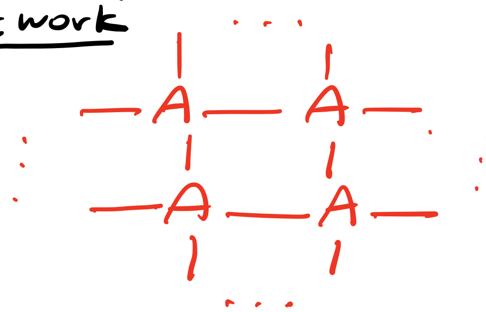
Towards Nontrivial Fixed Point of 2D Ising with Tensor RG

Tensor RG Approach

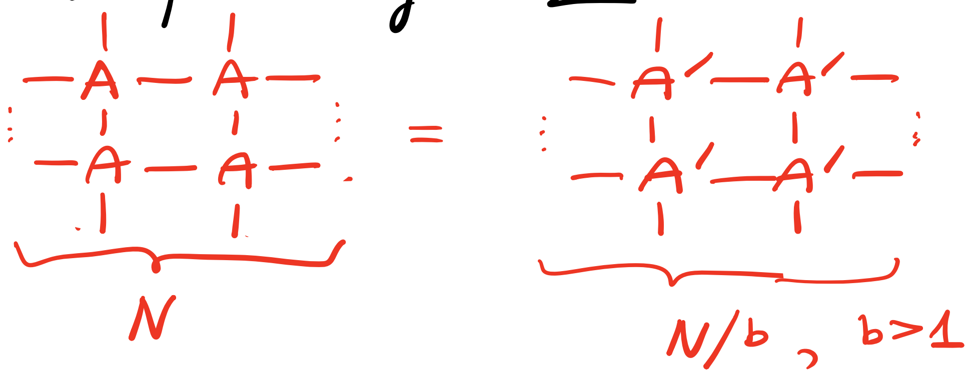
·) rewrite partition function of a lattice model as a tensor network

$$Z = \mathrm{Tr}_s\, e^{H} =$$

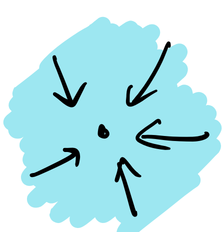


·) construct tensor RG map — coarse-graining transformations preserving Z

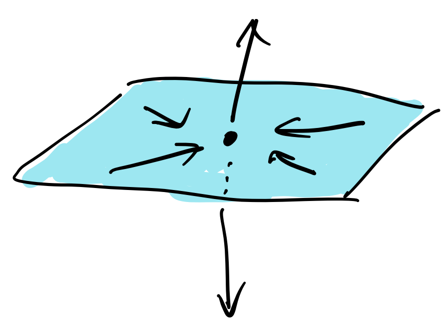
$A \to A'$:



·) understand phase diagram by studying tensor RG near fixed points



or



or...

STATUS

Done: neighborhoods of high-T and low-T fixed points

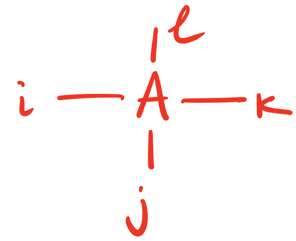
Kennedy, S.R. 2021, 2022

In progress nontrivial fixed point of 2d Ising ($\leftrightarrow$ critical point)

Long-term goal 3D

BASIC DEFINITIONS

Tensor

$A_{ijkl} =$ 

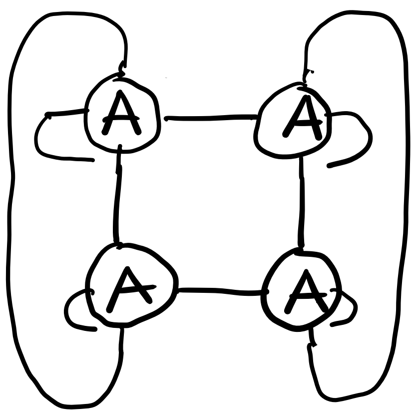
- table of real (or complex) numbers
- every index $i,j,k,l \in I$ index set
  often $I = \mathbb{N}$
- $|I| = \chi$ bond dimension

