The Matrix Dyson Equation in random matrix theory

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DISCUSSION AREA IN THE MATH FLOOR

"Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients."

Eugene Wigner, 1956

Nobel prize 1963

Plan of the lectures

1. Introduction, Wigner's vision, concepts, correlation functions, scales, WDM universality, Wigner vs. invariant ensembles.

2. Three step strategy, Dyson Brownian motion, Local law, Stieltjes transform, Helffer-Sjöstrand formula,

3. Models of increasing complexity, singularity structure of the density, motivations from physics: universal dichotomy in disordered quantum systems, band matrices.

4. Some ideas of proofs: moment vs. resolvent method. Schur formula, derivation of the Dyson equation. Large deviation estimates, Ward identity. Analysis of the Dyson equation, existence/uniqueness, self-energy operator, Perron-Frobenius theorem, 1/3-Hölder regularity. Matrix Dyson equation, symmetric polar decomposition.

LECTURE 1: INTRODUCTION

Basic question [Wigner]: What can be said about the statistical properties of the eigenvalues of a large random matrix? Do some universal patterns emerge?

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \Longrightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

N = size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

Wigner Ensemble: i.i.d. entries

$H = (h_{jk})$ real symmetric or complex hermitian $N \times N$ matrix

Entries are i.i.d. up to $h_{jk} = \bar{h}_{kj}$ (for $j < k$), with normalization

$$\mathbb{E} h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}.$$

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are of order one: (on average)

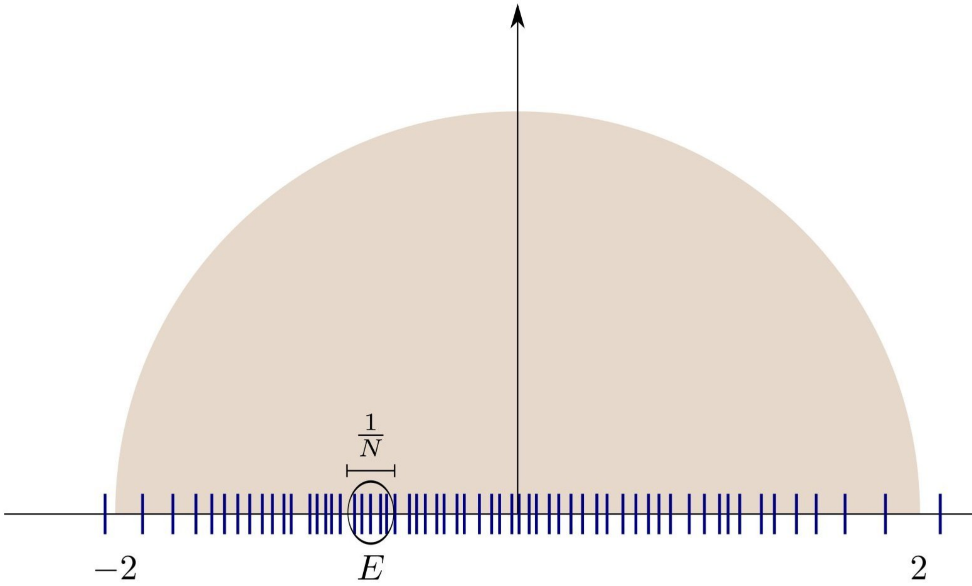
$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 = \mathbb{E}\frac{1}{N}\mathrm{Tr} H^2 = \frac{1}{N}\sum_{ij}\mathbb{E}|h_{ij}|^2 = 1$$

If h_{ij} is Gaussian, then GUE, GOE.

Global vs. local law

Global density: Semicircle Law

Typical ev. gap $\approx \frac{1}{N}$ (bulk)



- Does semicircle law hold just above this scale? ($\Longrightarrow$ local semicircle law)
- How do eigenvalues behave exactly on this scale? ($\Longrightarrow$ WDM universality)

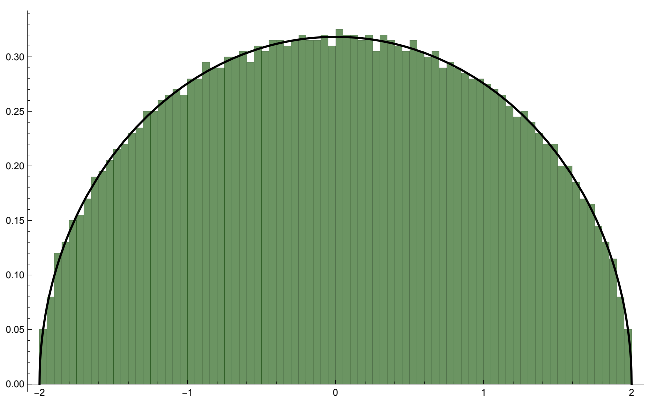
Wigner's revolutionary observation: The global density may be model dependent, but the gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric).

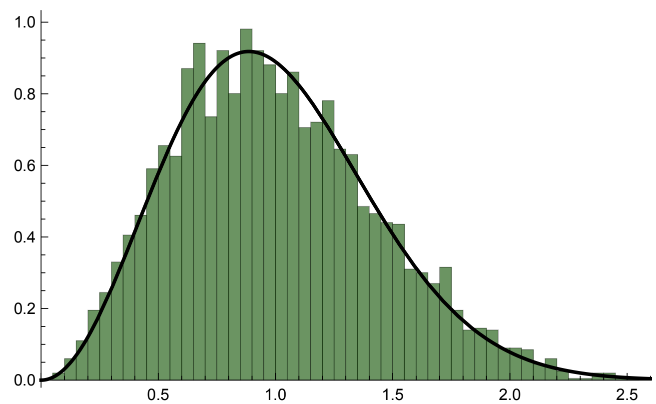
In particular, it can be determined from the Gaussian case (GUE/GOE).

Wigner surmise and level repulsion

$$\mathbb{P}_{GOE}\Big(N\varrho(\lambda_i)(\lambda_{i+1}-\lambda_i)=s+\mathrm{d}s\Big)\simeq\frac{\pi}{2}\,{\color{red}s}\,\mathrm{e}^{-\pi s^2/4}\,\mathrm{d}s$$
$$\mathbb{P}_{GUE}\Big(N\varrho(\lambda_i)(\lambda_{i+1}-\lambda_i)=s+\mathrm{d}s\Big)\simeq\frac{32}{\pi^2}\,{\color{red}s^2}\,\mathrm{e}^{-4s^2/\pi}\,\mathrm{d}s\,,$$

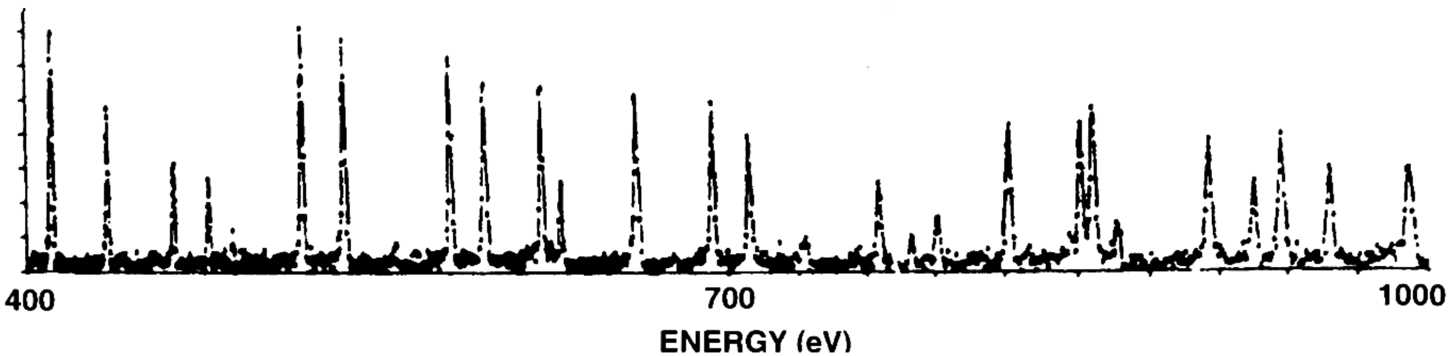
for λ_i in the bulk (repulsive correlation depending on the symmetry!)





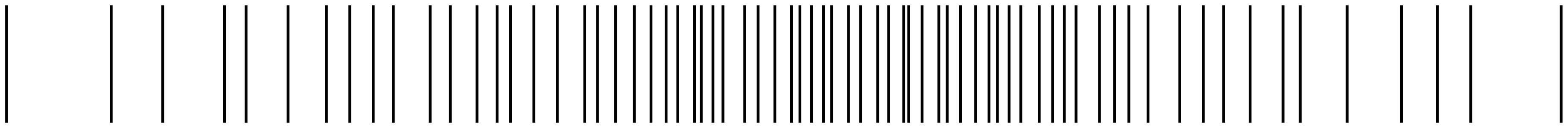
Side note: Wigner surmise guessed on a 2×2 matrix calculation and is only an approximation to the true gap distribution given by a Fredholm determinant of the Dyson's sine-kernel [Gaudin, Mehta, Dyson]

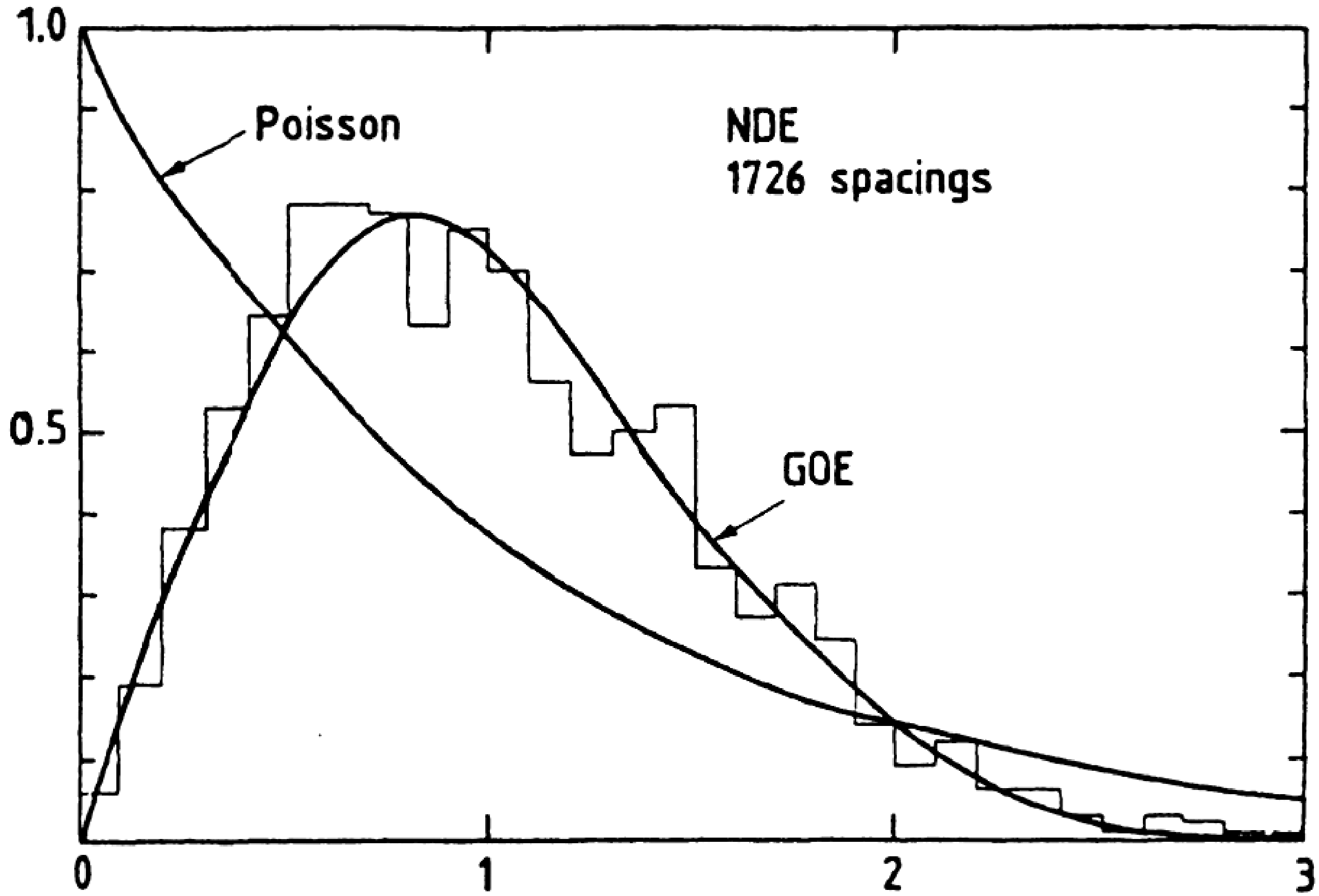
E. Wigner (1955): The excitation spectra of heavy nuclei have the same spacing distribution as the eigenvalues of GOE.
Experimental data for excitation spectra of heavy nuclei: (^{238}U)



Typical Poisson statistics:

Typical GOE random matrix eigenvalues





Level spacing (gap) histogram for different point processes.

NDE – Nuclear Data Ensemble, resonance levels of 30 sequences of 27 different nuclei.

