

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- Our approach also applies to gapless systems, if combined with regularity estimates on correlations (via RG).

  For example: universality of transport in graphene.

  - Stauber-Peres-Geim 2008: universality of σ_{11} for nonint. graphene.
  - GMP 2011: universality against short-range interactions.
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- Today. Interacting edge transport.

Edge transport in many-body quantum systems

Many-body systems

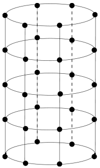
- Interacting lattice many-body Fermi system on cylinder Λ_L, $|\Lambda_L| = L^2$.
- Fock space Hamiltonian: $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$ with (ρ = spin, sublattice...)

$$\mathcal{H}_0 = \sum_{x,y}\sum_{\rho,\rho'} a^+_{x,\rho} H_{\rho\rho'}(x,y) a^-_{y,\rho'} \ , \quad \mathcal{V} = \sum_{x,y}\sum_{\rho,\rho'} v_{\rho\rho'}(x,y) a^+_{x,\rho} a^+_{y,\rho'} a^-_{y,\rho'} a^-_{x,\rho}$$

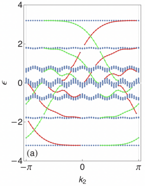
$H(x;y)$, $v(x;y)$ finite-ranged. Transl. inv.: $[H, T_1] = [v, T_1] = 0$.

- Hyp.: H has a bulk gap, and supports edge modes at the Fermi level.
  Gibbs state: $\rho_{\beta,L} = \mathcal{Z}^{-1} e^{-\beta(\mathcal{H}-\mu\mathcal{N})}$, with μ in a bulk spectral gap of H.

Lattice:

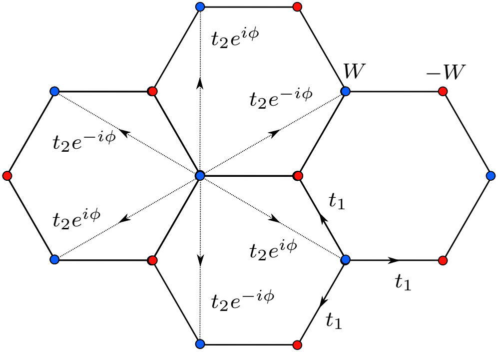


Spectrum of H:



Simplest example: Haldane model

- Haldane '88. Graphene-like model, zero-flux magnetic field.



- Free Hamiltonian: nn hopping + nnn hopping + staggered potential.

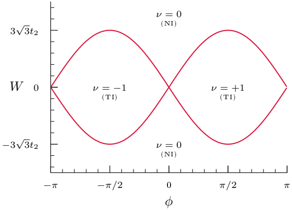
$$H_0 = t_1 \sum_{\langle x,y \rangle} |x\rangle\langle y| + \sum_{\langle\langle x,y \rangle\rangle} t_2(x,y)|x\rangle\langle y| + W\Big[\sum_{x\in A} |x\rangle\langle x| - \sum_{y\in B} |y\rangle\langle y|\Big]$$

Hall conductivity and edge modes

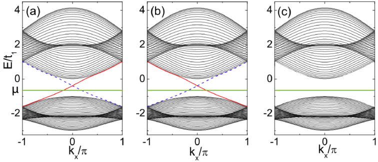
- For generic ϕ, W the spectrum of H_0 on the infinite lattice is gapped. The Hall conductivity on the infinite lattice is:

$$\sigma_{12} = \frac{\nu}{2\pi} , \qquad \nu = -1, 0, 1 .$$

Topological phase diagram:



Edge modes:



- A phase transition takes place at discont. of ν. The phase diag. is robust against many-body interactions, up to a renormalization of the curves. [Giuliani-Jauslin-Mastropietro-P. '16, '19.]

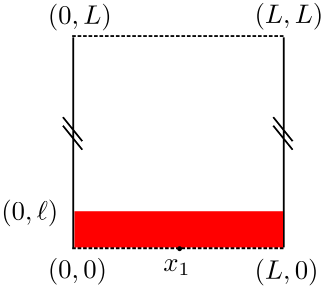
Edge response function

- Consider a slowly varying perturbation: $(0 < \eta, \theta \ll 1,\ t \leq 0)$

$$H(\eta t) := H + e^{\eta t}\mu(\theta x)\ , \qquad \mu(x) \text{ bump at } x = 0$$