- Hilbert-Schmidt condition

$$\|A\| = \Big(\sum_{i,j,k,l \in I} |A_{ijkl}|^2 \Big)^{1/2} < \infty$$

Tensor Network

periodic contraction of tensors

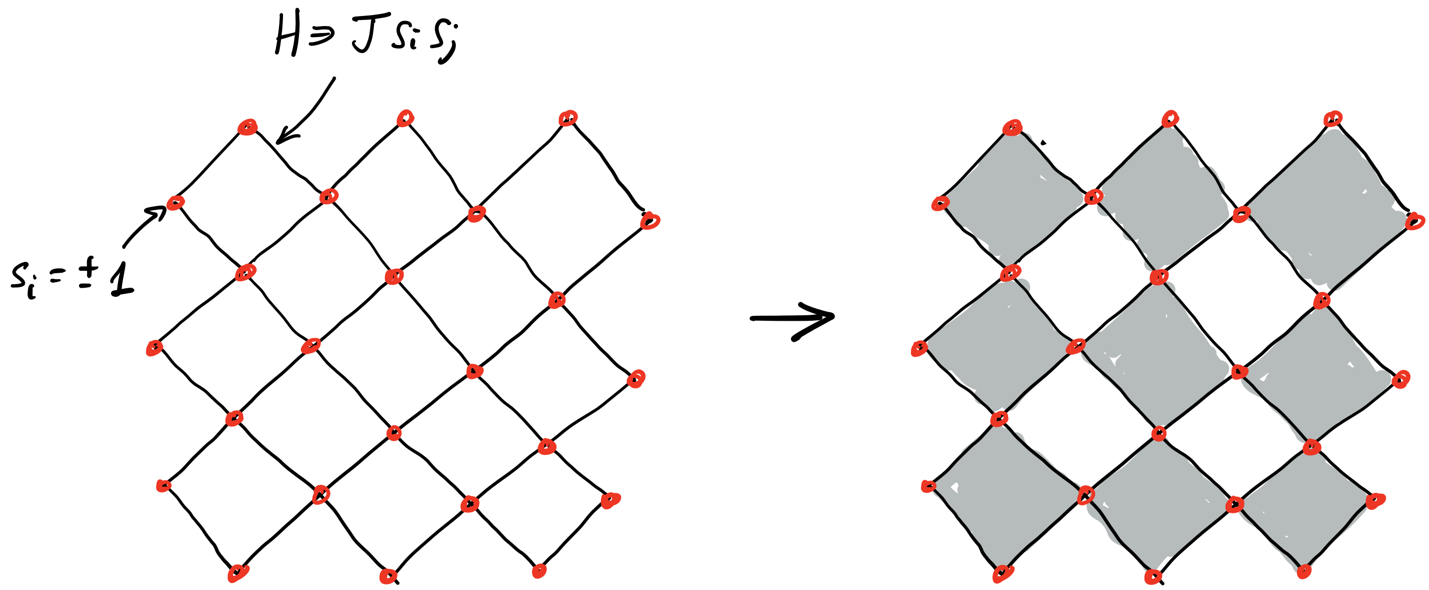
$Z(A, 2\times 2) =$ 

"partition function"

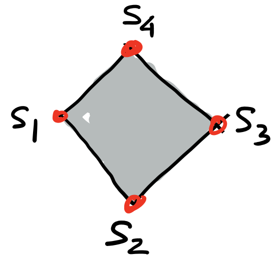
Analogously $Z(A, N\times N)$

Theorem [Kennedy, S.R.] Finite-volume partition function of any lattice spin model with finite range interactions can be written as a tensor network

Example 2D Ising



$$A_{s_1 s_2 s_3 s_4} =$$



$$= e^{J(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1)}$$

$s_i = \pm 1$

Tensor Network RG with rescaling factor b is a map $A \to A'$ s.t.

$$Z(A, N \times N) = Z\left(A', \frac{N}{b} \times \frac{N}{b}\right)$$

Cf. WILSON – KADANOFF RG

Map in the space of <u>Hamiltonians</u> $H[s] \to H'[t]$ s.t.

