The Spectral Gap and Low-Energy Spectrum in Mean-Field Quantum Spin Systems

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Joint work with Chokri Manai: arXive:2302.00465

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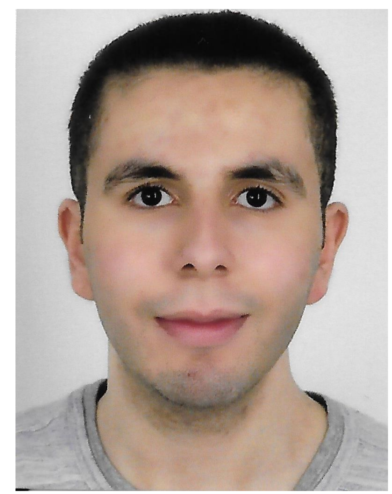

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Mean-field quantum spin systems

N qubits with mean-field interaction given by a symmetric polynomial $P : \mathbb{R}^3 \to \mathbb{R}$

$$H = N\, P\left(\tfrac{2}{N}\, \mathbf{S}\right) \quad \text{on} \quad \mathcal{H}_N = \bigotimes_{n=1}^{N} \mathbb{C}^2$$

$\mathbf{S} = \sum_{n=1}^{N} \mathbf{S}(n)$ the vector of **total spin**

$[S_x, S_y] = iS_z$ (and cyclically)

Prototype: $P(\mathbf{m}) = -\alpha m_x^2 - \beta m_y^2 - \gamma m_z$ $\alpha = 1, \beta = 0$: Quantum Curie-Weiss model

Why interesting? Effective descriptions for:

shape transitions of nuclei — Lipkin-Meshkov-Glick 65, . . .

interacting Bosons in a double well — Turbiner '88, . . . Cirac-Lewenstein-Mølmer-Zoller '98, . . .

quantum annealing — Bapst-Semerjian '12, . . .

. . .

Questions? Free energy, **low-energy spectrum**, effective dynamics, . . .

. . . , Fannes-Spohn-Verbeurre '80, . . . , Petz-Raggio-Verbeurre '88, Raggio-Werner '88, . . . , Ribeiro-Vidal-Mosseri '08, Cayes-Crawford-Ioffe-Levit '08, . . . , Landsmann-Moretti-van de Ven '20, . . . , Björnberg-Fröhlich-Ueltschi '20, . . .

Structure of mean-field spin systems

N qubits with mean-field interaction given by a symmetric polynomial $P : \mathbb{R}^3 \to \mathbb{R}$

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$\mathbf{S} = \sum_{n=1}^{N} \mathbf{S}(n)$ the vector of **total spin**

$[S_x, S_y] = iS_z$ (and cyclically)

H is block diagonal with respect to the decomposition into irreps of $SU(2)$ according to total spin:

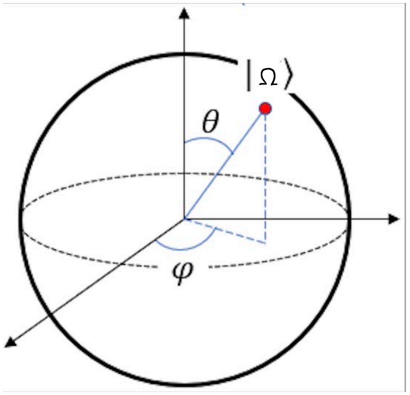
$$\mathcal{H}_N \equiv \bigoplus_{J=\frac{N}{2}-\lfloor\frac{N}{2}\rfloor}^{N/2} \bigoplus_{\alpha=1}^{M_{N,J}} \mathbb{C}^{2J+1}, \qquad M_{N,J} = \frac{2J+1}{N+1}\binom{N+1}{\frac{N}{2}+J+1}.$$

Block (J,α)**-Hamiltonian**: $H_{J,\alpha} = N\, P\Big(\tfrac{2}{N}\,\mathbf{S}\Big)$ with spin-J operator $\mathbf{S} = (S_x, S_y, S_z)^T$ on $\mathbb{C}^{2J+1}$.

cf. analysis of large J limit: Hepp-Lieb ’73, . . . , Michoel-Nachtergaele ’04-’05, . . . , Biskup-Chayes-Starr ’07, . . .

