CHARACTERISING EXTREMAL GIBBS STATES

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6 February 2023

Based on joint work with Jakob Björnberg, Jürg Fröhlich

Main object: Set of infinite-volume Gibbs states of classical & quantum spin systems

Here: spin $\frac{1}{2}$ XYZ model

Domain $\Lambda \Subset \mathbb{Z}^d$

Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^2$

Spin operators $(S_x^i)_{x \in \Lambda}^{i=1,2,3}$ s.t. $[S_x^1, S_y^2] = i\,\delta_{x,y}\, S_x^3$

etc...

Hamiltonian: $H_\Lambda = -\sum_{\|x-y\|=1} \left(J_1 S_x^1 S_y^1 + J_2 S_x^2 S_y^2 + J_3 S_x^3 S_y^3 \right)$

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Special cases:
- $J_1 = J_2 = 0$: Ising model
- $J_3 = 0$, $J_1 = J_2$: XY model
- $J_1 = J_2 = J_3$: Heisenberg model

Finite-volume Gibbs state:

$$\langle a \rangle_\Lambda = \frac{1}{Z_\Lambda} \operatorname{Tr} a \, e^{-\beta H_\Lambda}, \quad Z_\Lambda = \operatorname{Tr} e^{-\beta H_\Lambda}$$

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Infinite-volume Gibbs states:

Those linear functionals on the space of local observables that satisfy the KMS condition.

Remark: If $\Lambda_n = \{-n, \dots, n\}^d$ and $K^{(n)}_{\Lambda_n} = \sum_{X \subset \Lambda_n} \Phi^{(n)}_X$ is such that $\|\Phi^{(n)}_X\| \to 0$ as $n \to \infty$, then the cluster points of

$$\frac{1}{Z_{\Lambda_n}} \operatorname{Tr} \cdot\, e^{-\beta H_{\Lambda_n} + K^{(n)}_\Lambda}$$

satisfy the KMS condition.

<u>Known:</u> The set of Gibbs states is convex.

Further, it is a Choquet simplex: any element is given by a unique convex combination of extremal states.

<u>Also known:</u> The extremal states are ergodic and they have short-range correlations.

$$\boxed{\lim_{\|x\| \to \infty} \left(\langle a \tau_x b \rangle - \langle a \rangle \langle \tau_x b \rangle \right) = 0}$$

$$\boxed{\lim_{n \to \infty} \left\langle \left(\frac{1}{n^d} \sum_{x \in \{1, \dots, n\}^d} \tau_x a - \langle a \rangle \right)^2 \right\rangle = 0}$$

Difficult to prove: Explicit characterisation of the set of extremal Gibbs states for a given model.

Here: Translation invariant states.

Ferromagnetic couplings, $J_i \geq 0$ for $i = 1, 2, 3$

Let us review what to expect, and some rigorous results.

① Ising regime $J_3 > J_1, J_2$

If $d \geq 2$, one expects that there exists β_c s.t.

- For $\beta \leq \beta_c$: Unique Gibbs state
- For $\beta > \beta_c$: Exactly two extremal Gibbs states:

$$\langle \cdot \rangle_{\pm} = \lim_{h \to 0\pm} \lim_{n \to \infty} \frac{1}{Z_{\Lambda_n}} \operatorname{Tr} \cdot e^{-\beta H_{\Lambda_n} + h \sum_x S_x^3}$$

Fully proved when $J_1 = J_2 = 0$, i.e. classical Ising model

Proof of LRO for β large and $J_1, J_2 \ll J_3$ [Ginibre '69]

$J_1 = J_2 < J_3$ [Kennedy '85]

Exactly 2 tangent functionals to the pressure [Fröhlich, Rey-Bellet, U '01]

Sort of Fourier transform (suggested by Spencer)

$$\boxed{\phi(h)} = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^3} \right\rangle_{\beta H_\Lambda} \overset{(?)}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \lim_{\Lambda' \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^3} \right\rangle_{\beta H_{\Lambda'}}$$

$$= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^3} \right\rangle = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \sum_{a=\pm} \frac{1}{2} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^3} \right\rangle_a$$

$$= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \sum_{a=\pm} \frac{1}{2} e^{\left\langle i \frac{h}{|\Lambda|} \sum_x S_x^3 \right\rangle_a} = \sum_{a=\pm} \frac{1}{2} e^{ih \langle S_0^3 \rangle_a}$$

$$= \cos\Big(h \underbrace{\langle S_0^3 \rangle_+}_{\equiv m(\beta)} \Big) = \boxed{\cos\big(h\, m(\beta)\big)}$$

Question: If we know $\phi(h)$, what can we infer?