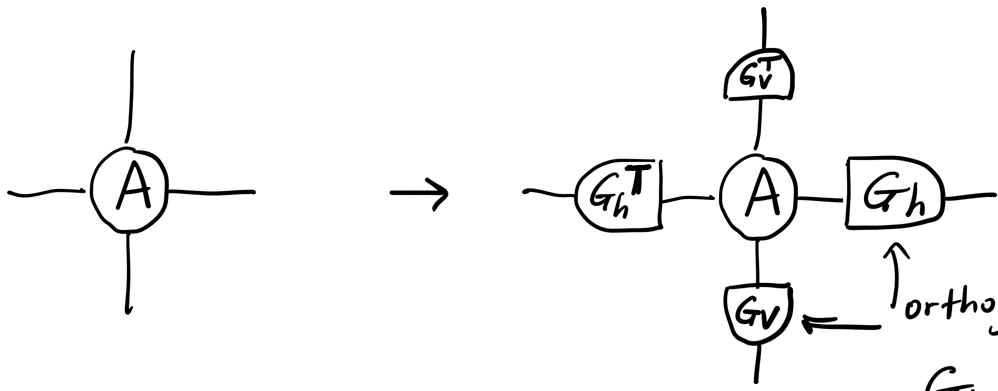
- $Z = \mathrm{Tr}_s\, e^{H[s]} = \mathrm{Tr}_t\, e^{H'[t]}$
- New spins t are functions of old spins s in a block

$$\boxed{\begin{matrix} s_1 & s_4 \\ s_2 & s_3 \end{matrix}} \to t$$

Tensor RG is a <u>more flexible</u> form of Wilson-Kadanoff

- tensor A' is not required to be positive (unlike $A \underset{\circ}{=} e^{H}$)
- relation between "new spins" (indices of A') and "old spins" (indices of A) can be more complicated than mere blocking
- preserves nearest-neighbor structure
- all this pays off in rigorous & numerical studies

Freedom to change basis
("gauge freedom")