Bloch coherent states

Spin J operators: $[S_x, S_y] = iS_z$ (and cyclically) $S_\pm = S_x \pm iS_y$ on Hilbert space $\mathbb{C}^{2J+1}$



Bloch coherent state: $\Omega = (\theta, \varphi), \quad 0 \le \theta \le \pi, \; 0 \le \varphi \le 2\pi$

$$|\Omega\rangle := \exp\left(\frac{\theta}{2}\left(e^{i\varphi}S_- - e^{-i\varphi}S_+\right)\right)|J\rangle$$

Properties:

Concentration: $|\langle\Omega'|\Omega\rangle|^2 = [\cos(\Delta(\Omega', \Omega)/2)]^{4J}$ Overcompleteness: $\frac{2J+1}{4\pi}\int d\Omega\,|\Omega\rangle\langle\Omega| = \mathbb{1}$

Symbols of a linear operator G on $\mathbb{C}^{2J+1}$

Lower: $g(\Omega) := \langle\Omega|G|\Omega\rangle$ Upper: $G = \frac{2J+1}{4\pi}\int d\Omega\, G(\Omega)\,|\Omega\rangle\langle\Omega|$

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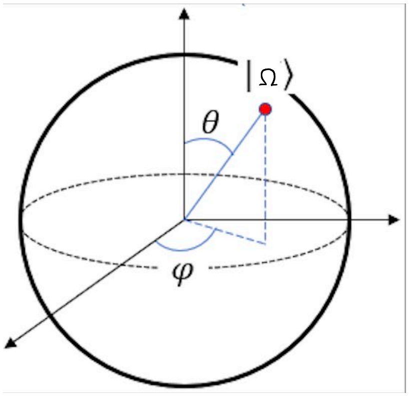

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Operator	$g(\Omega)$, (2.15)	$G(\Omega)$, (2.13)
S_z	$J\cos\theta$	$(J+1)\cos\theta$
S_x	$J\sin\theta\cos\varphi$	$(J+1)\sin\theta\cos\varphi$
S_y	$J\sin\theta\sin\varphi$	$(J+1)\sin\theta\sin\varphi$
$S_z^{\ 2}$	$J(J-\frac{1}{2})(\cos\theta)^2 + J/2$	$(J+1)(J+3/2)(\cos\theta)^2 - \frac{1}{2}(J+1)$
$S_x^{\ 2}$	$J(J-\frac{1}{2})(\sin\theta\cos\varphi)^2 + J/2$	$(J+1)(J+3/2)(\sin\theta\cos\varphi)^2 - \frac{1}{2}(J+1)$
$S_y^{\ 2}$	$J(J-\frac{1}{2})(\sin\theta\cos\varphi)^2 + J/2$	$(J+1)(J+3/2)(\sin\theta\cos\varphi)^2 - \frac{1}{2}(J+1)$

Lieb '73

General $J \to \infty$ limit:
Duffield '90

Semiclassics for mean-field spin systems

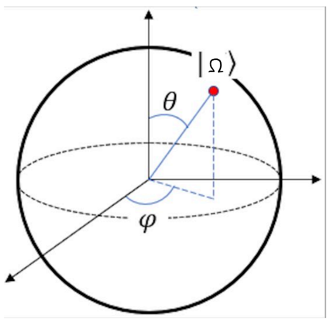
Block (J,α)-Hamiltonian: $H_{J,\alpha} = N\,P\left(\frac{2}{N}\,\mathbf{S}\right)$ with spin-J operator $\mathbf{S} = (S_x, S_y, S_z)^T$ on $\mathbb{C}^{2J+1}$.

Quantitative Duffield Theorem: Manai/W. '23

For some $C \in [0,\infty)$:

Lower: $$\sup_N \sup_{0\leq J\leq N/2} \sup_\Omega \left| \langle \Omega, J | H_{J,\alpha} | \Omega, J\rangle - N\,P\big(\tfrac{2J}{N}\,\mathbf{e}(\Omega)\big) \right| \leq C.$$

Upper: $$\sup_N \sup_{J,\alpha} \left\| H_{J,\alpha} - \frac{2J+1}{4\pi} \int d\Omega\; N\,P\big(\tfrac{2J}{N}\,\mathbf{e}(\Omega)\big)\, |\Omega, J\rangle\langle \Omega, J| \right\| \leq C,$$



Consistent **semiclassical symbol** $P : B_1 \to \mathbb{R}$ on B_1 parametrising the **phase space** of the mean-field system of N **qubits**!