- Edge current operator in a strip of width ℓ: $(1 \ll \ell \ll L)$

$$\mathcal{J}^{\ell}_{x_1} = \sum_{x_2 \leq \ell} j_{1,(x_1,x_2)} \qquad (j_{1,x} = \text{horiz. current density})$$



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- Let $\rho(t) =$ time-evolved state, $\rho(-\infty) = \rho_{\beta,L}$. Linear resp.: $(\beta, L \to \infty)$

$$\mathrm{Tr}\, \mathcal{J}^{\ell}_0 \rho(0) - \mathrm{Tr}\, \mathcal{J}^{\ell}_0 \rho(-\infty) = \int_{-\pi}^{\pi} \frac{dp}{(2\pi)}\, \hat{\mu}(p,0) G^{\ell}(\eta, \theta p) + \text{h.o.t.}$$

$$\boxed{G^{\ell}(\eta, p) = -i \lim_{a\to\infty} \lim_{\beta,L\to\infty} \int_{-\infty}^{0} dt\, e^{\eta t} \sum_{y: y_2 \le a} e^{ipy_1} \left\langle [n_y(t), \mathcal{J}^{\ell}_0] \right\rangle_{\beta,L}}$$

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Difficulties: control of real-time integral as $\eta \to 0^+$, gapless modes.

Remark: Order of $\eta, p \to 0^+$ limit matters. E.g.: $G^{\ell}(\eta, 0) = 0!$

Multi-channel Luttinger liquid

- Effective $1+1$ dimensional QFT for edge modes (scaling limit):

$$\begin{aligned} \mathcal{Z} &= \int D\psi\, e^{-S(\psi)} \\ S(\psi) &= \sum_{\omega} \int_{\mathbb{R}^2} d\underline{x}\, Z_\omega \psi^+_{\underline{x},\omega}(\partial_0 + i v_\omega \partial_1)\psi^-_{\underline{x},\omega} \\ &\quad + \sum_{\omega,\omega'} \lambda_{\omega,\omega'} Z_\omega Z_{\omega'} \int_{\mathbb{R}^2\times\mathbb{R}^2} d\underline{x} d\underline{y}\, n_{\underline{x},\omega} n_{\underline{y},\omega'} v(\underline{x}-\underline{y})\,. \end{aligned}$$

$\psi^{\pm}_{\underline{x},\omega}$ = Grassmann field, $\underline{x} = (x_0, x_1)$, ω = chirality (edge modes).

- Z_ω, v_ω chosen to correctly match the scaling of edge correlations.
- Elastic scattering hyp.: if k_F^ω is the Fermi momentum of the ω edge state,

$$(*) \qquad k_F^{\omega_1} - k_F^{\omega_2} = k_F^{\omega_3} - k_F^{\omega_4} \quad \text{only for edge modes equal in pairs.}$$

Generic, in absence of special sym. ($k_F^{\omega_1} \equiv k_F^{\omega_1}(\mu)$, μ = Fermi level).

Anomalous Ward identities

- The model is formally covariant under local chiral gauge transformations:

$$\psi^{\pm}_{\underline{x},\omega} \underset{\text{Jacobian 1}}{\longrightarrow} e^{\pm i\alpha_\omega(\underline{x})}\psi^{\pm}_{\underline{x},\omega} \quad \underset{\text{Formally!}}{\Longrightarrow} \quad \mathcal{Z}(A_\omega) = \mathcal{Z}(A_\omega + D_\omega \alpha_\omega)$$

with $D_\omega = \partial_0 + i v_\omega \partial_1$. Ward identity: $\langle \hat{n}_{\underline{p},\omega}\, ; \hat{n}_{-\underline{p},\omega'} \rangle = 0.$ (?)

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- The symmetry is broken by unavoidable regularizations, which produce anomalies in the WIs as cutoffs are removed. Correct result:

$$\langle \hat{n}_{\underline{p},\omega} \,; \hat{n}_{-\underline{p},\omega'} \rangle = T_{\omega,\omega'}(\underline{p}) \frac{1}{Z^2_{\omega'}} \frac{1}{4\pi|v_{\omega'}|} \frac{ip_0 + v_{\omega'}p_1}{-ip_0 + v_{\omega'}p_1}$$

$$\left(\frac{1}{T(\underline{p})}\right)_{\omega,\omega'} = \delta_{\omega,\omega'} + \frac{ip_0 + v_\omega p_1}{-ip_0 + v_\omega p_1} \frac{1}{4\pi|v_\omega|} \frac{1}{Z_\omega} \lambda_{\omega,\omega'} Z_{\omega'} \,.$$

- Similar relations can be found for other correlations, e.g. for the vertex function $\langle \hat{n}_{\underline{p},\omega} \,; \hat{\psi}^{-}_{\underline{k},\omega'} \,; \hat{\psi}^{+}_{\underline{k}+\underline{p},\omega'} \rangle$.