Zeros of the Riemann-zeta function (detour)

$$\delta_n = \widehat{\gamma}_{n+1} - \widehat{\gamma}_n, \qquad \widehat{\gamma}_n = \frac{1}{2\pi}\gamma_n \log \gamma_n, \qquad \zeta(\frac{1}{2} + i\gamma_n) = 0$$

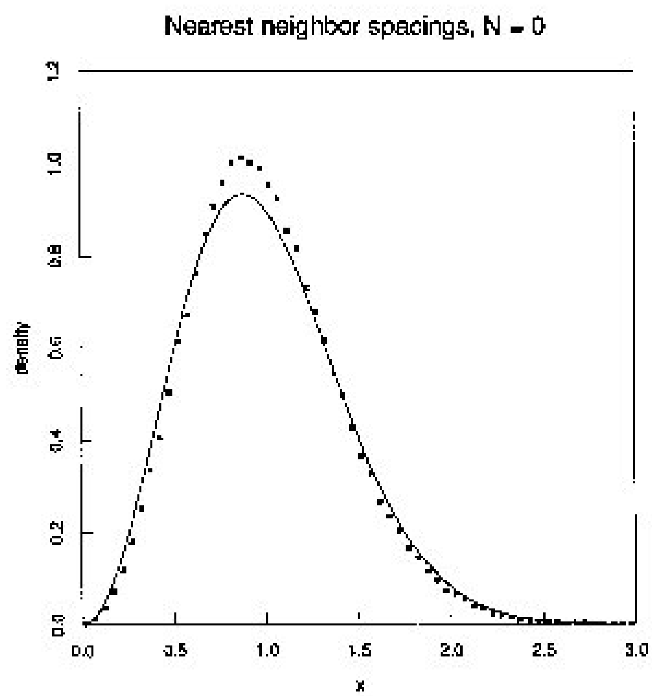


FIGURE 3

Probability density of the normalized spacings δ_n. Solid li GUE prediction. Scatter plot: empirical data based on ze γ_n, $1 \leqslant n \leqslant 10^5$.

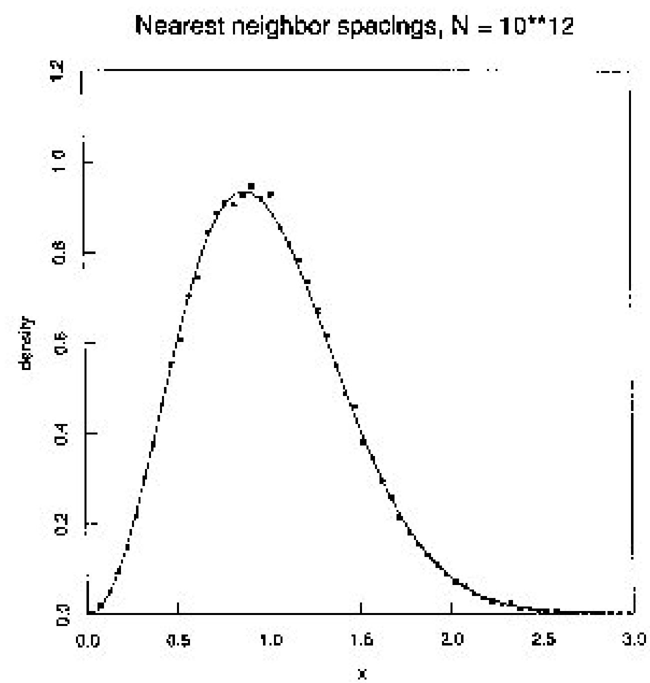


FIGURE 4

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} - 10^5$.

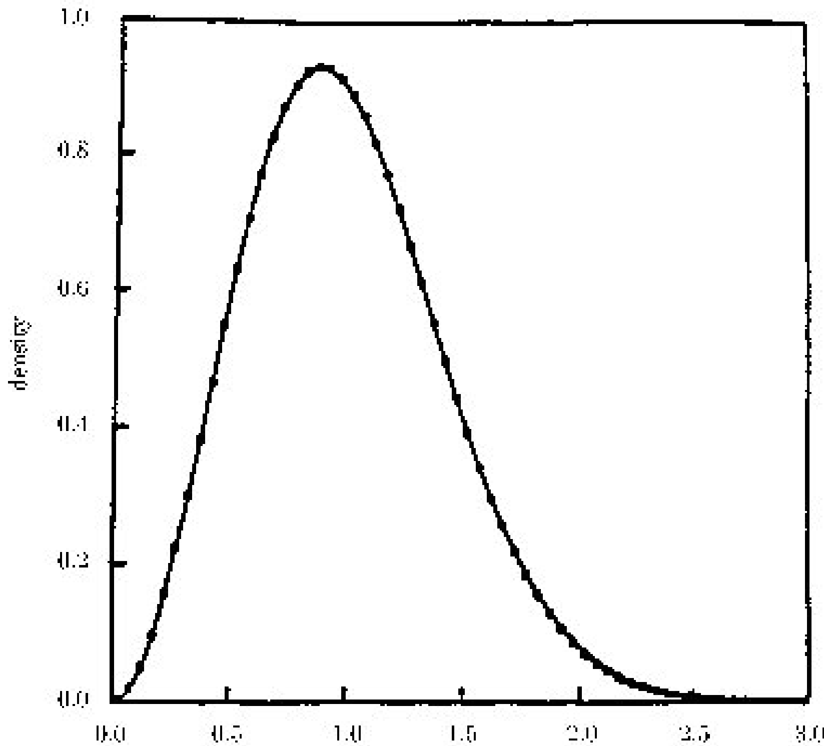


FIGURE 1. Nearest neighbor spacings among 70 million zeroes beyond the 10^{20}-th zero of zeta, verses μ_1(GUE).

Pair correlation functions (sine kernel)

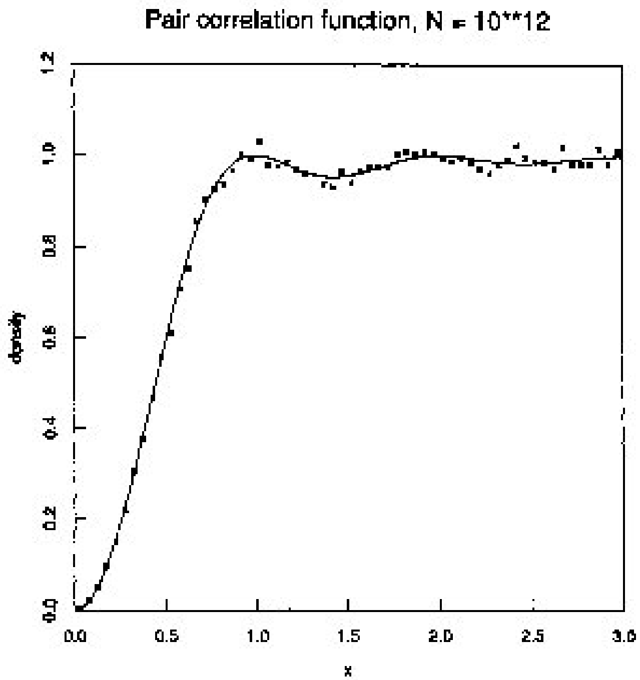


FIGURE 2

Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12}+1 \leqslant n \leqslant 10^{12}+10^5$.

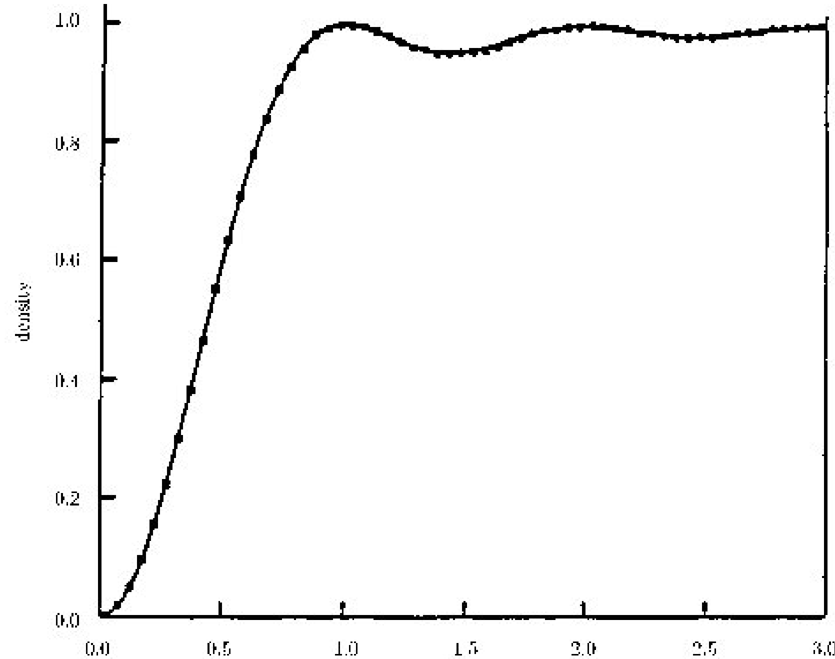


FIGURE 2. Pair correlation for zeros of zeta based on 8×10^6 zeros near the 10^{20}-th zero, versus the GUE conjectured density $1-\left(\frac{\sin \pi x}{\pi x}\right)^2$.

Sine kernel for correlation functions

Probability density of the eigenvalues: $p(x_1, x_2, \dots, x_N)$

The k-point correlation function is given by

$$p_N^{(k)}(x_1, x_2, \dots, x_k) := \int_{\mathbb{R}^{N-k}} p(x_1, \dots x_k, x_{k+1}, \dots, x_N) \mathrm{d}x_{k+1} \dots \mathrm{d}x_N$$

$k = 1$ point correlation function: density ϱ

Rescaled correlation functions at energy E (in the bulk, $\varrho(E) > 0$)

$$p_E^{(k)}(\mathbf{x}) := \frac{1}{[\varrho(E)]^k} p_N^{(k)}\Big(E + \frac{x_1}{N\varrho(E)}, E + \frac{x_2}{N\varrho(E)}, \dots, E + \frac{x_k}{N\varrho(E)}\Big)$$

Rescales the gap $\lambda_{i+1} - \lambda_i$ to O(1).

Local correlation statistics for GUE [Gaudin, Dyson, Mehta]

$$\lim_{N\to\infty} p_E^{(k)}(\mathbf{x}) = \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{k}, \qquad S(x) := \frac{\sin \pi x}{\pi x}$$

Special $k = 2$ case: Pair correlations.

$$\lim_{N\to\infty} \frac{1}{[\rho(E)]^2} p_N^{(2)}\left(E + \frac{x_1}{N\rho(E)}, E + \frac{x_2}{N\rho(E)}\right)$$

$$= \det\left\{S(x_i - x_j)\right\}_{i,j=1}^{2}, \qquad S(x) := \frac{\sin \pi x}{\pi x}, \quad |E| < 2$$

$$= 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2 \qquad \Longrightarrow \qquad \text{Level repulsion}$$

Wigner-Dyson-Mehta universality: Local statistics is universal in the bulk spectrum for any Wigner matrix; only symmetry type matters.

Solved for any symmetry class by the three step strategy
[Bourgade, E, Schlein, Yau, Yin: 2009-2014]

Related results:

[Johansson, 2000] Hermitian case with large Gaussian components

[Tao-Vu, 2009] Needs four moment matching.

(Similar development for the edge, for β-log gases and for many related models, such as sample covariance matrices, sparse graphs, regular graphs etc).

Invariant ensembles (detour)

Unitary ensemble: Hermitian matrices with density

$$\mathcal{P}(H)\mathrm{d}H \sim e^{-\mathrm{Tr}\, V(H)}\mathrm{d}H$$

Invariant under $H \to UHU^{-1}$ for any unitary U (GUE)
Joint density function of the eigenvalues is explicitly known

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\sum_j V(\lambda_j)}$$

classical ensembles $\beta = 1, 2, 4$ (orthogonal, unitary, symplectic symmetry classes; GOU, GUE, GSE for Gaussian case)

Correlation functions can be explicit computed via orthogonal polynomials due to the Vandermonde determinant structure.

large N asymptotic of orthogonal polynomials $\Longrightarrow$ local eigenvalue statistics indep of V. But density of e.v. depends on V.

Many previous results for classical invariant ensembles

Dyson (1962-76), Mehta (1960-) classical Gaussian ensembles via Hermite polynomials

General case by Deift etc. (1999), Pastur-Schcherbina (2008), Bleher-Its (1999), Deift etc (2000-, GOE and GSE), Lubinsky (2008)

All these results are limited to invariant ensembles. If the condition on Gaussian distribution is dropped (Wigner ensembles), the ensemble is not invariant.

The only result for the non-invariant case is by Johansson 2001, (Ben Arous-Peche, 2005) where sine kernel is proven for the Hermitian Wigner ensembles with a substantial Gaussian component:

$$H = \sqrt{1-t}H_0 + \sqrt{t}V, \qquad t > 0, \quad H_0 \text{ is Wigner } V \text{ is GUE}$$

Method: saddle point analysis on an explicit formula (Brezin-Hikami) valid only for Hermitian matrices.

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Summary of the first lecture

— We study eigenvalue statistics of large $N \times N$ random matrices.

— Two big families of ensembles (Gaussian is the only common)

== (a) Wigner (type): independent matrix elements

== (b) Invariant: exp$(-TrV(H))$dH – stat mechanics of log gases

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\sum_j V(\lambda_j)}$$
$$= \exp \beta \Big[\sum_{i<j} \log |\lambda_i - \lambda_j| - \sum_j V(\lambda_j) \Big]$$

— Normalization: $\lambda_j \sim O(1)$, gap $= \lambda_{i+1} - \lambda_i \sim 1/N$ (bulk)

— Scales: Global [O(1)], Mesoscopic, Microscopic $O(1/N)$.

— Wigner's fundamental vision: Global density may be model dependent, but microscopic statistics is universal, depends only on the symmetry type (β) [Wigner-Dyson-Mehta universality]

— Semicircle law holds only for Wigner matrices. Sample covariance matrices have different global density (Marchenko–Pastur law). For invariant ensembles, the density depends on V

— Local statistics (sine kernel, determinant structure of higher order correlation functions) were found in the 60's for GOE/GUE.