Semiclassics for mean-field spin systems

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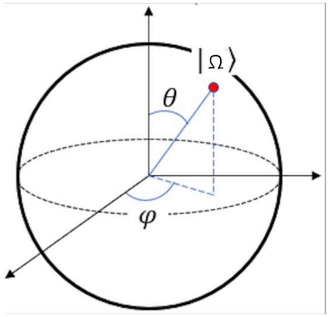

Immediate consequence of the above and the **Berezin-Lieb inequalities**: Berezin '72, Lieb '73

$$\lim_{N\to\infty} N^{-1} \ln \operatorname{Tr} \exp\left(-\beta N \mathrm{P}\left(\tfrac{2}{N}\mathbf{S}\right)\right) = \max_{r\in[0,1]} \left\{ I(r) - \beta \min_{\Omega\in S^2} \mathrm{P}\left(r\mathbf{e}(\Omega)\right)\right\}.$$

with the binary entropy $I(r) = -\frac{1+r}{2}\ln\frac{1+r}{2} - \frac{1-r}{2}\ln\frac{1-r}{2}$. Manai-W. '23

cf. Fannes-Spohn-Verbeurre '80

Low-energy spectra & spectral gap

Recipe from fluctuation theory:

Quadratic approximations at the minima $\mathbf{m}_0$ $(\mathbf{m}_1, \ldots, \mathbf{m}_L) \in B_1$ of P determine the low-energy spectra and spectral gap.

cf. Bose systems: Grech-Seiringer '13, Lewin-Nam-Serfaty-Solovej '15, ...

- Simple-minded Taylor:

$$P(\mathbf{m}) = P(\mathbf{m}_0) + \nabla P(\mathbf{m}_0) \cdot (\mathbf{m} - \mathbf{m}_0) + \frac{1}{2}(\mathbf{m} - \mathbf{m}_0) D_P(\mathbf{m}_0)(\mathbf{m} - \mathbf{m}_0) + \mathcal{O}\left((\mathbf{m} - \mathbf{m}_0)^3\right)$$

- Case $|\mathbf{m}_0| = 1$: $\nabla P(\mathbf{m}_0) = -|\nabla P(\mathbf{m}_0)| \mathbf{e}_{\mathbf{m}_0}$ does not vanish in general!
- Fluctuations for fixed J occur only in the angular directions:

  Local chart $\Phi : \operatorname{ran} Q_\perp \to T_{\mathbf{m}_0} S^2$ and its quadratic approximation to $P \circ \Phi$:

$$D_P^\perp(\mathbf{m}_0) := Q_\perp D_P(\mathbf{m}_0) Q_\perp + |\nabla P(\mathbf{m}_0)| Q_\perp, \qquad Q_\perp := \mathbb{1}_{\mathbb{R}^3} - \mathbf{e}_{\mathbf{m}_0}^T \mathbf{e}_{\mathbf{m}_0}.$$

Low-energy spectra & spectral gap

Theorem (Manai-W. '23)

*In case P has a **unique global minimum** at $\mathbf{m}_0 \in S^2$ at which $|\nabla P(\mathbf{m}_0)|, \det D_P^{\perp}(\mathbf{m}_0) > 0$, the lowest eigenvalues of H coincides with points in the set*

$$NP(\mathbf{m}_0) + (2k-1)|\nabla P(\mathbf{m}_0)| + (2m+1)\sqrt{\det D_P^{\perp}(\mathbf{m}_0)} + o(1)$$

where $m \in \mathbb{N}_0$ and $k = N/2 - J \in \mathbb{N}_0$ relates to the total spin J. In particular, the ground-state is unique and found at $J = N/2$, and the spectral gap is

$$\operatorname{gap} H = 2 \min \left\{ |\nabla P(\mathbf{m}_0)|, \sqrt{\det D_P^{\perp}(\mathbf{m}_0)} \right\} + o(1).$$

- Surprisingly simple formula for the gap! (cf. physicist's recipe with DOS).
- **Quantum Curie-Weiss model:** $P(\mathbf{m}) = -m_x^2 - \gamma m_z$

  Paramagnetic phase $\gamma > 2$: $\mathbf{m}_0 = (0,0,1)^T$ $\operatorname{gap} H = 2\sqrt{\gamma(\gamma-2)}$