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- Idea: use the exact relations to characterize scaling limit of lattice model.

Main result: interacting edge transport

- We consider $\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{V}$, transl. inv. in the direction of the edge, with $\mathcal{H}_0$ displaying arbitrarily many edge modes, under the assumption $(*)$.

Theorem (V. Mastropietro, M. P. - Comm. Math. Phys. 2022)

For $|\lambda|$ small, the $\beta, L \to \infty$ edge conductance is, for $\underline{p} = (\eta, p)$ and $|\underline{p}| \ll 1$:

$$G^\ell(\underline{p}) = \sum_\omega r_\omega(\underline{p}) \frac{v_\omega p}{-i\eta + v_\omega p} \frac{sgn(v_\omega)}{2\pi} + o(1)$$

where

$$r_\omega(\underline{p}) = \Big(\big(1 + \frac{1}{4\pi |v|} \Lambda\big) \frac{1}{1 + \frac{1}{4\pi|v|} \omega(\underline{p}) \Lambda} \Big)_{\omega\omega}$$

with: $v_\omega \equiv v_\omega(\lambda)$, $v = diag(v_\omega)$, $\Lambda_{\omega\omega'} = O(\lambda)$, $\omega(\underline{p}) = diag\Big(\frac{-i\eta + v_\omega p_1}{i\eta + v_\omega p_1}\Big)$.
In particular,

$$\lim_{\ell \to \infty} \lim_{p \to 0} \lim_{\eta \to 0^+} G^\ell(\underline{p}) = \sum_\omega \frac{sgn(v_\omega)}{2\pi} \, .$$

Remarks

- Combined with the universality of the Hall conductivity [GMP17], the result implies the stability of the bulk-edge duality against interactions.
- Previous work on interacting edge modes:
  - Antinucci-Mastropietro-P. 2018: one edge mode. Chiral Luttinger liquid universality class.
  - Mastropietro-P. 2018: two counterpropagating edge modes, spin transport. Helical Luttinger liquid universality class.
- Main technical tools:
  - Rigorous RG analysis of the edge correlations, scaling limit.

    [Gawedzki, Kupiainen, Feldman, Magnen, Rivasseau, Sénéor, Benfatto, Gallavotti, Balaban, Knörrer, Trubowitz, Brydges, Slade...]
  - Lattice Ward identities, implied by lattice conservation laws. Put nontrivial constraints between scaling limit and lattice model.
  - Anomalous Ward identities, for the effective QFT. Implications: exact expressions for correlations, vanishing of the beta function.

Sketch of the proof

Euclidean response function

- The proof starts by a rigorous Wick rotation from real to imaginary times of the transport coefficient.

  We have:

$$G^{\ell}(\eta, p) = \lim_{\beta, L \to \infty} \int_0^{\beta} ds\, e^{-i\eta s} \sum_{y \in \Lambda_L} e^{ipy_1} \langle n_y(-is)\, ; \mathcal{J}_0^{\ell} \rangle_{\beta, L}$$

  In contrast to real-time correlations, imaginary-time correlations can be estimated efficiently via convergent expansions and multiscale analysis.

- $G^{\ell}(\eta, p)$ extends to a function on $\mathbb{R} \times \mathbb{S}^1$.

  We are interested in the $(\eta, p) \to (0^+, 0)$ limit. Recall:

$$G^{\ell}(\eta, 0) = 0\ .$$

  That is, the response to a constant perturbation is trivial. Other limit?

- We construct $G^{\ell}(\eta, p)$ via a rigorous RG analysis. The response function is actually discontinuous at zero.