— WDM universality has recently been proven for both families (including general β-ensembles, not only for the classical cases, $\beta = 1, 2, 4$ that originate from a matrix model).

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LECTURE 2: Three-step strategy

1. Local density law down to scales $\gg 1/N$

   (Needed in entry-wise form, i.e. control also matrix elements G_{ij} the resolvent $G(z) = (H - z)^{-1}$ and not only TrG)

2. Use local equilibration of Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component

3. Use perturbation theory to remove the tiny Gaussian component.

Steps 2 and 3 need Step 1 as an input but are considered standard since very general theorems are available. [E-Schlein-Yin-Yau], ... most recent: [Landon-Yau]

Step 1 is model dependent.

Formulation of the local law via resolvents

Resolvent: $G(z) = (H - z)^{-1}$, at spectral parameter $z \in \mathbb{C}_+$.

Claim: Resolvent is asymptotically deterministic for $\eta := \text{Im}\, z \gg \frac{1}{N}$.

Theorem: There exists a deterministic matrix $M_{xy}(z)$ such that

$$\Big| G_{xy}(z) - M_{xy}(z) \Big| \le \frac{N^{\varepsilon}}{\sqrt{N\eta}} \qquad \text{(Entrywise law)}$$

$$\Big| \frac{1}{N}\text{Tr} BG(z) - \frac{1}{N}\text{Tr} BM(z) \Big| \le \frac{N^{\varepsilon}}{N\eta}\|B\| \qquad \text{(Average law)}$$

Note that

$$\frac{1}{N}\text{Im Tr} G(E + i\eta) = \frac{1}{N}\sum_i \frac{\eta}{|\lambda_i - E|^2 + \eta^2} = \frac{1}{N}\sum_i \delta_\eta(\lambda_i - E)$$

so eigenvalue density on scale η is identified via the average law.
Behind this formalism: Stieltjes transform.

Stieltjes transform

Def: Let μ be a probability measure on $\mathbb{R}$. Its Stieltjes transform at spectral parameter $z \in \mathbb{H}$ is given by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x - z}$$

Easy facts:

- $z \to m_\mu(z)$ is analytic in $\mathbb{H}$ with image in $\mathbb{H}$:

$$\operatorname{Im} m(z) = \eta \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{|x - E|^2 + \eta^2}, \qquad z = E + i\eta$$

- $i\eta\, m_\mu(i\eta) \to -1$ as $\eta \to \infty$
- $|m(z)| \leq \frac{1}{\operatorname{Im} z}$;
- These 3 properties characterize the Stieltjes transform (i.e. for any such function $m(z)$ there is a prob. measure s.t. $m = m_\mu$).

Message: $m_\mu(E+i\eta)$ resolves the measure μ around E on scale η

$$\frac{1}{\pi}\,\mathrm{Im}\, m_\mu(z) := (h_\eta \star \mu)(E) = \int_{\mathbb{R}} h_\eta(x-E)\mathrm{d}\mu(x)$$

where h_μ is an approximate delta fn. on scale η

$$h_\mu(x) := \frac{1}{\pi}\,\frac{\eta}{x^2+\eta^2}, \qquad \int h_\mu(x)\mathrm{d}x = 1$$

Inversion formula holds

$$\lim_{\eta\to 0+} \frac{1}{\pi}\,\mathrm{Im}\, m_\mu(E+i\eta) = \mu(E)\mathrm{d}E$$

(weak convergence).

Similarly to the Fourier transform, the pointwise convergence of the Stieltjes tr. characterizes weak convergence of prob. measures:

$$\mu_N \rightharpoonup \mu \qquad \text{iff} \qquad m_{\mu_N}(z) \to m_\mu(z) \quad \forall z \in \mathbb{H}$$

More precise results on speed of convergence are also available.

Resolvents and Stieltjes transform

Obvious fact: The trace of the resolvent G of a hermitian matrix H is the Stieltjes transform of its empirical spectral density:

$$\varrho_N(E) := \frac{1}{N}\sum_{\alpha=1}^{N}\delta(\lambda_\alpha - E), \qquad \frac{1}{N}\mathrm{Tr}G(z) = \frac{1}{N}\sum_\alpha \frac{1}{\lambda_\alpha - z} = m_{\varrho_N}(z)$$

Clearly the limit

$$\lim_{\eta\to 0+} \operatorname{Im} m_{\varrho_N}(E+i\eta)$$

may not exist, but its expectation may exist

$$\lim_{\eta\to 0+} \mathbb{E}\operatorname{Im} m_{\varrho_N}(E+i\eta) = \varrho(E)$$

and gives the density of states (DOS). Moreover, even without expectation, we can hope that

$$m_N(E+i\eta) \approx m_\varrho(E), \qquad \eta \gg \frac{1}{N}$$

holds with very high probability. This tells us that the eigenvalues are uniformly distributed down to the smallest possible scales $\eta \gg 1/N$.

Functional calculus via resolvent

Express a $f(H)$, a function of a Hermitian H, in terms of resolvent?

Let $\chi \in C^2[-2,2]$ be a cutoff function on $\mathbb{R}$ s.t. $\chi \equiv 1$ on $[-1,1]$.

Then (Stokes/Green)

$$f(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{i\eta f''(\sigma)\chi(\eta) + i(f(\sigma) + i\eta f'(\sigma))\chi'(\eta)}{\tau - \sigma - i\eta} \mathrm{d}\sigma \mathrm{d}\eta. \tag{1}$$

and use it for $\tau = H$. Observe resolvent on the RHS, i.e. $f(H)$ can be written as an integral of the resolvent.

The numerator is $\partial_{\bar{z}} \tilde{f}$, where $\tilde{f}(x+iy) := (f(x) + iyf'(x))\chi(y)$ is an almost analytic extension of f.

[this formalism can be used to build up spectral theorem for unbounded self-adjoint operators]

Helffer-Sjöstrand functional calculus

$$f(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{i\eta f''(\sigma)\chi(\eta) + i(f(\sigma) + i\eta f'(\sigma))\chi'(\eta)}{\tau - \sigma - i\eta} \mathrm{d}\sigma \mathrm{d}\eta. \tag{2}$$

Helffer-Sjöstrand formula states that for a real valued f

$$\int_{\mathbb{R}} f(\tau)\nu(\mathrm{d}\tau) = -\frac{1}{2\pi}\Big(L_1 + L_2\Big) \tag{3}$$

with

$$L_1 = \int_{\mathbb{R}^2} \Big[\eta f''(\sigma)\chi(\eta) + f'(\sigma)\chi'(\eta)\Big] \mathrm{Im}\, m(\sigma + i\eta) \mathrm{d}\sigma \mathrm{d}\eta$$

$$L_2 = \int_{\mathbb{R}^2} \eta f'(\sigma)\chi'(\eta) \mathrm{Re}\, m(\sigma + i\eta) \mathrm{d}\sigma \mathrm{d}\eta$$

where $m(z) = m_\nu(z)$ is the Stieltjes transform of ν.

Use it for $\nu = \varrho_N - \varrho = \frac{1}{N}\sum_\alpha \delta_{\lambda_\alpha} - \varrho$ to get estimate on linear statistics

$$\frac{1}{N}\sum_\alpha f(\lambda_\alpha) - \int_{\mathbb{R}} f(\tau)\nu(\mathrm{d}\tau)$$

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Summary of previous lectures

— We study local eigenvalue statistics of large Hermitian matrices

— Density is model dependent, local statistics is not [Wigner]

— Three step strategy – local law is the model dependent input

— Local law: there exists a deterministic matrix $M(z)$, $\|M\| \sim O(1)$, that very well approximates the resolvent $G(z) = (H-z)^{-1}$ in the mesoscopic regime $\eta = \mathrm{Im}\, z \gg 1/N$

$$G_{xy} \approx M_{xy}, \qquad \frac{1}{N}\mathrm{Tr} BG \approx \frac{1}{N}\mathrm{Tr} BM$$

(with very high probabiity).

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LECTURE 3: Models of increasing complexity

- Wigner matrix: i.i.d. entries, $s_{ij} := \mathbb{E}|h_{ij}|^2$ are constant ($= \frac{1}{N}$).
  (Density = semicircle; $G \approx$ diagonal, $G_{xx} \approx G_{yy}$)
  [E-Schlein-Yau-Yin, 2009–2011], [Tao-Vu, 2009]

- Generalized Wigner matrix: indep. entries, $\sum_j s_{ij} = 1$ for all i.
  (Density = semicircle; $G \approx$ diagonal, $G_{xx} \approx G_{yy}$)
  [E-Yau-Yin, 2011], [E-Knowles-Yau-Yin, 2012]

- Wigner type matrix: indep. entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \approx$ diagonal, $G_{xx} \not\approx G_{yy}$)
  [Ajanki-E-Krüger, 2015]

- Correlated Wigner matrix: correlated entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \not\approx$ diagonal)
  [Ajanki-E-Krüger '15-'16] [Che '16], [E-Krüger-Schröder '17], [Alt-E-Krüger-Schröder '18]

Other extensions of the original Wigner model

- Invariant ensembles: $P(H) \sim \text{exp}\Big[-\beta N \text{Tr} V(H)\Big]$

  Deift et. al., Pastur-Shcherbina, Bourgade-E-Yau, Bekerman-Guionnet-Figalli, etc.

- Low moment assumptions, heavy tails

  Johansson, Guionnet-Bordenave, Götze-Naumov-Tikhomirov, Benaych-Peche, Aggarwal

- Deformed models, general expectation

  O'Rourke-Vu, Lee-Schnelli-Stetler-Yau, He-Knowles-Rosenthal

- Sparse matrices, Erdős-Rényi and d-regular graphs

  E-Knowles-Yau-Yin, Huang-Landon-Yau, Bauerschmidt-Huang-Knowles-Yau, etc.

- Band matrices

  Fyodorov-Mirlin, Disertori-Pinson-Spencer, Schenker, Sodin, E-Knowles-Yau, T. Shcherbina, Bourgade-E-Yau-Yin, E-Bao etc.

Many other directions and references are left out, apologies...

OVERVIEW OF RESULTS

- Properties of the limiting (self-consistent) density of states (DOS), special focus on their singularity structure;

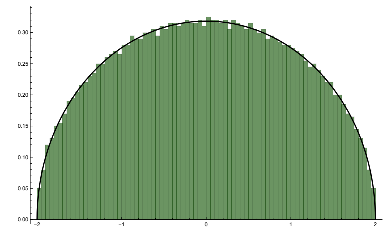
- Local laws, i.e. approximating $G(z)$ with a deterministic quantity $M(z)$ down to the smallest possible scale $\eta = \operatorname{Im} z \gg 1/N$;

- Universality of the eigenvalue statistics on the scale $1/N$.

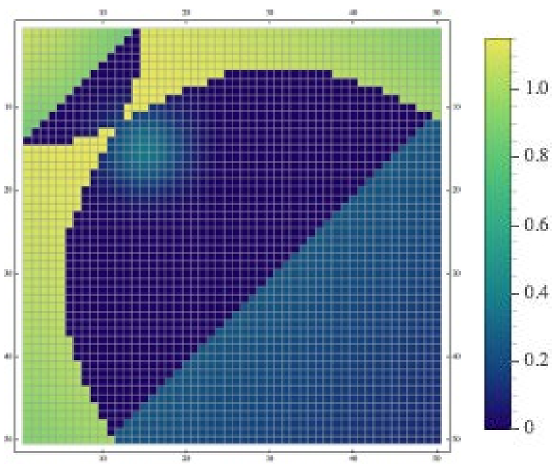
Most results will be presented only informally; the precise statement of the local law will be given later.

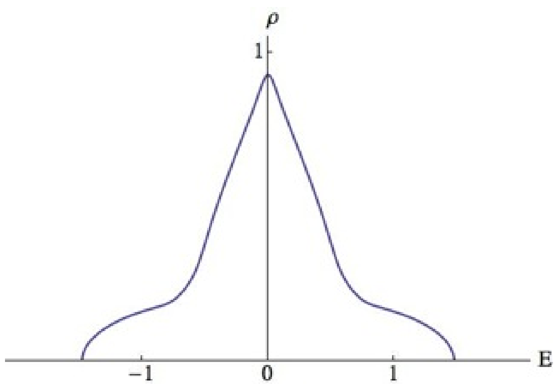
Variance profile and self-consistent density of states (DOS)

$$\sum_j s_{ij} = 1 \qquad \Longleftrightarrow$$



General variance profile $s_{ij} = \mathbb{E}|h_{ij}|^2$: not the semicircle any more.