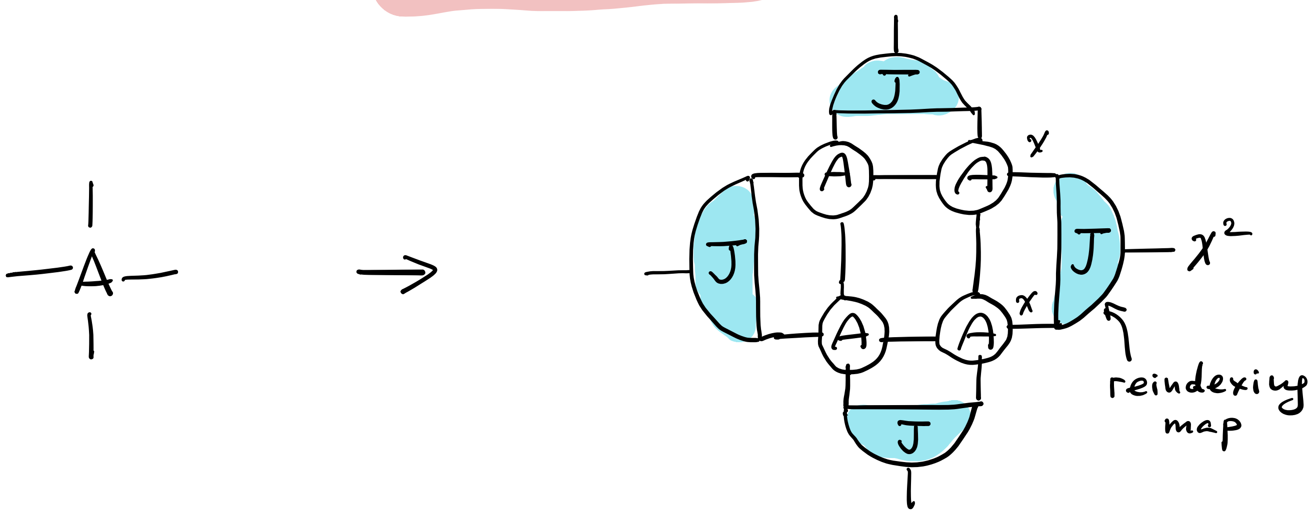


orthogonal transformations

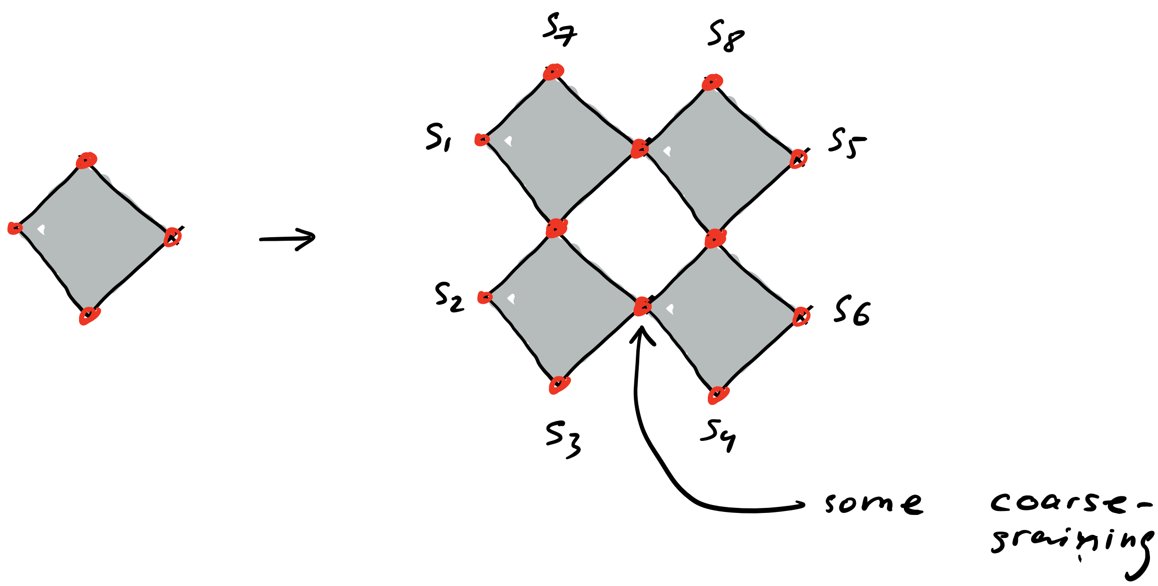
$$G_h \, G_h^T = \mathbb{1}$$
$$G_v \, G_v^T = \mathbb{1}$$

- tensor network preserved
- some tensor elements may become negative after rotation

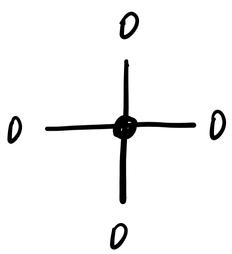
SIMPLE RG



For the Ising model:

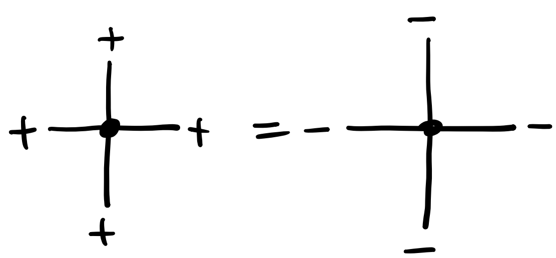


Simple fixed points:

A_{HT}: 

$= 1$ High-T fixed point

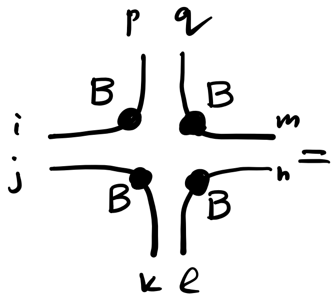
$A_{LT} = A^{(+)} \oplus A^{(-)}$ Low-T fixed point