Singular and regular contributions

- A rigorous RG analysis gives the following splitting, setting $\underline{p} = (\eta, p)$:

$$G^{\ell}(\underline{p}) = \underbrace{\big(\vec{Z}_0, D^{\rm rel}(\underline{p})\vec{Z}_1\big)}_{\text{scaling limit}} \quad + \quad \underbrace{R^{\ell}(\underline{p})}_{\text{irrelevant terms}}$$

  - $(\vec{A}, \vec{B}) = \sum_{\omega} A_{\omega} B_{\omega}$ (sum over edge modes at $x_2 = 0$)
  - $D^{\rm rel}_{\omega,\omega'}(\underline{p}) = \langle \hat{n}_{\underline{p},\omega} \, ; \hat{n}_{-\underline{p},\omega'} \rangle^{\rm rel}$ (discontinuous at $\underline{p} = (0,0)$)
  - $Z_{\mu,\omega}$ are model dep. parameters (dressing of current and density)
  - $R^{\ell}(\underline{p})$ is continuous at $\underline{p} = \underline{0}$.

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- From $G^{\ell}(\eta, 0) = 0$ and continuity of $R^{\ell}(\underline{p})$, we determine $R^{\ell}(\underline{0})$. We get:

$$\boxed{\lim_{\ell\to\infty} \lim_{p\to 0} \lim_{\eta\to 0^+} G^{\ell}(\underline{p}) = \big(\vec{Z}_0, \mathcal{A}\vec{Z}_1\big)}$$

with:

$$\mathcal{A} := \lim_{p\to 0} \lim_{\eta\to 0^+} D^{\rm rel}(\underline{p}) - \lim_{\eta\to 0^+} \lim_{p\to 0} D^{\rm rel}(\underline{p}) \,.$$

Vertex Ward Identities [Neglecting x_2 labels]

- Conservation of lattice current: $\partial_t n_x(t) + \mathrm{div}_x j_x(t) = 0$. Implication:

$$\sum_{\mu=0,1} p_\mu \langle \mathbf{T}\, \hat{j}_{\mu,\underline{p}}\, ; \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle = \langle \mathbf{T}\, \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}} \rangle - \langle \mathbf{T}\, \hat{a}^-_{\underline{k}+\underline{p}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle \,.$$

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$$\sum_{\mu=0,1} p_\mu \langle \mathbf{T}\, \hat{j}_{\mu,\underline{p}}\, ; \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle = \langle \mathbf{T}\, \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}} \rangle - \langle \mathbf{T}\, \hat{a}^-_{\underline{k}+\underline{p}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle \,.$$

- A similar (anomalous) WI holds for the effective QFT:

$$\langle \hat{n}_{\underline{p},\omega}\, ; \hat{\psi}^-_{\underline{k},\omega'}\, ; \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle = \frac{T_{\omega,\omega'}(\underline{p})}{Z_{\omega'}(-i\eta + v_{\omega'} p)} \big(\langle \hat{\psi}^-_{\underline{k},\omega'} \hat{\psi}^+_{\underline{k},\omega'} \rangle - \langle \hat{\psi}^-_{\underline{k}+\underline{p},\omega'} \hat{\psi}^+_{\underline{k}+\underline{p},\omega'} \rangle \big) \,.$$

- For $\underline{p}$ small and for $\underline{k}' = \underline{k} - \underline{k}_F^\omega$ small, comparison via RG:

$$\langle \mathbf{T}\, \hat{j}_{\mu,\underline{p}}\, ; \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}+\underline{p}} \rangle \simeq \sum_{\omega'} Z_{\mu,\omega'} \langle \hat{n}_{\underline{p},\omega'}\, ; \hat{\psi}^-_{\underline{k},\omega}\, ; \hat{\psi}^+_{\underline{k}+\underline{p},\omega} \rangle \,, \quad \langle \mathbf{T}\, \hat{a}^-_{\underline{k}} \hat{a}^+_{\underline{k}} \rangle \simeq \langle \hat{\psi}^-_{\underline{k},\omega} \hat{\psi}^+_{\underline{k},\omega} \rangle$$

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Two eqs. for QFT correlations! Impose constraints on ren. parameters:

$$\lim_{p\to 0} \lim_{\eta \to 0^+} T^T(\underline{p}) \vec{Z}_0 = \vec{Z} \; , \qquad \lim_{\eta \to 0^+} \lim_{p\to 0} T^T(\underline{p}) \vec{Z}_1 = v \vec{Z} \; .$$

Plugging in $G = (\vec{Z}_0, \mathcal{A}\vec{Z}_1)$, universality (remarkably) follows. ■

Conclusions and open problems

- Edge response function for interacting quantum Hall systems.
  The method allows to prove the emergence of the multi-channel Luttinger model as effective QFT, and to prove quantization of transport.

- Validity of linear response from quantum dynamics?
  For gapped systems, Wick rotation and cluster expansion techniques can be used to prove convergence of real-time expansions.
  [Greenblatt, Lange, Marcelli, P. 2022]
  Extension to gapless models?