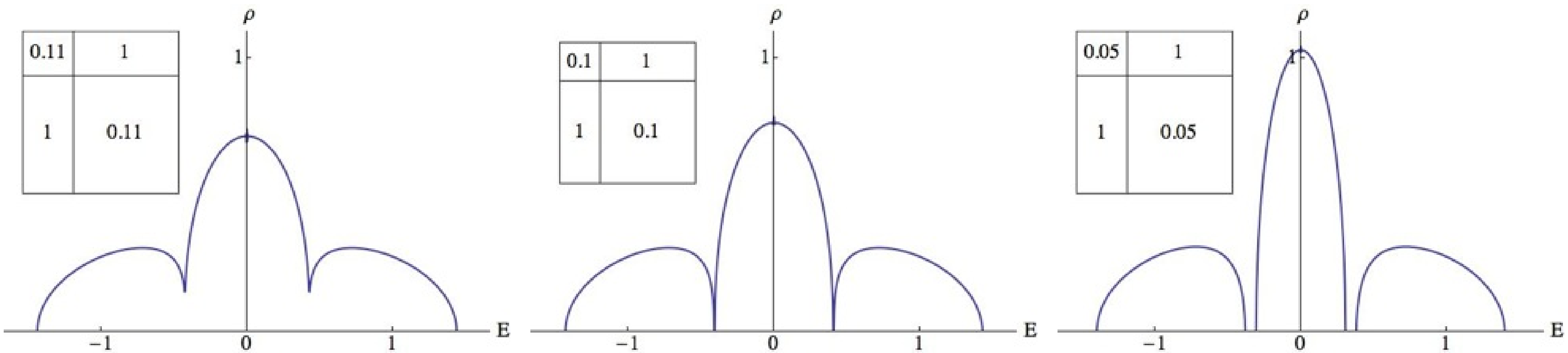




$\sum_j s_{ij} \neq$ const $\qquad \Longrightarrow \qquad$ Density of states

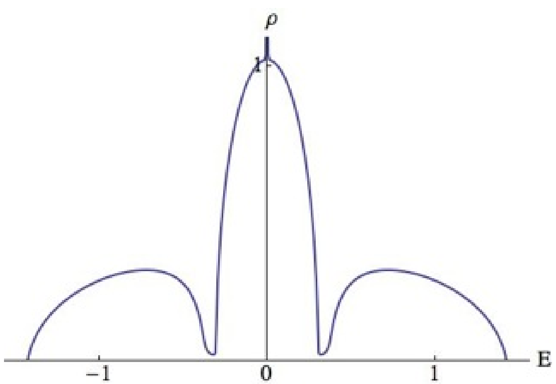
Features of the DOS for Wigner-type matrices

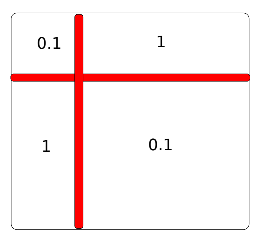
1) Support splits via cusps:



(Matrices in the pictures represent the variance matrix)

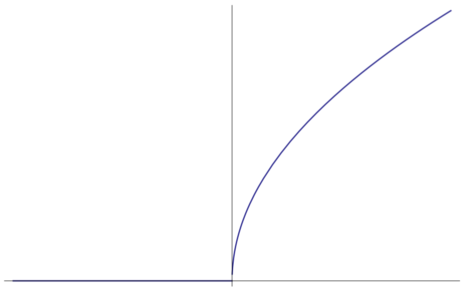
2) Smoothing of the S-profile avoids splitting ($\Rightarrow$ single interval)



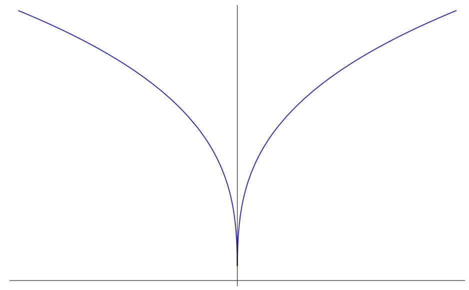


DOS of the same matrix as above but discontinuities in S are regularized

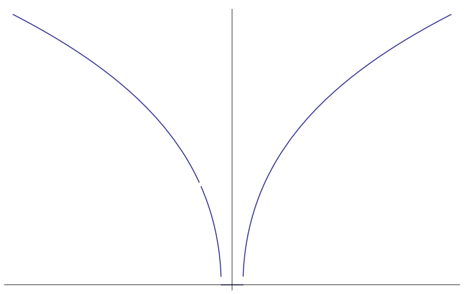
Universality of the DOS singularities for Wigner-type models



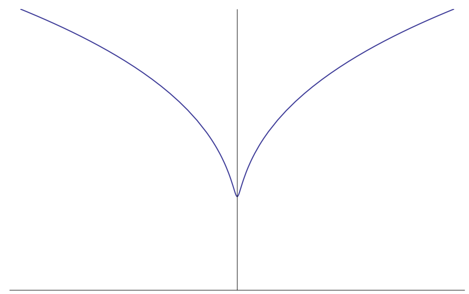
Edge, $\sqrt{E}$ singularity



Cusp, $|E|^{1/3}$ singularity



Small-gap



Smoothed cusp

$$\frac{(2+\tau)\tau}{1+(1+\tau+\sqrt{(2+\tau)\tau})^{2/3}+(1+\tau-\sqrt{(2+\tau)\tau})^{2/3}} \qquad \frac{\sqrt{1+\tau^2}}{(\sqrt{1+\tau^2}+\tau)^{2/3}+(\sqrt{1+\tau^2}-\tau)^{2/3}-1}-1$$

$$\tau := \frac{|E|}{\text{gap}}, \qquad \tau := \frac{|E|}{(\text{minimum })^{1/3}}$$

Main theorems on local laws and universality (informally)

Theorem [Ajanki-E-Krüger, 2014] Let $H = H^*$ be a Wigner-type matrix with general variance profile $c/N \leq s_{ij} \leq C/N$. Then optimal local law (including edge) and bulk universality hold.

Theorem [Ajanki-E-Krüger, 2016, E-Krüger-Schröder 2017]
Let $H = H^*$ be correlated

$$H = A + \frac{1}{\sqrt{N}} W$$

where A is deterministic, W is random with $\mathbb{E}W = 0$ and polynomial decay of correlation:

$$\mathrm{Cov}\Big(\phi(W_A), \psi(W_B)\Big) \leq \frac{C(\phi, \psi)}{\Big[1 + \mathrm{dist}(A, B)\Big]^s}; \qquad s > 12,$$

for any subsets A, B of the set of index pairs and a matching bound on higher cumulants. Assume

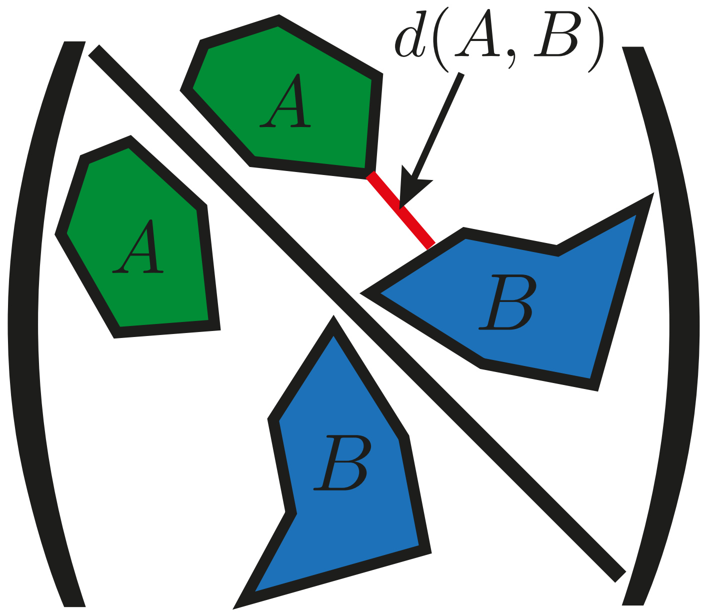
$$\mathbb{E}|\mathbf{u}^* W \mathbf{v}|^2 \geq c\|\mathbf{u}\|^2\|\mathbf{v}\|^2 \qquad \forall \mathbf{u}, \mathbf{v}.$$

Then optimal local law and bulk universality hold.

Distance for index sets in the condition on the correlation decay

$$\mathrm{Cov}\Big(\phi(W_A), \psi(W_B)\Big) \le \frac{C(\phi,\psi)}{\Big[1 + \mathrm{dist}(A,B)\Big]^s}$$

for any $A, B \subset S \times S$ assumes the usual metric on the set $S = \{1, 2, \ldots, N\}$ of indices. Here $W_A = \{W_{ij} \, : \, (i,j) \in A\}$.



Local law and "usual" corollaries

There exists a deterministic matrix M, with $\|M\| = O(1)$ s.t.

$$\Big|G_{ij}(z) - M_{ij}(z)\Big| \lesssim \frac{1}{\sqrt{N\mathrm{Im}\, z}}$$
$$\Big|\frac{1}{N}\mathrm{Tr}BG(z) - \frac{1}{N}\mathrm{Tr}BM(z)\Big| \lesssim \frac{\|B\|}{N\eta}$$

with very high prob.

- Delocalization of bulk eigenvectors
- Rigidity of bulk eigenvalues
- Wigner-Dyson-Mehta universality in the bulk

Remark: Very recently all the same results have been extended to the edge (Tracy-Widom universal statistics)
[Landon-Yau], [Alt-E-Krüger-Schröder]

Delocalization

Let $\mathbf{u}$ be a bulk eigenvector, $H\mathbf{u} = \lambda\mathbf{u}$, $\varrho(\lambda) \geq c > 0$, then

$$\max_i |u(i)| \leq \frac{N^\varepsilon}{\sqrt{N}} \|\mathbf{u}\|$$

for any $\varepsilon > 0$ with very high probability. Shows that the system is in the delocalized regime.

Ex: Prove $|u(x)|^2 \leq \eta \max_E \operatorname{Im} G_{xx}(E + i\eta)$ via spectral theorem.

Rigidity

For any E, let $k(E)$ be the index of the corresponding quantile, i.e.

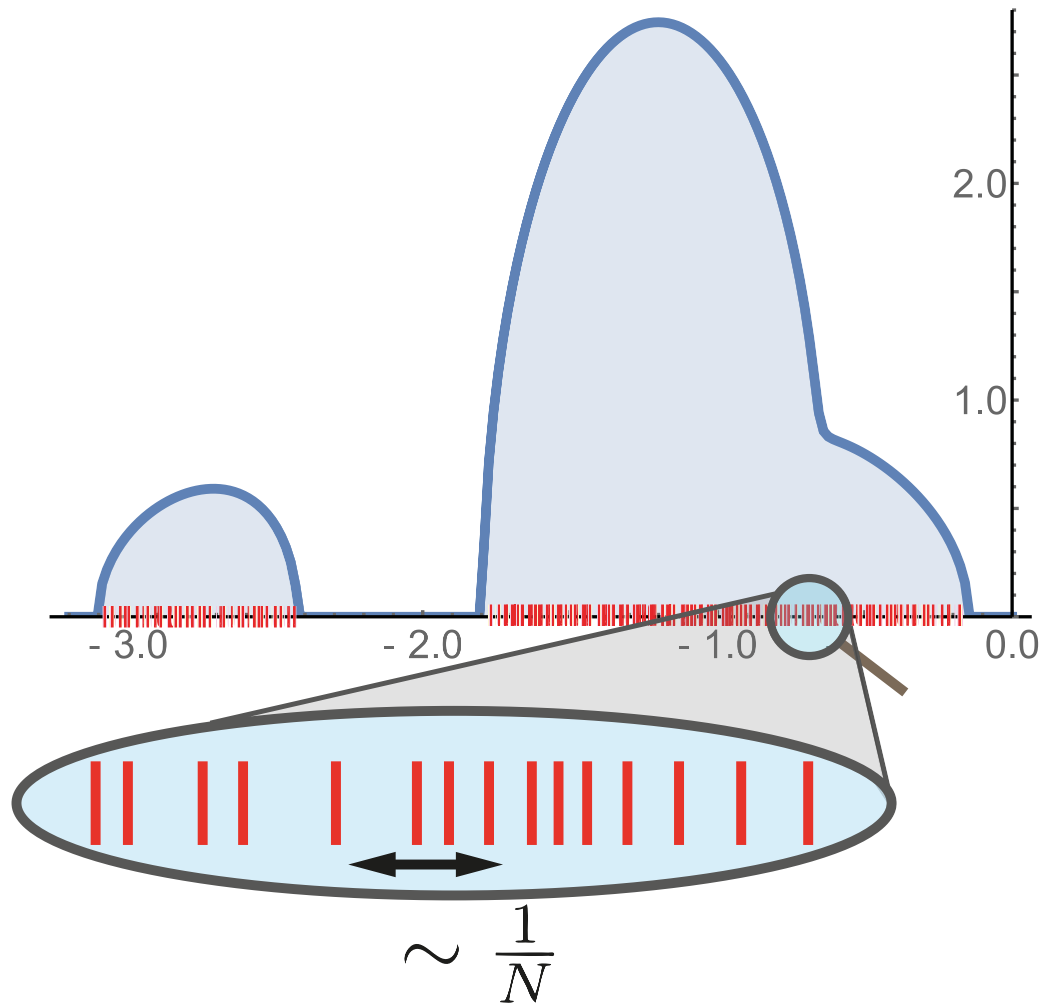
$$\int_{-\infty}^{E} \varrho(x) \mathrm{d}x = \frac{k(E)}{N}$$

Then for any bulk energy, $\varrho(E) \geq c > 0$,

$$\left| \lambda_{k(E)} - E \right| \leq \frac{N^\varepsilon}{N}$$

with very high probability, i.e. eigenvalues rigidly stick.

Bulk universality: sine-kernel statistics holds on the level of individual eigenvalues:



III. MOTIVATIONS

- Physics: nuclear physics (Wigner), disordered quantum systems

- Statistics: Wishart ensemble, sample covariance matrices, Tracy Widom law

- Wireless communication; channel capacity [Tulino-Verdu], [Hachem-Loubaton-Najim], [Alt-E-Krüger]

- Neural networks, ODE's with random coefficients. [Chalker-Mehlig], [E-Kruger-Renfrew]

- Quantum chaos, Quantum unique ergodicity [Bourgade-Yau]

Quantum systems

Quantum mechanical system is described by

- a state space $\ell^2(\Sigma)$ (e.g. $\Sigma = \{\uparrow, \downarrow\}$ for a spin, or $\Sigma = \mathbb{Z}^3$ for an electron in a metallic lattice);
- an $\Sigma \times \Sigma$ symmetric (selfadjoint) matrix (operator) H, the Hamilton operator;
- Matrix elements $H_{x,x'}$ describe quantum transition rates from x to x'.
- Eigenvalues of H (real) are the energy levels of the system
- Time evolution is given by $\psi_t = e^{itH}\psi_0$

Disordered: if $H_{x,x'}$ is random $\Longrightarrow$ Random matrix.