  Ferromagnetic phase $0 \leq \gamma < 2$: $\mathbf{m}_0^{\pm} = (\pm\sqrt{1-\gamma^2/4}, 0, \gamma/2)^T$

$$|\nabla P(\mathbf{m}_0^{\pm})| = 2, \quad \det D_P^{\perp}(\mathbf{m}_0^{\pm}) = 4 - \gamma^2$$

Low-energy spectra & spectral gap

Theorem (Manai-W. '23)

*In case P has a **finite number of global minima** at $\mathbf{m}_1, \ldots, \mathbf{m}_L \in S^2$ at which $|\nabla P(\mathbf{m}_l)|, \det D_P^\perp(\mathbf{m}_l) > 0$, the lowest eigenvalues of H coincides with points in the set*

$$NP(\mathbf{m}_l) + (2k-1)|\nabla P(\mathbf{m}_l)| + (2m+1)\sqrt{\det D_P^\perp(\mathbf{m}_l)} + o(1)$$

where $l \in \{1, \ldots, L\}$, $m \in \mathbb{N}_0$ and $k = N/2 - J \in \mathbb{N}_0$ relates to the total spin J.

- **Quantum Curie-Weiss model:** $P(\mathbf{m}) = -m_x^2 - \gamma m_z$

  Paramagnetic phase $\gamma > 2$: $\mathbf{m}_0 = (0,0,1)^T$ $\quad$ $\operatorname{gap} H = 2\sqrt{\gamma(\gamma-2)}$

  Ferromagnetic phase $0 \leq \gamma < 2$: $\mathbf{m}_0^\pm = (\pm\sqrt{1-\gamma^2/4}, 0, \gamma/2)^T$

$$|\nabla P(\mathbf{m}_0^\pm)| = 2, \quad \det D_P^\perp(\mathbf{m}_0^\pm) = 4 - \gamma^2$$

  All low-energy levels are almost doubly degenerate with gap $4\sqrt{1-\gamma^2/2}$ separating the lowest doublets.

Low-energy spectra & spectral gap

Compare to the case of **vanishing gap** if the minimum of P is in the interior of B_1.

Theorem (Manai-W. '23)

In case the ***unique global minimum*** $\mathbf{m}_0 \in B_1$ *is at* $0 < |\mathbf{m}_0| < 1$ *with* $D_P(\mathbf{m}_0) > 0$*. Then the ground state is contained in a subspace with total spin* J *with* $|J - N|\mathbf{m}_0|/2| \leq \mathcal{O}(\sqrt{N})$ *and*

$$E_0(H) = E_0(H_{J,\alpha}) = NP(\mathbf{m}_0) + |\mathbf{m}_0|\sqrt{\det D_P^{\perp}(\mathbf{m}_0)} + o(1).$$

For any J *with* $|J - N|\mathbf{m}_0|/2| \leq o(\sqrt{N})$ *the ground-state energy* $E_0(H_{J,\alpha})$ *is still given by the above formula.*

Controlling fluctuations

For simplicity, **unique minimizers** at $\mathbf{m}_0 = \mathbf{e}_z \in S^2$.

Overall strategy: Investigate spectra of each block $H_{J,\alpha}$ separately.

Subspace of $\mathbb{C}^{2J+1}$ at minimizing direction is spanned by z-basis:

$$\mathcal{H}_J^K = \operatorname{span}\left\{ |N/2 - k\rangle \in \mathbb{C}^{2J+1} \,\middle|\, k \in \{0,,1,\ldots,K\}\right\}$$

- $\left\|(2S_z/N - 1)P_J^K\right\| \leq K/N.$
- Size of **fluctuation operators** $\xi \in \{x, y\}$: $\left\|\sqrt{\frac{2}{N}} S_\xi \, P_J^K\right\| \leq C_J \sqrt{K}.$

Quadratic approximation of P at $\mathbf{m}_0$:

$$Q_J(\mathbf{m}_0) := N\, P(\mathbf{m}_0) + 2\left(\mathbf{S} - \tfrac{N}{2}\mathbf{m}_0\right) \cdot \nabla P(\mathbf{m}_0) + \tfrac{2}{N}\mathbf{S} \cdot D_P^\perp(\mathbf{m}_0)\mathbf{S} \quad \text{on } \mathbb{C}^{2J+1}.$$

$$\left\|(H_{J,\alpha} - Q_J(\mathbf{m}_0))P_J^{K_N}\right\| = o(1) \qquad \text{as long as } K_N = o(N^{1/3}) \text{ and } J \geq N/2 - K_N.$$

Controlling fluctuations

Abbreviate *by* $|\nabla P| := |\nabla P(\mathbf{m}_0)|$ and by $\omega_{x/y}$ the eigenvalues of $D_P^\perp(\mathbf{m}_0)$.