- Two-terminal conductance? Backscattering should destroy universality, and disorder should restore it. [Kane-Fisher-Polchinski].

- Fractional quantization...?

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- Fractional quantization...?
- **Thank you!**

Wick rotation

- Let us consider, for $\eta > 0$:

$$\int_{-\infty}^{0} dt\, e^{t\eta}\, \langle [A(t)\,, B] \rangle_{\beta,L}$$

A and B are extensive, finite-ranged observables: $O = \sum_{X \subset \Lambda_L} O_X$.

- Approximate η by $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}$, s.t. $|\eta - \eta_\beta| \leq \frac{2\pi}{\beta}$. Thus,

$$\Big| \int_{-\infty}^{0} dt\, e^{t\eta}\, \langle [A(t)\,, B] \rangle_{\beta,L} - \int_{-\infty}^{0} dt\, e^{t\eta_\beta}\, \langle [A(t)\,, B] \rangle_{\beta,L} \Big|$$

$$\leq \int_{-\infty}^{0} dt\, |e^{t\eta} - e^{t\eta_\beta}| \big\| [A(t)\,, B] \big\|$$

$$\leq \frac{C}{\beta} \int_{-\infty}^{0} dt\, e^{t\eta} L^2 t^{2+1} \to 0 \qquad \text{as } \beta \to \infty.$$

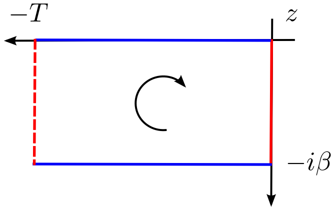
The last inequality follows from the Lieb-Robinson bound:

$$\|[A_X(t), B_Y]\| \leq C_{AB} e^{vt - c\,\mathrm{dist}(X,Y)}$$

Wick rotation

- We analytically continue to imaginary times. We have, by KMS:

$$\int_{-\infty}^{0} dt\, e^{t\eta_\beta} \langle [A(t)\,, B]\rangle = \int_{-\infty}^{0} dt \left(e^{t\eta_\beta} \langle A(t)B\rangle - e^{(t-i\beta)\eta_\beta}\langle A(t-i\beta)B\rangle \right)$$
$$= i\int_{0}^{\beta} dt\, e^{-it\eta_\beta}\langle A(-it)B\rangle_{\beta,L}$$



- Errors (dotted red) estimated via bounds on Euclidean correlations:

$$|\langle A(T-it)B\rangle_{\beta,L}| \leq \langle A(-it)A(-it)^*\rangle_{\beta,L}^{1/2} \langle B^* B\rangle_{\beta,L}^{1/2}$$

Grassmann QFT

- Grassmann representation of the Euclidean QFT:

$$\mathcal{Z}_{\beta,L} = \mathbb{E}_g\big(e^{V(\psi)}\big)$$

where:

  - $\psi \equiv \psi^{\pm}_{\mathbf{x}}$ is a complex Grassmann field, for $\mathbf{x} = (x_0, x) \in [0,\beta) \times \Lambda_L$
  - $\mathbb{E}_g$ is a Gaussian integration, with propagator:

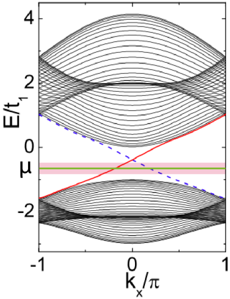
$$\mathbb{E}_g(\psi^+_{\mathbf{x}}\psi^-_{\mathbf{y}}) = \frac{1}{\beta} \sum_{k_0 \in \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})} e^{ik_0(x_0-y_0)} \frac{1}{-ik_0 + H - \mu}(x,y) =: g(\mathbf{x},\mathbf{y}) \, .$$

  - $V(\psi)$ is a quartic interaction:

$$V(\psi) = \lambda \int_{[0,\beta)^2} dx_0 dy_0 \sum_{x,y\in\Lambda_L} \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}} \delta(x_0 - y_0) v(x-y) \, .$$

Reduction to an effective $1d$ model

- Integration of bulk degrees of freedom. Write $g = g_1 + g_2$, and correspondingly $\psi = \psi_1 + \psi_2$. g_2 : energies away from μ.