Universality conjecture for disordered quantum systems:

A disordered quantum system with sufficient complexity exhibits one of the following two behaviors:

- Insulator (typically at strong disorder; for simplicity $\Sigma = \mathbb{Z}^d$)
  - Localized eigenvectors, i.e. $\exists \Sigma' \subset \Sigma$, $|\Sigma'| \ll |\Sigma|$ s.t.

$$\sum_{x \in \Sigma \setminus \Sigma'} |\psi(x)|^2 \ll 1.$$

  - exp. offdiag decay of the resolvent

$$|G_{xx'}| \le C e^{-|x-x'|/\ell}, \qquad \ell \text{ localization length}$$

  - lack of transport

$$\sup_{t \in \mathbb{R}} \sum_x x^2 |\psi_t(x)|^2 \le C < \infty; \qquad \psi_t = e^{itH}\psi_0$$

  - nearby eigenvalues are independent (Poisson local spectral statistics).

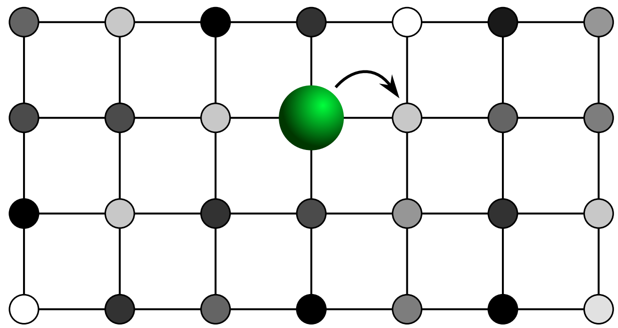
- Conductor (typically at weak disorder)
  - Delocalized eigenvectors
  - non-integrable decay of the resolvent
  - transport (via quantum diffusion)
  - eigenvalues are strongly correlated (Wigner-Dyson local statistics)

At first sight, localization is surprising (Anderson). Still, mathematically it is more accessible (Fröhlich-Spencer, Aizenman-Molchanov, Minami, ...).

Anderson metal-insulator quantum phase transition

Two popular models to study the dichotomy

(1) Random Schrödinger operators: in lattice box $\Sigma := [1, L]^d \cap \mathbb{Z}^d$



In $d = 1$ it corresponds to a narrow band matrix with i.i.d. diagonal:

$$H = \begin{pmatrix} v_1 & 1 & & & & \\ 1 & v_2 & 1 & & & \\ & 1 & \ddots & & & \\ & & & \ddots & 1 & \\ & & & 1 & v_{L-1} & 1 \\ & & & & 1 & v_L \end{pmatrix}$$

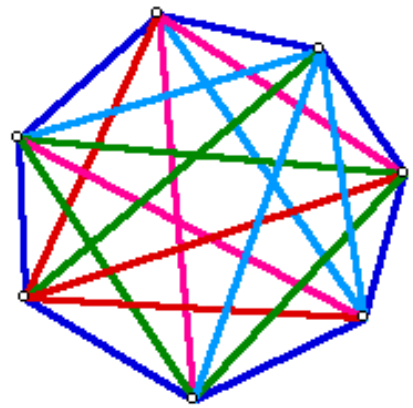
Follows insulator behavior

(2) Mean field Wigner random matrices: On the set $\Sigma = \{1, 2, \ldots, N\}$

$$H = (h_{xy}), \qquad H = H^* \qquad \mathbb{E}h_{xy} = 0.$$

entries are identically distributed and independent up to symmetry.

$$H = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \ldots & h_{NN} \end{pmatrix}$$



H models a mean-field hopping mechanism with random quantum transition rates. No spatial structure.

Follows conductor behavior

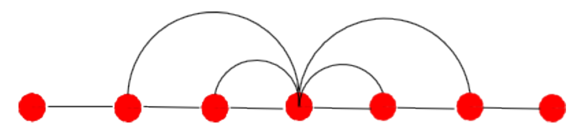
Band matrices: interpolation between Anderson and Wigner

$\Sigma = [1, L]^d \cap \mathbb{Z}^d$ lattice box

Entries of $H = H^*$ are indep but not identically distr. Wigner type!

$$\mathbb{E}|h_{xy}|^2 = \frac{1}{W^d} f\Big(\frac{|x-y|}{W}\Big)$$

W is the bandwidth (interaction range)



$$H = \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ 0 & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix}$$

Band matrix in $d = 1$ physical dim.

$W = O(1)$ [$\sim$ Anderson] $\longleftrightarrow$ $W = L$ [Wigner]

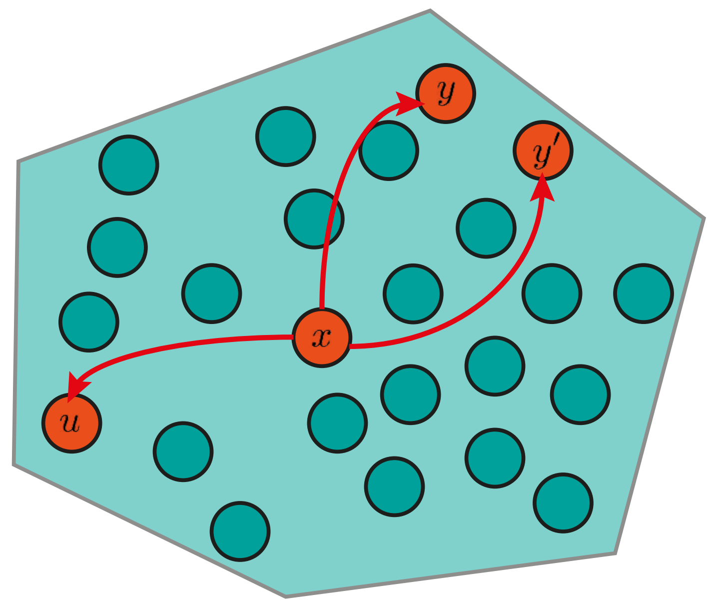
"Facts" from physics: Transition occurs at

$W \sim L^{1/2}$	$(d = 1)$	SUSY [Fyodorov-Mirlin, 1991]
$W \sim \sqrt{\log L}$	$(d = 2)$	RG scaling [Abrahams, 1979]
$W \sim W_0(d)$	$(d \geq 3)$	extended states conjecture

Mean field quantum Hamiltonian with correlation

Equip the configuration space Σ with a metric to have "nearby" states.

It is reasonable to allow that h_{xy} and $h_{xy'}$ are correlated if y and y' are close with a decaying correlation as $\mathrm{dist}(y, y')$ increases.



Non-trivial spatial structure changes the density of states.

There are many other natural models leading to correlated structures. Now we turn to some proofs.

LECTURE 4: Some proof ideas.

There are two basic methods to study RM eigenvalue statistics, i.e. for understanding the empirical ev. measure $\mu_N(\mathrm{d}x) = \frac{1}{N}\sum_i \delta(\lambda_i - x)$

– Moment method. Find $\frac{1}{N}\mathbb{E}\mathrm{Tr}H^k = \mathbb{E}\int x^k \mu_N(\mathrm{d}x)$.

– Resolvent method: Use that resolvent = St. transform of $\mu_N(\mathrm{d}x)$

$$\frac{1}{N}\mathrm{Tr}G(z) = \frac{1}{N}\mathrm{Tr}\frac{1}{H-z} = \int_{\mathbb{R}} \frac{\mu_N(\mathrm{d}x)}{x-z}, \qquad z = E + i\eta$$

and use many convenient properties of the resolvent.

Moment method is suitable for global laws and for extreme edges, even for complicated models involving many matrices or free probability, but inefficient for local information inside the spectrum.

We will follow the resolvent method and we first informally present it for Wigner type matrices.

Def: [Wigner type matrix] H is an $N \times N$ hermitian random matrix

$$\mathbb{E}h_{ij} = 0, \qquad s_{ij} := \mathbb{E}|h_{ij}|^2$$

The matrix $S := (s_{ij})$ defines an operator on $\mathbb{C}^N$.

Def: [Quadratic vector equation] Given $z \in \mathbb{H} := \{\text{Im}\, z > 0\}$, consider

$$-\frac{1}{m_i} = z + (Sm)_i, \qquad i = 1, 2, \dots N, \qquad [QVE]$$

We will see that the QVE has a (unique) solution in the upper half plane denoted by $\mathbf{m}(z) = (m_1(z), \dots m_N(z)) \in \mathbb{H}^N$.

Key relation [informally] $G = \frac{1}{H-z}$ is close to $M = \text{diag}(\mathbf{m})$;

$$G_{ii}(z) \approx m_i(z), \qquad G_{ij}(z) \approx 0, \quad (i \neq j) \quad N \to \infty$$

$$\frac{1}{N}\mathrm{Tr}G = \frac{1}{N}\sum_i G_{ii}(z) \approx \frac{1}{N}\sum_i m_i(z) =: \langle \mathbf{m} \rangle \quad (N \to \infty)$$

Thus, to compute the density of states for H, we need to solve

$$-\frac{1}{m_i} = z + (Sm)_i, \qquad i = 1, 2, \ldots N, \qquad [QVE]$$

for every $z \in \mathbb{H}$, compute

$$m(z) := \langle \mathbf{m}(z) \rangle := \frac{1}{N}\sum_i m_i(z)$$

and find its inverse Stieltjes transform.

For Wigner matrices, $s_{ij} = \frac{1}{N}$, everything is simpler: we have

$$-\frac{1}{m_i} = z + (S\mathbf{m})_i = z + \langle \mathbf{m} \rangle, \qquad \forall i$$

thus $m_i = \langle \mathbf{m} \rangle$ and $-\frac{1}{\langle \mathbf{m} \rangle} = z + \langle \mathbf{m} \rangle$, which gives the semicircle law.

For general S there is no eq. for $\langle \mathbf{m} \rangle$, one has to solve the QVE.

Structure of the proof of the local law

Step 1: Probabilistic step (derivation of the QVE)

Prove that $g_i := G_{ii}$ approximately satisfied the QVE, i.e.

$$-\frac{1}{g_i} = z + (S\mathbf{g})_i + d_i, \qquad (*)$$

for some small (random) error vector $\mathbf{d} = (d_i)$.

Step 2: Deterministic step (Stability of the QVE)

Consider (*) as a small perturbation of the QVE:

$$-\frac{1}{m_i} = z + (S\mathbf{m})_i \quad \text{and} \quad (*) \qquad \Longrightarrow \|\mathbf{m} - \mathbf{g}\| \lesssim \|\mathbf{d}\|$$

Key question: stability of the QVE in an appropriate norm/space.

Next, we will make Step 1. plausible before we enter Step 2.

Step 1. Derivation of the QVE (Dyson equation)

There are two methods for deriving the QVE:

(i) Schur complement formula

(ii) Cumulant expansion

Here we present (i) since it is conceptually simpler (but less general)

Schur complement formula: Use the block-decomposition

$$H = \begin{pmatrix} h_{11} & [\mathbf{a}^1]^* \\ \mathbf{a}^1 & H^{[1]} \end{pmatrix}, \qquad [\mathbf{a}^1]^* = (h_{21}, h_{31}, \ldots, h_{N1})$$

$$G_{11} = \Big(\frac{1}{H-z}\Big)_{11} = \frac{1}{h_{11} - z - [\mathbf{a}^1]^* \frac{1}{H^{[1]}-z}\mathbf{a}^1}$$

Holds for any $1 \to i$

$$\frac{1}{G_{ii}} = h_{ii} - z - \sum_{j,k\neq i} h_{ij} G^{[i]}_{jk} h_{ki}, \qquad G^{[i]} = G^{[i]}(z) = \frac{1}{H^{[i]} - z}$$

$h_{ij}h_{ki}$ is independent of $G^{[i]}_{jk}$, so concentration estimate holds:

$$\sum_{j,k\neq i} h_{ij} G^{[i]}_{jk} h_{ki} \approx \mathbb{E}_i \sum_{j,k\neq i} h_{ij} G^{[i]}_{jk} h_{ki} = \sum_{j\neq i} s_{ij} G^{[i]}_{jj} \approx \sum_j s_{ij} G_{jj}$$

thus we get the perturbed QVE

$$-\frac{1}{G_{ii}} = z + \sum_j s_{ij} G_{jj} + d_i$$

The main part of the error d_i comes from the fluctuation and

$$|d_i| \lesssim \frac{1}{\sqrt{N\eta}} \quad \text{with very high prob.}$$

Remark: Here S is the self-energy operator/matrix since

$$g_i \approx \frac{1}{-(Sg)_i - z}, \qquad g_i := G_{ii} = \left(\frac{1}{H-z}\right)_{ii}$$

Quadratic large deviation bound to estimate d_i

Let $B \in \mathbb{C}^{M \times M}$ be a fixed matrix ($M = N - 1$, $B = G^{[i]}$ in appl.)
Let $\mathbf{a} \in \mathbb{C}^M$ be centered random vector with indep. components.

Assume "mean-field" with high moment condition

$$\mathbb{E}|\sqrt{M} a_i|^p \le C_p,$$

independently of M. Clearly

$$\mathbb{E}\mathbf{a}^* B \mathbf{a} = \sum_i B_{ii} \mathbf{E}|a_i|^2$$

Question: How close is the (random) quadratic form $\mathbf{a}^* B \mathbf{a}$ to its expectation in very high probability sense?