② XY regime $J_1 = J_2 > J_3$

$d=2$: All Gibbs states respect the continuous symmetry, $\forall \beta$ [Mermin, Wagner '66; Dobrushin, Shlosman '75; Fröhlich, Pfister '81]

Does this imply that the Gibbs state is unique?

$d \geq 3$: One expects that these are extremal Gibbs states:

For $\vec{a} \in \mathbb{S}^1$:

$$\langle \cdot \rangle_{\vec{a}} = \lim_{h \to 0+} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \langle \cdot \rangle_{\beta H_\Lambda - h \sum_{x \in \Lambda} (a_1 S_x^1 + a_2 S_x^2)}$$

Distinct states when $\beta > \beta_c$

LRO proved when $J_3 = 0$, β large [Dyson, Lieb, Simon '78]

The sort of Fourier transform

$$\boxed{\phi(h)} = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^1} \right\rangle_{\beta H_\Lambda} \overset{(?)}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \lim_{\Lambda' \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^1} \right\rangle_{\beta H_{\Lambda'}}$$

$$= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^1} \right\rangle = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \int_{\mathbb{S}^1} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^1} \right\rangle_{\vec{a}} \, d\vec{a}$$

$$= \lim_{\Lambda \Uparrow \mathbb{Z}^d} \int_{\mathbb{S}^1} e^{\left\langle i \frac{h}{|\Lambda|} \sum_x S_x^1 \right\rangle_{\vec{a}}} \, d\vec{a} = \int_{\mathbb{S}^1} e^{ih \langle S_0^1 \rangle_{\vec{a}}} \, d\vec{a}$$

$$= \int_{\mathbb{S}^1} e^{ih \langle \vec{a} \cdot \vec{S}_0 \rangle_{\vec{e}_3}} \, d\vec{a} = \int_{\mathbb{S}^1} e^{ih\, a_3 \langle S_0^3 \rangle_{\vec{e}_3}} \, d\vec{a}$$

$$\equiv m(\beta)$$

$$= \boxed{\sum_{k \geq 0} \frac{1}{(k!)^2} \left(i h\, m(\beta) \right)^{2k}}$$

(a Bessel function)

③ Heisenberg regime $J_1 = J_2 = J_3$

$d = 2$: Unique Gibbs state $\forall \beta$?

$d \geq 3$: One expects that these are extremal Gibbs states:

For $\vec{a} \in \mathbb{S}^2$:

$$\langle \cdot \rangle_{\vec{a}} = \lim_{h \to 0+} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \langle \cdot \rangle_{\beta H_\Lambda - h \sum_{x \in \Lambda} \vec{a} \cdot \vec{S}_x}$$

Distinct states when $\beta > \beta_c$

The sort of Fourier transform: One finds

$$\boxed{\phi(h)} = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \left\langle e^{i \frac{h}{|\Lambda|} \sum_x S_x^1} \right\rangle_{\beta H_\Lambda}$$

$$= \int_{\mathbb{S}^2} e^{i h a_3 \underbrace{\langle S_0^3 \rangle_{\vec{e}_3}}_{\equiv m(\beta)}} d\vec{a} = \boxed{\frac{\sin(h\, m(\beta))}{h\, m(\beta)}}$$

We found:

Ising: $\phi(h) = \cos(h\, m(\beta))$

XY: $\phi(h) = \sum_{k \geq 0} \frac{1}{(k!)^2} (i\, h\, m(\beta))^{2k}$

Heisenberg: $\phi(h) = \frac{\sin(h\, m(\beta))}{h\, m(\beta)}$

$m(\beta)$ depends on microscopic details (dimension, strength of interactions, ...)

The function $\phi(h)$ depends otherwise on nature of symmetry breaking

It can be calculated on complete graphs [Björnberg, Fröhlich, U '20]

Connection with loop models

Some quantum (& classical) spin systems can be represented by loop models [Tóth '93; Aizenman, Nachtergaele '94; U '13]

The sort of Fourier transform can be expressed as expectation of lengths of loops.

The latter have Poisson-Dirichlet distribution [Goldschmidt, U, Windridge '11] for the conjecture; [Schramm '05; Berestycki, Kozma '15] for rigorous results on the complete graph.

$\longrightarrow$ Alternate way to calculate $\Phi(h)$.

Conclusion

- Still many open questions about set of Gibbs states in such models as XYZ spin systems.
- Spencer's Fourier transform could help! Convenient on complete graph & with loop models
- What exactly do we learn from knowing this function?

$\boxed{\text{GRAZIE}}$