- On the subspace $\mathcal{H}_J^{K_N}$ with $J \geq N/2 - K_N$:

$$\begin{aligned} Q_J(\mathbf{e}_z) - N\,P(\mathbf{m}_0) &= |\nabla P|\,(N - 2S_z) + \frac{2}{N}\left(\omega_x S_x^2 + \omega_y S_y^2\right) \\ &= |\nabla P|\,\frac{N^2 - 4S_z^2}{2N} + \frac{2}{N}\left(\omega_x S_x^2 + \omega_y S_y^2\right) + o(1) \\ &= |\nabla P|\,\left[\frac{N}{2} - \frac{2J(J+1)}{N}\right] + \frac{2}{N}\left[(\omega_x + |\nabla P|)S_x^2 + (\omega_y + |\nabla P|)S_y^2\right] + o(1) \\ &= |\nabla P|\,\left[\frac{N}{2} - J - 1\right] + D_N + o(1) \end{aligned}$$

Note: $\left[\sqrt{\tfrac{2}{N}}S_x, \sqrt{\tfrac{2}{N}}S_y\right] = i\,2S_z/N = i\,(1 + o(1))$

and hence D_N is represents a **harmonic oscillator** for large N.

- On the subspace $\left(\mathcal{H}_J^{K_N}\right)^\perp$: Since $P(\mathbf{m}) \geq P(\mathbf{m}_0) + c\,|m_z - 1|$ for some $c > 0$:

$$H_{J,\alpha} \geq C + NP(\mathbf{m}_0) + c|2S_z - N|$$

by the quantitative Duffield's theorem with some C. The last term causes a shift of K_N.

Further comments on the proof

1. Two-step approximation method and **Schur complement analysis**

$$\mathcal{H}_J^{K_N} \longrightarrow \begin{cases} \text{Eigenspace of limiting harmonic oscillator} \\ D = \omega_x L_x^2 + \omega_y L_y^2 \quad \text{on } \ell^2(\mathbb{N}_0) \\ \text{projected down to } \mathcal{H}_J^{K_N} \subset \mathbb{C}^{2J+1}. \end{cases}$$

Limiting fluctuation operators on $\ell^2(\mathbb{N}_0)$ in the canonical basis:

$$\langle k|L_x|k'\rangle = i^{k'-k}\langle k|L_y|k'\rangle = \sqrt{\frac{\max\{k,k'\}}{2}}\,\delta_{|k-k'|=1}.$$

2. Case of finitely many minima: Controlling **tunnelling** through a macroscopic barrier by separating patches through **concentration estimates** for coherent states

$$\mathcal{H}_J^{K_N} \longrightarrow \bigvee_{l=1}^{L} \mathcal{H}_J^{K_N}(\mathbf{m}_l)$$

3. Proof generalises from polynomials to mean-field operators with smooth upper symbol $h \in C^2$:

$$H = \bigoplus_{J=\frac{N}{2}-\lfloor\frac{N}{2}\rfloor}^{N/2} \bigoplus_{\alpha=1}^{M_{N,J}} H_{J,\alpha} \qquad H_{J,\alpha} = \frac{2J+1}{4\pi}\int d\Omega\, N\, P\big(\tfrac{2J}{N}\,\mathbf{e}(\Omega)\big)\,\big|\Omega,J\rangle\langle\Omega,J\big|$$

For details see: arXive:2302.00465

Summary

Semiclassical criteria & expressions for spectral gaps in mean-field systems of N spin-$\frac{1}{2}$:

$$\text{If } |\mathbf{m}_0| = 1: \qquad \operatorname{gap} H = 2 \min \left\{ |\nabla P(\mathbf{m}_0)|, \sqrt{\det D_P^{\perp}(\mathbf{m}_0)} \right\} + o(1).$$

Complete description of low-energy spectral for gapped systems in terms of a limiting (harmonic oscillator) operator! See: arXive:2302.00465

Related results & questions:

- Corrections to the free energy also for $\beta < \infty$
- Spectral gap in the Lebowitz-Penrose limit
- Generalization spin–$\frac{1}{2}$ to spin-s
- Semiclassical time evolutions Fröhlich-Knowles-Lenzmann '07
- ...