Answer: The variance tells it, since one can prove

$$\mathbb{E}\Big|\mathbf{a}^* B \mathbf{a} - \mathbb{E}\mathbf{a}^* B \mathbf{a}\Big|^p \lesssim \Big[\mathbb{E}\Big|\mathbf{a}^* B \mathbf{a} - \mathbb{E}\mathbf{a}^* B \mathbf{a}\Big|^2\Big]^{p/2}$$

$\Longrightarrow$ bound with very high probability (after Markov's ineq.)

The variance can be computed ($\Longrightarrow$ Exercise) and if B is a resolvent of a hermitian matrix

$$B = \frac{1}{T-z}, \qquad T = T^*, \quad z = E + i\eta$$

then we find ($\Longrightarrow$ Exercise)

$$\mathbb{E}\Big|\mathbf{a}^* B\mathbf{a} - \mathbb{E}\mathbf{a}^* B\mathbf{a}\Big|^2 \le \frac{C}{M\eta}\frac{1}{M}\mathrm{Im}\,\mathrm{Tr} B$$

Here we used the Ward identity for any hermitian matrix T

$$\sum_j \Big|\Big(\frac{1}{T-z}\Big)_{ij}\Big|^2 = \frac{1}{\mathrm{Im}\, z}\mathrm{Im}\Big(\frac{1}{T-z}\Big)_{ii}$$

Applying this to $M = N-1$, $B = G^{[i]}$ we get the fluctuation error

$$|d_i| \lesssim \frac{1}{\sqrt{N\eta}} \quad \text{with very high prob.}$$

After Step 2 (stability), we conclude the local law in the form:

$$|G_{ii} - m_i| \lesssim \frac{1}{\sqrt{N\eta}} \quad \text{with very high prob.}$$

Summary so far

Local law with resolvent method: (for Wigner type matrices) find a deterministic approximation $m_i(z)$ to $G_{ii}(z)$ as $\text{Im}\, z \ll 1$ and $N \gg 1$:

$$\Big| G_{ii}(z) - m_i(z) \Big| \lesssim \frac{1}{\sqrt{N \text{Im}\, z}} \qquad \text{with high prob.}$$

QVE (vector Dyson equation)

$$-\frac{1}{m_i} = z + (S\mathbf{m})_i, \qquad \text{Im}\, m_i > 0$$

Self-consistent density: $\varrho(E) = \langle \text{Im}\, \mathbf{m}(E + i0) \rangle = \frac{1}{N} \sum_i \text{Im}\, m_i$.

Two steps:

1. Prove that $g_i = G_{ii}$ satisfies

$$-\frac{1}{g_i} = z + (S\mathbf{g})_i + d_i, \qquad |d_i| \lesssim \frac{1}{\sqrt{N\eta}} \qquad (*)$$

(Schur formula + Large deviation)

2. Stability analysis of QVE: (*) and the QVE

$$-\frac{1}{m_i} = z + (S\mathbf{m})_i$$

imply

$$\|\mathbf{g} - \mathbf{m}\| \lesssim \|\mathbf{d}\|$$

Linear stability operator: $L = 1 - \mathbf{m}^2 S$ needs to be inverted.

Step 2: Analysis and stability of the Dyson equation

$$-\frac{1}{\mathbf{m}} = z + \mathbf{a} + S\mathbf{m}, \qquad z \in \mathbb{H}, \quad \mathbf{m} \in \mathbb{H}^N$$

where $S = S^t$ is symmetric, bounded $\|S\| \leq C$ matrix with positive entries, $s_{ij} \geq 0$, and $\mathbf{a} \in \mathbb{R}^N$ external source. We set $\mathbf{a} = 0$ here.

Notation: For any function f and vector $\mathbf{m}$ we write

$$f(\mathbf{m}) := \Big(f(m_1), f(m_2), \ldots, f(m_N) \Big) \in \mathbb{C}^N$$

in particular

$$\frac{1}{\mathbf{m}} = \Big(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_N} \Big)$$

Similarly, for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ vectors, their (entrywise) product is

$$\mathbf{uv} := \Big(u_1 v_1, u_2 v_2, \ldots, u_N v_N \Big)$$

and

$$\mathbf{u} \leq \mathbf{v} \quad \Longleftrightarrow \quad u_i \leq v_i$$

VI.1. Existence, uniqueness, representation

$$-\frac{1}{\mathbf{m}(z)} = z + S\mathbf{m}(z), \qquad z \in \mathbb{H}, \qquad \mathbf{m} : \mathbb{H} \to \mathbb{H}^N$$

Thm [Folklore]: (i) The QVE has a unique solution.
(ii) For each i, there is a unique prob measure ν_i on $\mathbb{R}$ s.t.

$$m_i(z) = \int_{\mathbb{R}} \frac{\nu_i(\mathrm{d}\tau)}{\tau - z}$$

Proof. Fixpoint argument for the map

$$\Phi(\mathbf{u})(z) := -\frac{1}{z + S\mathbf{u}(z)}$$

(i) Φ maps an appropriate bounded subset into itself
(ii) contraction in a specific metric.

The right metric, for some small parameter δ, is

$$d_\delta(\mathbf{u},\mathbf{v}) := \sup_{z\in\mathbb{H}_\delta} \max_{i\le N} D(u_i,v_i), \qquad D(\zeta,\omega) := \frac{|\zeta-\omega|^2}{(\mathrm{Im}\,\zeta)(\mathrm{Im}\,\omega)}$$

$$H_\delta := \left\{ z\in\mathbb{H} \; : \; \mathrm{Im}\, z \ge \delta, |z| \le \delta^{-1} \right\}$$

Relation to the hyperbolic metric on $\mathbb{H}$: $D(\zeta,\omega) = 2(\cosh(\zeta,\omega) - 1)$.

The representation as a Stieltjes tr. follows from the facts

- Starting from a constant function, every iterate is analytic on $\mathbb{H}$, hence so is their limit, on compact sets

- $i\eta m_k(i\eta) \to -1$ can be checked directly since ($\eta = \mathrm{Im}\, z$)

  $|m_k| \le \dfrac{1}{\mathrm{Im}\, z}$ from the Im-part of QVE: $\dfrac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|^2} = \eta + S\mathrm{Im}\,\mathbf{m} \ge \eta$

  then $|z\mathbf{m}+1| = |\mathbf{m}S\mathbf{m}| \le \|S\|\|\mathbf{m}\|^2 \to 0,$ as $\mathrm{Im}\, z \to \infty$.

VI.2. Bounds on the solution

Two norms

$$\|\mathbf{u}\|_2^2 := \frac{1}{N}\sum_i |u_i|^2, \qquad \|\mathbf{u}\| := \|\mathbf{u}\|_\infty := \max_i |u_i|$$

We have three bounds:

– ℓ^2-bound useful in the bulk, if $s_{ij} \sim \frac{1}{N}$

– Unconditional ℓ^2-bound away from zero.

– ℓ^∞-bound if S is $\frac{1}{2}$-Hölder continuous.

ℓ^2-bound in the bulk

Proposition. Suppose *flatness* [for simplicity]

$$\sup_{x,y} s_{xy} \sim \inf_{x,y} s_{xy} > 0, \quad \text{i.e.} \quad c\langle \mathbf{v} \rangle \le S\mathbf{v} \le C\langle \mathbf{v} \rangle, \quad \forall \mathbf{v} \ge 0$$

Then

$$\|\mathbf{m}\|_2 \lesssim 1, \qquad \|\mathbf{m}(z)\| \lesssim \frac{1}{\varrho(z) + \mathrm{dist}(z, \mathrm{supp}\varrho)},$$

$$\varrho(z) \lesssim \mathrm{Im}\,\mathbf{m}(z) \lesssim \varrho(z)\|\mathbf{m}(z)\|^2$$

where

$$\varrho(z) = \langle \mathrm{Im}\,\mathbf{m}(z) \rangle \quad \text{harmonic extension of } \varrho(\tau)$$

Remark: $A \sim B$ means A/B and B/A are bounded uniformly in the constants hidden in the conditions.

Proof. Take Im-part of the QVE

$$\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|^2} = \eta + S\operatorname{Im}\mathbf{m} \gtrsim \langle \operatorname{Im}\mathbf{m}\rangle \qquad (*)$$

by the lower bound $S\mathbf{v} \gtrsim \langle \mathbf{v}\rangle$. Then

$$\operatorname{Im}\mathbf{m} \gtrsim |\mathbf{m}|^2\langle \operatorname{Im}\mathbf{m}\rangle \Longrightarrow \langle \operatorname{Im}\mathbf{m}\rangle \gtrsim \langle |\mathbf{m}|^2\rangle\langle \operatorname{Im}\mathbf{m}\rangle \Longrightarrow \|\mathbf{m}\|_2^2 \lesssim 1.$$

Also,

$$|\mathbf{m}|^2\langle \operatorname{Im}\mathbf{m}\rangle \lesssim \operatorname{Im}\mathbf{m} \le |\mathbf{m}| \Longrightarrow |\mathbf{m}| \lesssim \frac{1}{\langle \operatorname{Im}\mathbf{m}\rangle}$$

Furthermore, from Stieltjes tr. representation

$$|\mathbf{m}| \lesssim \frac{1}{\operatorname{dist}(z, \operatorname{supp}\varrho)}$$

Assume $|z| \lesssim 1$, then from QVE

$$\frac{1}{|m_i|} \le |z| + \Big|\sum_j s_{ij} m_j\Big| \lesssim 1 + \|\mathbf{m}\|_1 \le 1 + \|\mathbf{m}\|_2 \lesssim 1$$

From (*), we have $\langle \operatorname{Im}\mathbf{m}\rangle \lesssim \operatorname{Im}\mathbf{m}$, and $\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|^2} \lesssim \eta + \langle \operatorname{Im}\mathbf{m}\rangle \lesssim \langle \operatorname{Im}\mathbf{m}\rangle$.

The saturated self-energy F operator, unconditional ℓ^2 bound

Taking the Im-part of the QVE, we have

$$\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|^2} = \eta + S(\operatorname{Im}\mathbf{m}), \quad \text{equivalently} \quad \frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|} = \eta|\mathbf{m}| + |\mathbf{m}|S\Big(|\mathbf{m}|\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|}\Big)$$

Define the positivity preserving operator $F = F(z)$ as

$$F(\cdot) := |\mathbf{m}|S\Big(|\mathbf{m}|\cdot\Big), \qquad \text{i.e.} \quad (Fw)_i = |m_i|\sum_j s_{ij}|m_j|w_j$$

Trivial bound $|\mathbf{m}| \le \eta^{-1}$ implies that F is bounded.

Perron-Frobenius implies

$$\exists \mathbf{f} \in \mathbb{R}^N, \quad \mathbf{f} \ge 0, \quad F\mathbf{f} = \|F\|\mathbf{f}$$

Scalar multiply Im-part of QVE by $\mathbf{f}$ and use symmetry of F

$$\Big\langle \mathbf{f}, \frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|}\Big\rangle = \eta\langle \mathbf{f}, |\mathbf{m}|\rangle + \Big\langle \mathbf{f}, F\,\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|}\Big\rangle = \eta\langle \mathbf{f}, |\mathbf{m}|\rangle + \|F\|\Big\langle \mathbf{f}, \frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|}\Big\rangle$$

After rearranging

$$\|F\| = 1 - \eta \frac{\langle \mathbf{f} |\mathbf{m}| \rangle}{\left\langle \mathbf{f} \frac{\mathrm{Im}\mathbf{m}}{|\mathbf{m}|} \right\rangle} < 1 \qquad (*)$$

Key bound (*) gives a finite control on F uniformly in (small) η.

From QVE

$$\|\mathbf{m}\|_2 \le \frac{1}{|z|}\Big(1 + \|\mathbf{m}S\mathbf{m}\|_2\Big) \le \frac{1}{|z|}\Big(1 + \Big\| \, |\mathbf{m}|S|\mathbf{m}| \, \Big\|_2\Big) = \frac{1}{|z|}\Big(1 + \| \, F\mathbf{1} \, \|_2\Big)$$

so we get an unconditional ℓ^2-bound away from zero:

$$\|\mathbf{m}\|_2 \le \frac{2}{|z|}$$

NO assumption on S (apart from symmetry and nonnegative kernel)

From ℓ^2 to max-bound under Hölder regularity of S (Skip)

Proposition. Let S be (piecewise) $\frac{1}{2}$-Hölder continuous and

$$s_{ij} = \frac{1}{N} S\Big(\frac{i}{N}, \frac{j}{N}\Big), \qquad c \le S(x,y) \le C$$

Then for $|z| \lesssim 1$ we have

$$|\mathbf{m}(z)| \sim 1, \qquad \mathrm{Im}\, \mathbf{m} \sim \langle \mathrm{Im}\, \mathbf{m} \rangle = \varrho$$

(all components of $|\mathbf{m}|$ and $\mathrm{Im}\, \mathbf{m}$ are comparable).

Proof. Subtract the i-th and j-th components of the QVE

$$\frac{1}{|m_i|} - \frac{1}{|m_j|} \le \sum_k |s_{ik} - s_{jk}||m_k| \le \|\mathbf{m}\|_2 \Big(N \sum_k |s_{ik} - s_{jk}|^2 \Big)^{1/2}$$

Using $\|\mathbf{m}\|_2 \lesssim 1$ and Hölder, we get

$$\frac{1}{|m_i|} \le \frac{1}{|m_j|} + C\sqrt{\frac{|i-j|}{N}}$$

Take reciprocal and sum up

$$\frac{1}{N}\sum_i \left[\frac{1}{\left|\frac{1}{m_j}\right| + C\sqrt{\frac{|i-j|}{N}}}\right]^2 \le \|\mathbf{m}\|_2 \lesssim 1$$

and

$$\frac{1}{N}\sum_i \left[\frac{1}{\left|\frac{1}{m_j}\right| + C\sqrt{\frac{|i-j|}{N}}}\right]^2 \gtrsim \frac{1}{N}\sum_i \left[\frac{1}{\left|\frac{1}{m_j}\right|^2 + C\frac{|i-j|}{N}}\right] \gtrsim \log|m_j|$$

Combining these inequalities, we have

$$\|\mathbf{m}\| \lesssim 1$$

The comparability of $\operatorname{Im} m_i$ components follow from $s_{ij} \sim C/N$

$$\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|^2} = \eta + S(\operatorname{Im}\mathbf{m}) \qquad \Longrightarrow \operatorname{Im}\mathbf{m} \sim \eta + S(\operatorname{Im}\mathbf{m})$$

and

$$S\operatorname{Im}\mathbf{m} \sim \langle \operatorname{Im}\mathbf{m}\rangle$$

VI.3. The stability operator of the QVE

The stability of the QVE against small perturbations plays a key role in showing that the solution of QVE approximates the resolvent.