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- ψ_2 is integrated out via convergent exp.: [Brydges-Battle-Federbush]

$$\mathbb{E}_g\big(e^{V(\psi)}\big) = \mathbb{E}_{g_1}\mathbb{E}_{g_2}\big(e^{V(\psi_1+\psi_2)}\big) = \mathbb{E}_{g_1}\big(e^{V_{\mathrm{eff}}(\psi_1)}\big)\ .$$

The field ψ_1 can be parametrized in terms of a truly $1+1$ dim. field:

$$\psi_{1,\underline{k}}(x_2) = \sum_{\omega} \xi^{\omega}_{k_1}(x_2)\varphi_{\omega,\underline{k}},$$

where $\xi^{\omega}_{k_1}(x_2)$ is the eigenstate of the ω-edge mode and:

$$\mathbb{E}_{\varphi}(\varphi^{+}_{\omega,\underline{k}}\varphi^{-}_{\omega',\underline{k}}) = \delta_{\omega,\omega'}\hat{g}_{\omega}(\underline{k})$$

$$\hat{g}_{\omega}(\underline{k}) = \frac{\chi(|\varepsilon_{\omega}(k_1) - \mu| \leq \delta)}{-ik_0 + \varepsilon_{\omega}(k_1) - \mu}$$

Massless propagator: close to k^{ω}_F, $\varepsilon_{\omega}(k_1) - \mu \simeq v_{\omega}(k_1 - k^{\omega}_F)$.

Multiscale integration

- We end up with a (complicated, but explicit) $1d$ effective theory:

$$\mathbb{E}_g\big(e^{V(\psi)}\big) = \int \nu(d\varphi) e^{\mathcal{V}(\varphi)}$$

where $\nu = \prod_\omega \nu_\omega$ and ν_ω has propagator $g_\omega(\underline{k})$.

- The massless $1d$ field is decomposed in scales:

$$\varphi_\omega = \sum_{h=h_\beta}^{0} \varphi_\omega^{(h)} \qquad\qquad g_\omega^{(h)}(\underline{k}) \simeq \frac{1}{Z_{\omega,h}} \frac{\chi(\|\underline{k} - \underline{k}_F^\omega\| \sim 2^h)}{-ik_0 + v_{\omega,h}(k_1 - k_F^\omega)}$$

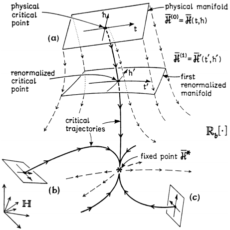
and integrated iteratively: (Gallavotti-Nicolò tree expansion)

$$\mathbb{E}_{\varphi^{(h_\beta)}+\ldots+\varphi^{(0)}}\Big(e^{\mathcal{V}\big(\varphi^{(h_\beta)}+\ldots+\varphi^{(0)}\big)}\Big) = \mathcal{Z}_h \mathbb{E}_{\varphi^{(h_\beta)}+\ldots+\varphi^{(h)}}\Big(e^{\mathcal{V}^{(h)}\big(\sqrt{Z_h}\varphi^{(\leq h)}\big)}\Big)$$

where $(\mathcal{V}^{(h)}, Z_h, v_h)$ solve a discrete recursion equation. In particular:

$$\mathcal{V}_4^{(h)}(\xi) = \sum_{\omega,\omega'} \lambda_{\omega,\omega',h} \int dx_0 \sum_{x_1} \xi^+_{\underline{x},\omega} \xi^-_{\underline{x},\omega} \xi^+_{\underline{x},\omega'} \xi^-_{\underline{x},\omega'}$$

RG flow



- The marginal direction associated to $Z_{h,\omega}$, $v_{h,\omega}$ and to the effective couplings $\lambda_{h,\omega,\omega'}$ is controlled thanks to a key aspect of integrability:

$$\lambda_{h,\omega,\omega'} = \lambda_{h+1,\omega,\omega'} + \beta^{\lambda}_{h+1,\omega,\omega'} \qquad \beta^{\lambda}_{h+1,\omega,\omega'} = O(\lambda^2_{h+1} 2^{\theta h})$$

  (asymptotic) vanishing of the beta function.

  Proof based on a generalization of the method of [Benfatto-Mastropietro]

- Flow of the running coupling constants, as $h \to -\infty$:

$$\boxed{\lambda_{h,\omega,\omega'} = C_{\omega,\omega'}\lambda + O(\lambda^2)\ , \qquad Z_{h,\omega} \sim 2^{-h\eta_\omega \lambda^2}\ , \qquad v_{h,\omega} - v_\omega = O(\lambda^2)\ .}$$