$= 1$

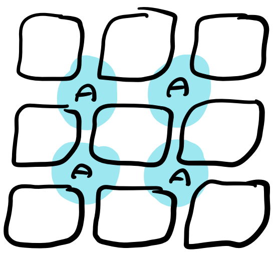
Corner-Double-Line (CDL) Tensors

$A_{ij,kl,mn,pq} =$

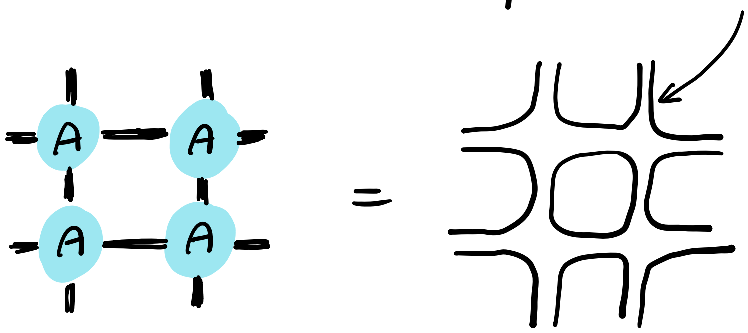


$= B_{ip}\, B_{qm}\, B_{jk}\, B_{ln}$

- Partition function is trivial:



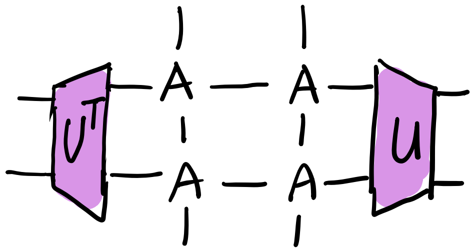
- Yet simple RG passes angular correlations further:



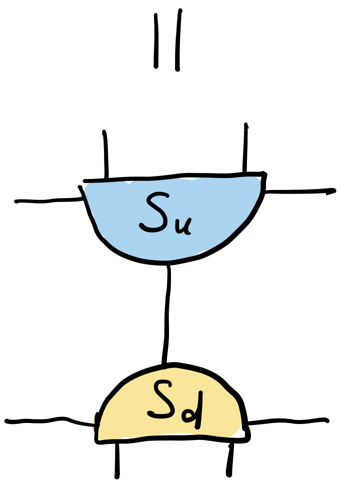
- Need a better RG map

Disentanglers

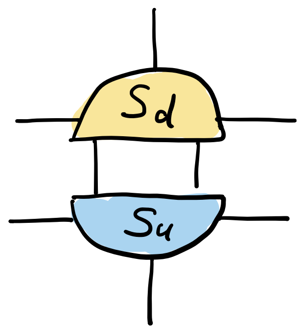
Evenbly-Vidal 2014
Kennedy, S.R. 2021

1) 