The key point is uniformity in $\eta \to 0$. Similarly to the boundedness of $\mathbf{m}$ above, η-dependent bounds are easy to get, but are useless for local laws or studying properties of the density since both are $\eta \approx 0$ phenomena.

We first introduce the stability operator in the context of proving regularity of $\mathbf{m}(z)$ down to the real axis (showing regularity of DOS in the bulk).

Later we will show how the same operator appears in the random matrix theory.

Regularity of $\mathbf{m}(z)$ down to the real axis

$$-\frac{1}{\mathbf{m}(z)} = z + S\mathbf{m}(z) \quad \Longrightarrow \quad m_i(z) = \int_{\mathbb{R}} \frac{\nu_i(\mathrm{d}\tau)}{\tau - z},$$

Thm: The generating measure has a density $\nu_i(\mathrm{d}\tau) = \nu_i(\tau)\mathrm{d}\tau$ that is uniformly Hölder 1/3 continuous. Its support is independent of i:

$$\Sigma = \mathrm{supp}\, \nu_i$$

and has finitely many intervals. Away from the boundary of Σ the density $\nu : \mathbb{R} \setminus \partial\Sigma \to \mathbb{R}_+^N$ is real analytic.

At the boundary points $\tau_0 \in \partial\Sigma$, it has either "cusp" or "edge" singularity. No other singularity can occur.

Since $\nu_k(\tau) = \frac{1}{\pi}\mathrm{Im}\, m_k(\tau + i0)$, all these statements rely on the regularity of $m(z)$ down to the real axis. We will demonstrate it by proving the Hölder-$\frac{1}{3}$ regularity.

Proof of the Hölder regularity

$$-\frac{1}{\mathbf{m}(z)} = z + S\mathbf{m}(z) \quad \Longrightarrow \quad \frac{\partial_z \mathbf{m}}{\mathbf{m}^2} = 1 + S\partial_z \mathbf{m}$$

$$\Longrightarrow \quad (1 - \mathbf{m}^2 S)\partial_z \mathbf{m} = \mathbf{m}^2 \qquad i.e. \quad \partial_z \mathbf{m} = (1 - \mathbf{m}^2 S)^{-1} \mathbf{m}^2$$

so $\mathbf{m}$ is the solution to an analytic ($\mathbb{C}^N$-valued) ODE, hence analytic, as long as the inverse above exists and $\|\mathbf{m}\| \le C$.

Main Lemma: [Proof later] For any $z \in \mathbb{H}$, $|z| \le C$ we have

$$\left\|(1 - \mathbf{m}^2 S)^{-1}\right\| \lesssim \frac{1}{\langle \mathrm{Im}\, \mathbf{m}(z)\rangle^2}$$

$$\mathbf{m}(z) \text{ is analytic} \quad \Longrightarrow \quad |\partial_z \mathrm{Im}\, \mathbf{m}| = \frac{1}{2}|\partial_z \mathbf{m}| \lesssim \frac{1}{\langle \mathrm{Im}\, \mathbf{m}\rangle^2} \sim \frac{1}{(\mathrm{Im}\, \mathbf{m})^2}$$

using $|\mathbf{m}| \sim 1$ and $\mathrm{Im}\, \mathbf{m}_i \sim \langle \mathrm{Im}\, \mathbf{m}\rangle$. After integration, we get

$$\sup_{z \neq z' \in \mathbb{H}} \frac{\|\mathrm{Im}\, \mathbf{m}(z) - \mathrm{Im}\, \mathbf{m}(z')\|}{|z - z'|^{1/3}} < \infty$$

Spectral gap for positive operators

Definition: For a hermitian matrix T, the spectral gap $\text{Gap}(T) \geq 0$ is the difference between the two largest eigenvalues of $|T|$. If $\|T\|_2$ is a degenerate ev. then the gap is zero.

Lemma [Spectral gap of T] Let T have non-negative integral kernel, $t_{ij} = t_{ji} \geq 0$ and let $\mathbf{h}$ be a Perron-Frobenius eigenfunction. Then

$$\text{Gap}(T) \geq \left(\frac{\|\mathbf{h}\|_2}{\|\mathbf{h}\|_\infty} \right)^2 \varepsilon, \qquad \varepsilon := \inf_{i,j} t_{ij}$$

Proof. $\Longrightarrow$ Exercise

Spectral gap of F

Lemma Recall $F = |\mathbf{m}|S\big(|\mathbf{m}|\cdot\big)$. For $|z| \le C$ we have
(i) The spectral radius $\|F\| \sim 1$ is a non degenerate eigenvalue
(ii) The corresponding ℓ^2-normalized non-negative eigenfunction

$$\mathbf{f}(z) \sim 1$$

(iii) F has uniform spectral gap

$$\mathrm{Gap}(F) \sim 1$$

(i) We have seen $\|F\| \le 1$. The (easier) lower bound follows from $F_{ij} = |m_i|s_{ij}|m_j| \gtrsim 1/N$ (recall $|\mathbf{m}| \sim 1$).

(ii) follows from $\mathbf{f} = (\|F\|)^{-1}F\mathbf{f} \sim \langle \mathbf{f} \rangle$

(iii) follows for the previous lemma and that $\|\mathbf{f}\|_\infty \sim \|\mathbf{f}\|_2$.

Boundedness of the stability operator

Goal: $$\left\|(1-\mathbf{m}^2 S)^{-1}\right\| \lesssim \frac{1}{\langle \operatorname{Im} \mathbf{m}(z)\rangle^2}$$

The key info is that this holds uniformly even as $\eta \to 0$. Write

$$(1-\mathbf{m}^2 S)\mathbf{w} = \mathbf{w} - \frac{\mathbf{m}^2}{|\mathbf{m}|} F\Big(\frac{\mathbf{w}}{|\mathbf{m}|}\Big) = e^{2i\varphi}|\mathbf{m}|\Big(e^{-2i\varphi} - F\Big)|\mathbf{m}|^{-1}\mathbf{w}, \qquad \mathbf{m} = e^{i\varphi}|\mathbf{m}|$$

Note that $\sin\varphi \sim \operatorname{Im}\mathbf{m}$ since $|\mathbf{m}| \sim 1$.

F has real spectrum, so $e^{-2i\varphi} - F$ should be invertible if $\sin 2\varphi \neq 0$.

In fact, it is invertible if $\sin\varphi \neq 0$, since $F \geq -1 + c$.

This intuition is true, but the proof is more complicated for non constant φ. We first present the simple case.

Mechanism for stability I. Generalized Wigner

In this case $m_x = m$ constant, the Stieltjes tr. of the semicircle:

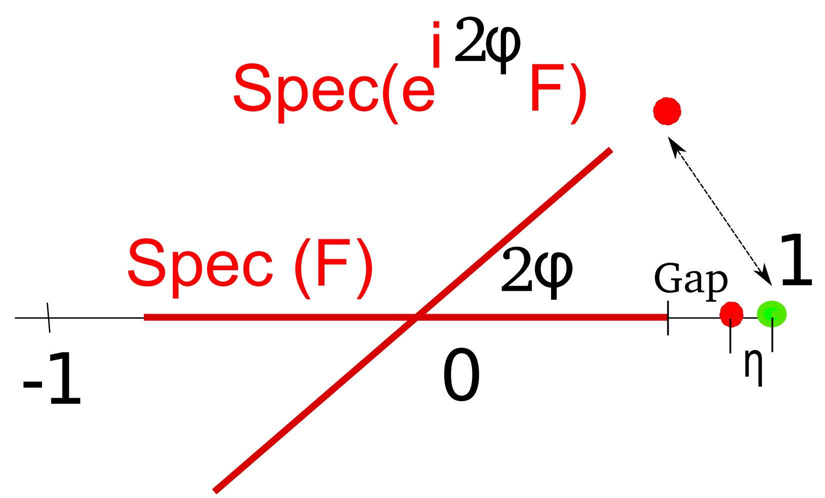
$$|m(z)| \le 1 - c\eta, \qquad \text{Im}\, m(z) \approx \varrho(E), \qquad z = E + i\eta$$

$$1 - m^2 S = 1 - e^{2i\varphi} F, \qquad m = |m|\, e^{i\varphi}, \quad F = |m| S |m| = |m|^2 S$$

F is symmetric, Spec$(F) \subset (-1, 1)$

In the bulk $\varphi \sim \text{Im}\, m \ne 0$

$$\Big\| \frac{1}{1 - m^2 S} \Big\| = \Big\| \frac{1}{1 - e^{2i\varphi} F} \Big\| \le \frac{C}{\varphi} \sim \frac{C}{\varrho}$$



For the edge analysis (when $\varphi \approx 0$) use the gap, the isolated eigenspace $F\mathbf{f} = |m|^2 \mathbf{f}$, with $\mathbf{f} = 1$, is treated separately.

Mechanism for stability II. Wigner-type

Stability Lemma. If T is hermitian, $T\mathbf{f} = \|T\|_2\mathbf{f}$, $\|T\|_2 \le 1$, then

$$\left\|\frac{1}{U-T}\right\| \le \frac{C}{\mathrm{Gap}(T)\,\Big|1-\|T\|_2\langle \mathbf{f}, U\mathbf{f}\rangle\Big|}, \qquad \text{for any } U \text{ unitary}$$

In our case $T = F$, $\|F\|_2 \le 1$, $U = e^{-2i\boldsymbol{\varphi}} = (|\mathbf{m}|/\mathbf{m})^2$ and

$$\Big|1-\|T\|_2\langle \mathbf{f}, U\mathbf{f}\rangle\Big| \ge \mathrm{Re}\Big[1-\langle |\mathbf{m}|^{-2}\mathbf{m}^2\mathbf{f}^2\rangle\Big] \gtrsim \langle \mathrm{Im}\,\mathbf{m}\rangle^2$$

since $\mathrm{Im}\, m_i \sim \langle \mathrm{Im}\,\mathbf{m}\rangle$ and $\mathbf{f} \sim 1$.

Thus, we have stability (albeit weaker than before)

$$\left\|\frac{1}{1-\mathbf{m}^2 S}\right\| \sim \left\|\frac{1}{e^{-2i\boldsymbol{\varphi}}-F}\right\| \le \frac{C}{\langle \mathrm{Im}\,\mathbf{m}\rangle^2}$$

Proof of the lemma: Need a lower bound on $\|(U-T)\mathbf{w}\|_2$ for any $\mathbf{w}$. Split $\mathbf{w} = \langle \mathbf{f}, \mathbf{w}\rangle\mathbf{f} + P\mathbf{w}$ and separate into three regimes depending on the relative size of $\Big|1-\|T\|_2\langle \mathbf{f}, U\mathbf{f}\rangle\Big|$ to $\|P\mathbf{w}\|_2$ and $\|PUf\|_2$.

Summary of the QVE [Ajanki-E-Krüger]

$$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}, \qquad \text{Im}\, \mathbf{m} \geq 0$$

• Existence, uniqueness, representation as a Stieltjes transform of a compactly supported generating measure

• Solution is bounded, $|\mathbf{m}| \sim 1$ and Im $m_i \sim \langle \text{Im}\, \mathbf{m} \rangle$.

• Solution is Hölder-1/3, real analytic apart from a few points on the real line (edges, cusps)

• Stability operator has bounded inverse in the bulk and away from the spectrum

• More careful analysis extends everything to the edges and cusps

Matrix Dyson equation

For any $z \in \mathbb{C}_+$, consider the equation (we set $A = \mathbb{E}H = 0$)

$$-\frac{1}{M} = z + \mathcal{S}[M], \qquad M = M(z) \in \mathbb{C}^{N\times N} \tag{4}$$

with the self-energy "super-operator"

$$\mathcal{S}[R] := \mathbb{E}\Big[HRH\Big], \qquad \mathcal{S} : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$$

We do not need the specific form of $\mathcal{S}$, we use only that

(i) $\mathcal{S}$ is selfadjoint w.r.t HS scalar product:

$$\langle R, \mathcal{S}[T]\rangle = \langle \mathcal{S}[T], R\rangle, \qquad \langle A, B\rangle := \frac{1}{N}\mathrm{Tr} A^* B$$

(ii) $\mathcal{S}$ is positivity preserving; $\mathcal{S}[R] \geq 0$ for any $R \geq 0$.

Fact: [Girko, Pastur, Wegner, Helton-Far-Speicher]
If $\mathcal{S}$ is self-adjoint and positivity preserving, then the MDE

$$-\frac{1}{M} = z + \mathcal{S}[M], \qquad M = M(z) \in \mathbb{C}^{N\times N}$$

has a unique solution with $\operatorname{Im} M \geq 0$ and it is a Stieltjes transform of a matrix-valued measure

$$M(z) = \int \frac{V(\tau)\mathrm{d}\tau}{\tau - z}, \qquad z \in \mathbb{C}_+$$

Proof: fixed point argument as before.