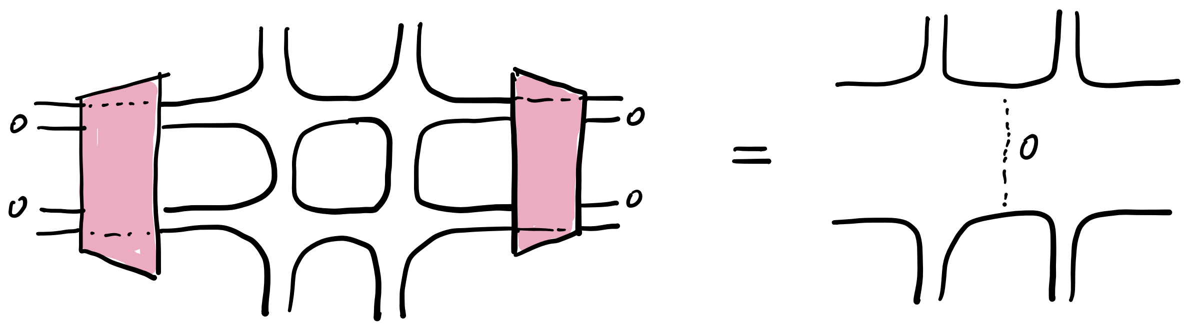
$$UU^T = \mathbb{1}$$

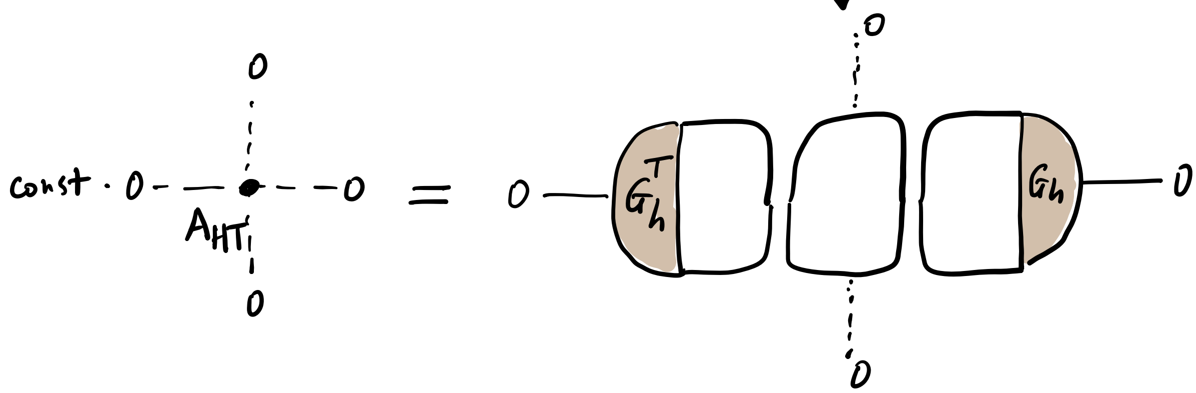
2) 

3) Recombination

$A' =$ 

Rationale CDL $\rightarrow$ A_{HT} in one step





Using disentanglers, we proved

Theorem 1 (High-T, Kennedy, S.R. '21)

$\exists$ Tensor RG map in a complex neighborhood

$$A = A_{HT} + \delta A \,, \quad \|\delta A\| < \varepsilon$$

$$\chi = \infty$$

s.t.

•) $A \to A' = \nu(A) \cdot [A_{HT} + \delta A']$

•) $\nu(A)$, $\delta A'$ are analytic in δA

•) $\nu(A) = 1 + O(\delta A^2)$

$$\|\delta A'\| < \gamma \cdot \|\delta A\| \qquad (\gamma < 1)$$

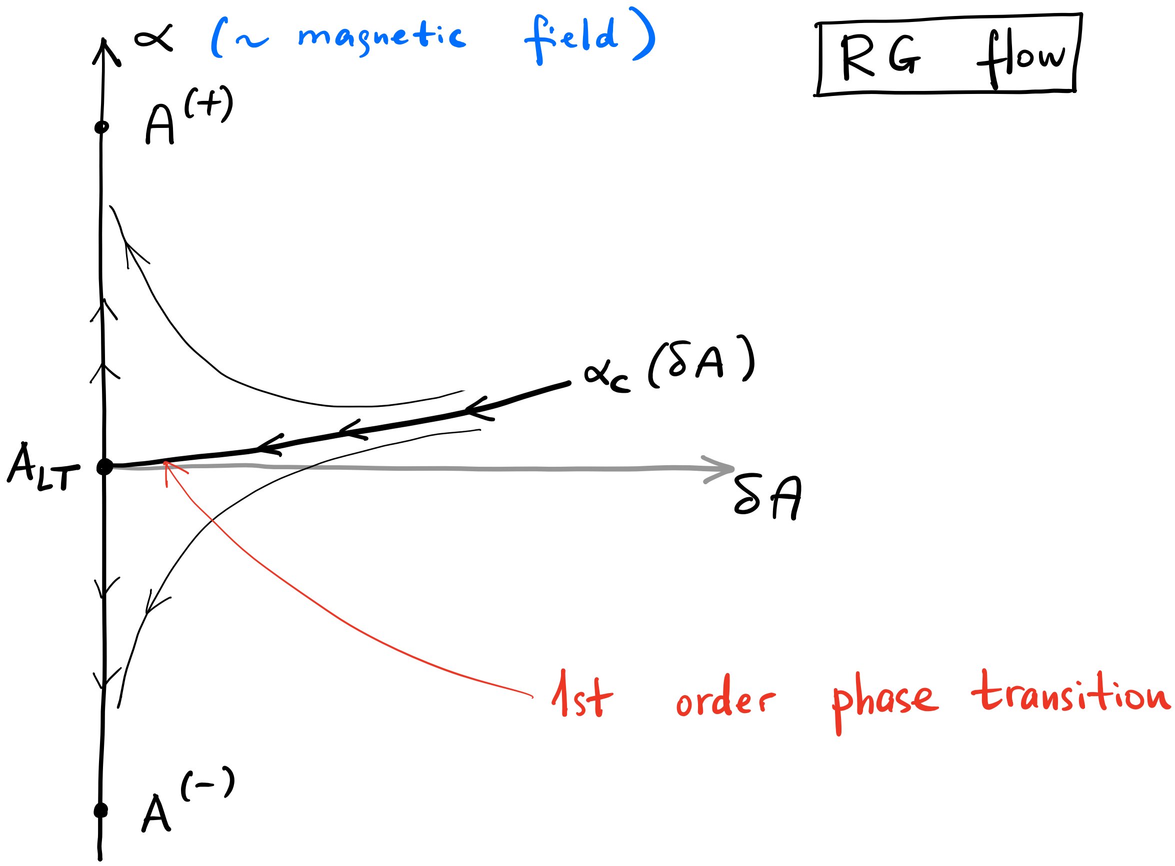
Theorem 2 (Low-T, Kennedy, S.R '22)

$\exists$ Tensor RG map defined for tensors

$$A = \alpha A^{(+)} + (1-\alpha) A^{(-)} + \delta A$$

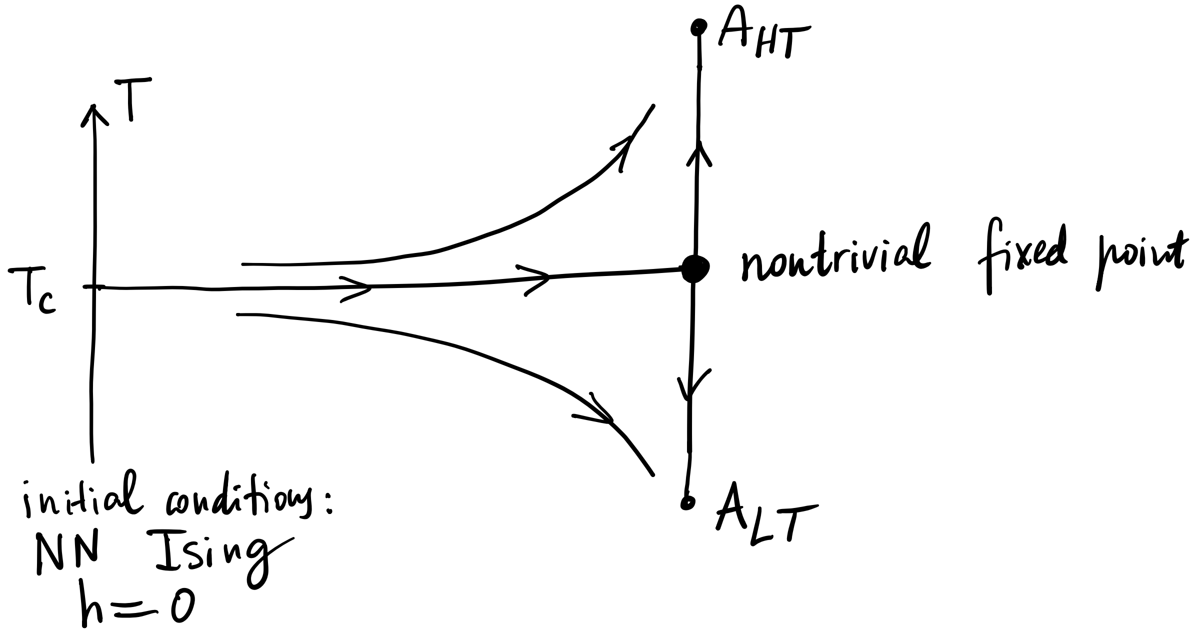
s.t. ...

$\boxed{\text{RG flow}}$



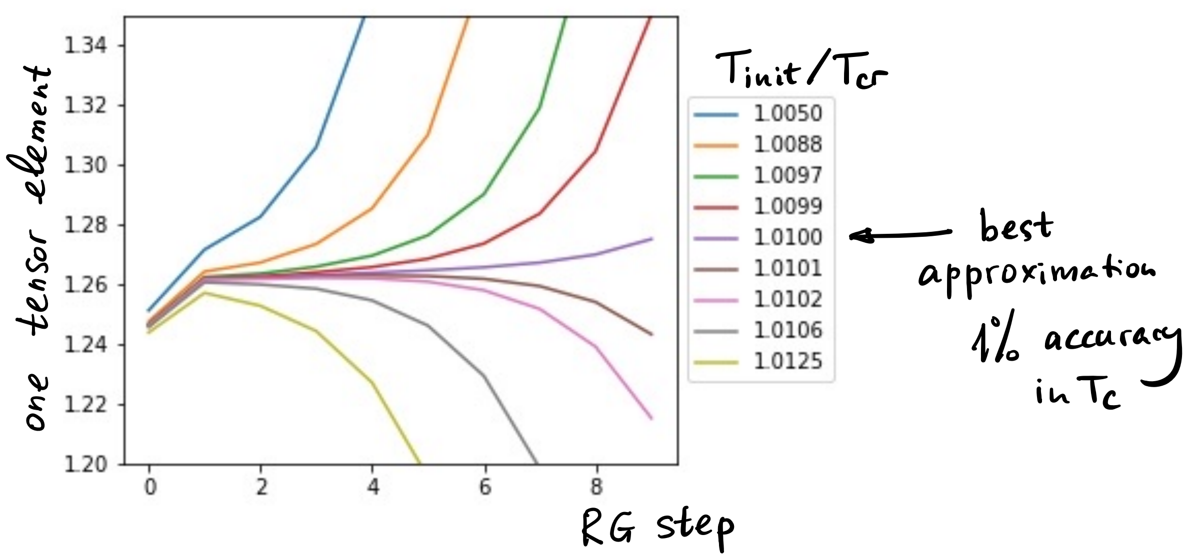
Towards nontrivial fixed point

Expect:

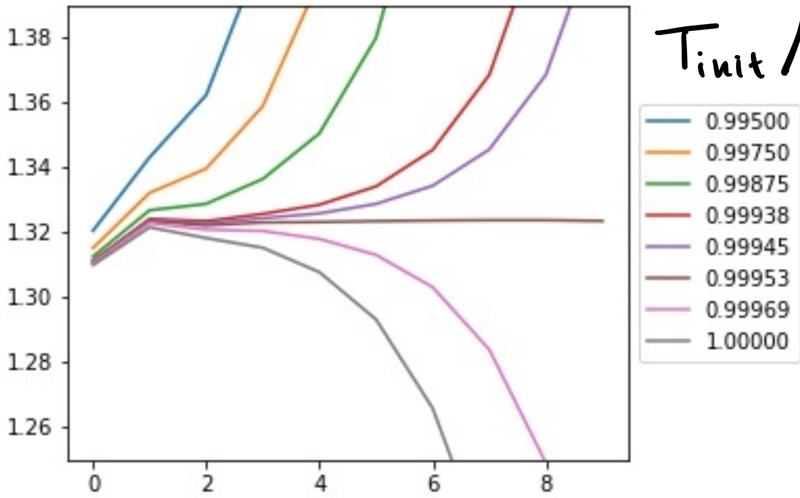


Numerical experiments (truncating to finite χ)

$\chi = 3$



$\chi = 4$



T_{init}/T_c

10^{-3} accuracy

χ	T_c accuracy
3	10^{-2}
4	10^{-3}
5	$5 \cdot 10^{-4}$
6	10^{-5}
...	

Conjecture This numerical algorithm gives a HS tensor $A_\chi \to A_*$ $(\chi \to \infty)$

Strategy for constructing A_*

- Start at χ_0, find A_{χ_0}
- Study RG map near A_{χ_0} — should be contracting except along one direction
- Use a contraction argument to show that A_* exists near A_{χ_0}

Computer-assisted proof, like Lanford's proof of Feigenbaum's conjectures [Lanford '82]