Define the density of states

$$\varrho(\tau) := \langle V(\tau)\rangle = \frac{1}{N}\mathrm{Tr}V(\tau), \qquad \tau \in \mathbb{R}$$

and the stability (super)operator $\mathcal{L}^{-1}$ where

$$\mathcal{L} := I - M\mathcal{S}[\cdot]M = I - \mathcal{C}_M\mathcal{S}$$

where $\mathcal{C}_M$ is the "sandwiching" operator $\mathcal{C}_M[T] := MTM$.

Results on the MDE

Theorem [Ajanki-E-Krüger 2016]

Assume *flatness*: $c\langle R\rangle \le S[R] \lesssim C\langle R\rangle$ for any $R \ge 0$. Then

(i) ϱ is compactly supported, $\frac{1}{100}$-Hölder continuous
(ii) ϱ is real analytic away from the edges.
(iii) $V(\tau) \gtrsim \varrho(\tau)$
(iv) M and M^{-1} are bounded away from the edges

$$\|M(z)\| \lesssim \frac{1}{\varrho(z) + \mathrm{dist}(z, \mathrm{supp}\ \varrho)}, \qquad \|M(z)^{-1}\| \lesssim 1 + |z|$$

(v) Stability operator is bounded in the spectral norm

$$\|(I - \mathcal{C}_{M(z)}\mathcal{S})^{-1}\|_{sp} \lesssim \frac{1}{\Big[\varrho(z) + \mathrm{dist}(z, \mathrm{supp}\ \varrho)\Big]^{100}}$$

Theorem [Alt-E-Krüger 2018] Hölder exponent improved to $1/3$ (optimal), stability around edges and cusps.

Some proofs for MDE

Main difficulty: non-commutativity. We present some instances.

1) Taking the Im part of the MDE

$$-\mathrm{Im}\,\frac{1}{M} = \frac{1}{M^*}\mathrm{Im}\,M\frac{1}{M} = \mathrm{Im}\,z + \mathcal{S}[\mathrm{Im}\,M] \geq \mathrm{Im}\,z$$

yields the trivial bound

$$\|M(z)\| \leq \frac{1}{\eta}, \qquad \eta = \mathrm{Im}\,z$$

2) Compact support for ϱ and bounds on M are similar to QVE.

3) Key question: What is the proper saturated self-energy operator, analogue of $|m|S\Big(|m|\cdot\Big)$, that was the basis of the stability analysis (that implied regularity but also local law for random matrix etc.)

Matrix stability operator

Lemma: $M = M(z)$ be the solution to MDE, then

$$\left\| \frac{1}{1 - M\mathcal{S}[\cdot]M} \right\|_{sp} = \left\| \frac{1}{1 - \mathcal{C}_M \mathcal{S}} \right\|_{sp} \lesssim \frac{1}{[\varrho(z) + \mathrm{dist}(z, \mathrm{supp}(\varrho)]^{100}}$$

Key: find the "right" symmetrization $\mathcal{F}$ of $\mathcal{C}_M\mathcal{S}$ despite the noncommutative matrix structure (M is even not normal, so even $|M|$ is problematic)

Need the analogue of

$$\mathbf{m} = e^{i\varphi}|\mathbf{m}|, \qquad F = |\mathbf{m}|S\big(|\mathbf{m}| \cdot \big), \qquad |1 - \mathbf{m}^2 S| = |e^{-2i\varphi} - F|$$

We needed that F is symmetric (for spectral analysis of $U - F$), positivity preserving (for Perron-Frobenius), and

$$\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|} = \eta|\mathbf{m}| + |\mathbf{m}|S\Big(|\mathbf{m}|\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|}\Big)$$

Bringing to matrix Perron-Frobenius form

Try to write the equation for $\operatorname{Im} M$

$$\operatorname{Im} M = \eta M^* M + M^* \mathcal{S}(\operatorname{Im} M) M$$

with some Q as (intuition: $|Q|$ "$\approx$" $|M|^{-1/2}$)

$$\frac{1}{Q}(\operatorname{Im} M)\frac{1}{Q^*} = \frac{1}{Q} M^* \frac{1}{Q^*} Q^* \mathcal{S}\Big[Q\,\frac{1}{Q}(\operatorname{Im} M)\frac{1}{Q^*}\,Q^*\Big] Q \frac{1}{Q} M \frac{1}{Q^*}$$

i.e. ($\eta = 0$)

$$X = Y^* Q^* \mathcal{S}[QXQ^*] QY, \quad \text{with} \quad X := \frac{1}{Q}(\operatorname{Im} M)\frac{1}{Q^*}, \quad Y := \frac{1}{Q} M \frac{1}{Q^*}$$

i.e.

$$X = Y^* \mathcal{F}[X] Y, \qquad \text{with} \quad \mathcal{F}[\cdot] := Q^* \mathcal{S}[Q \cdot Q^*] Q$$

Notice that $X = \operatorname{Im} Y$ and if Y is unitary, then X and Y commute,

$$X = \mathcal{F}[X]$$

so Perron-Frobenius applies and $\mathcal{F}$ is bounded. Also: $\mathcal{F}^* = \mathcal{F}$!

Now we write the stability operator in terms of $\mathcal{F}$. For any R

$$M\mathcal{S}[R]M = QYQ^*\mathcal{S}[Q\frac{1}{Q}R\frac{1}{Q^*}Q^*]QYQ^* = QY\mathcal{F}\Big[\frac{1}{Q}R\frac{1}{Q^*}\Big]YQ^*$$

so

$$R - M\mathcal{S}[R]M = Q\Big(1 - Y\mathcal{F}[\cdot]Y\Big)\Big[\frac{1}{Q}R\frac{1}{Q^*}\Big]Q^*$$

Thus

$$I - \mathcal{C}_M\mathcal{S} = \mathcal{K}_Q(I - \mathcal{C}_Y\mathcal{F})\mathcal{K}_Q^{-1}, \qquad \mathcal{K}_Q[R] := QRQ^*$$

i.e. assuming $Q \sim 1$, we have

$I - \mathcal{C}_M\mathcal{S}$ is stable $\Longleftrightarrow$ $I - \mathcal{C}_Y\mathcal{F}$ is stable

All we need is a symmetric polar decomposition of $M = QYQ^*$.

Symmetric Polar Decomposition of M if $\mathrm{Im}\, M > 0$

Goal: $M = QYQ^*$, $\quad Y$ unitary, $\quad |Q| \sim 1$

Explicitly: use that $M = A + iB$ and $B > 0$ to write

$$M = \sqrt{B}\Big(\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}} + i\Big)\sqrt{B}$$

and make the middle factor unitary by dividing its absolute value:

$$M = \sqrt{B}WYW\sqrt{B} =: QYQ^*$$

$$W := \left[1 + \Big(\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}}\Big)^2\right]^{\frac{1}{4}}, \quad Y := \frac{\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}} + i}{W^2}$$

In the regime, where $c \le B \le C$ and $\|A\| \le C$, we have

$$Q = \sqrt{B}W \sim 1$$

Another form of $\mathcal{F}$:

$$\mathcal{F} = \mathcal{K}_Q^* \mathcal{S} \mathcal{K}_Q = \mathcal{C}_W \mathcal{C}_{\sqrt{\mathrm{Im}\, M}} \mathcal{S} \mathcal{C}_{\sqrt{\mathrm{Im}\, M}} \mathcal{C}_W$$

Thus Krein-Rutman applied on the super-operator level to

$$\mathcal{F} = \mathcal{C}_W \mathcal{C}_{\sqrt{\mathrm{Im} M}} \mathcal{S} \mathcal{C}_{\sqrt{\mathrm{Im} M}} \mathcal{C}_W$$

implies that $\mathcal{F}$ has a unique, HS-normalized eigenmatrix F with e.v. $\|\mathcal{F}\| \leq 1$ and a spectral gap.

Noncommutative generalization of the Stability Lemma

$$\left\| \frac{1}{1 - \mathcal{C}_M \mathcal{S}} \right\|_{sp} \lesssim \left\| \frac{1}{\mathcal{U} - \mathcal{F}} \right\|_{sp} \lesssim \frac{1}{\mathrm{Gap}(\mathcal{F}) \, \left| 1 - \|\mathcal{F}\| \langle F, \mathcal{U}(F) \rangle \right|}$$

with $\mathcal{U} = \mathcal{C}_U$, then we prove that (noncommutative!)

$$|1 - \|\mathcal{F}\| \langle F, UFU \rangle | \geq c, \qquad \mathrm{Gap}(\mathcal{F}) \geq c$$

with some $c = c(\varrho) > 0$ if $\rho = \langle \mathrm{Im}\, M \rangle > 0$ (i.e. in the bulk). $\square$

Basic structure of the proof of the local law

Back to random matrices, recall the local law for correlated case.

$G(z) = (H - z)^{-1}$, where $H = H^*$ has a correlation structure.

Theorem [AEK, EKS] In the bulk spectrum, $\varrho(\Re z) \geq c$, we have

$$|G_{xy}(z) - M_{xy}(z)| \lesssim \frac{1}{\sqrt{N \, \text{Im}\, z}}, \qquad \left|\frac{1}{N}\text{Tr}G(z) - \frac{1}{N}\text{Tr}M(z)\right| \lesssim \frac{1}{N \, \text{Im}\, z}$$

with very high probability.

M is given by the solution of the MDE

$$-\frac{1}{M} = z + \mathcal{S}[M], \qquad \text{Im}\, M \geq 0, \quad \text{Im}\, z > 0$$

M is typically not diagonal, so G has nontriv off-diagonal component.

Derivation of MDE: Cumulant Expansion

$$G(z) := (H - z)^{-1} \qquad I + zG = HG$$

Write it as

$$I + (z + \mathcal{S}[G])G = D \qquad \text{with} \quad D := HG + \mathcal{S}[G]G$$

Note that MDE is equivalent to the same eq. with $D = 0$

$$I + (z + \mathcal{S}[M])M = 0 \qquad (MDE)$$

Need to show that D is small then G approx. satisfies MDE.

In the Gaussian case, a simple integration by parts suffices:

$$\mathbb{E}D = \mathbb{E}\Big[HG + \mathcal{S}[G]G\Big] = \mathbb{E}\Big[- \widetilde{\mathbb{E}}[\widetilde{H}G\widetilde{H}]G + \mathcal{S}[G]G\Big] = 0$$

and higher moments $\mathbb{E}|D|^p$ are small by a similar argument.

In the general case, one can use either a full cumulant expansion or resolvent expansion

Proof of the local law

$$I + (z + \mathcal{S}[M])M = 0, \qquad I + (z + \mathcal{S}[G])G = D$$

Subtracting from each other:

$$(I - M\mathcal{S}[\cdot]M)[G - M] = MD + M\mathcal{S}[G - M](G - M)$$

With the linear stability operator (acting on the space of matrices)

$$\mathcal{L} := 1 - M\mathcal{S}[\cdot]M$$

we have (in appropriate norms and with "good" test-matrices B)

$$\|G - M\| \lesssim \|\mathcal{L}^{-1}\|\|D\|, \qquad \mathrm{Tr}(G - M)B \approx \mathrm{Tr} MD(\mathcal{L}^{-1})^*(B)$$

(quadratic corrections neglected here).

Operator norm is too strong; the good norm is

$$\|A\|_p^p := \sup_{\|\mathbf{x}\|=\|\mathbf{y}\|=1} \mathbb{E}|\langle \mathbf{x}, A\mathbf{y}\rangle|^p$$

More generally, the following facts can be established in the bulk

$$\|\mathcal{L}^{-1}\| \le C \qquad \|D\| \lesssim \frac{1}{\sqrt{N\eta}}, \qquad \frac{1}{N}\mathrm{Tr}DA \lesssim \frac{1}{N\eta}\|A\| \qquad (*)$$

The last two bounds hold with very high probability. The bound for $\frac{1}{N}\mathrm{Tr}AD$ requires an extra mechanism, called *fluctuation averaging*.

Then the entrywise and averaged optimal laws follow immediately:

$$\max_{x,y}|(G-M)_{xy}| \lesssim \frac{1}{\sqrt{N\eta}}, \qquad \frac{1}{N}\mathrm{Tr}(G-M)B \lesssim \frac{1}{N\eta}\|B\| \qquad (**)$$

We thus need to establish (*)

The bounds on D are obtained with probabilistic techniques (cumulant expansion) and have not been discussed.

We focused only on bounding the deterministic quantity $\|\mathcal{L}^{-1}\|$

Dyson equations and their stability operators

Name	Dyson Eqn	For	Stab. op	Feature
Wigner $\mathbb{E}\vert h_{ij}\vert^2 = s_{ij} = \frac{1}{N}$	$-\frac{1}{m} = z + m$	$m \approx \frac{1}{N}\mathrm{Tr}G$	$\frac{1}{1-m^2\vert e\rangle\langle e\vert}$	$m = m_{sc}$ is explicit
Gen. Wigner $\sum_j s_{ij} = 1$	$-\frac{1}{m} = z + m$	$m \approx \frac{1}{N}\mathrm{Tr}G$	$\frac{1}{1-m^2 S}$	Split S as $S^\perp + \vert e\rangle\langle e\vert$
Wigner-type s_{ij} arbitrary	$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}$	$m_x \approx G_{xx}$	$\frac{1}{1-\mathbf{m}^2 S}$	$\mathbf{m}$ to be determined
Corr. Wigner $\mathbb{E}h_{xy}h_{uw} \not\asymp \delta_{xw}\delta_{yu}$	$-\frac{1}{M} = z + \mathcal{S}[M]$	$M_{xy} \approx G_{xy}$	$\frac{1}{1-M\mathcal{S}[\cdot]M}$	Matrix eq. Super-op

- Gen. Wigner could be studied via a scalar equation only (in practice a vector eq. is also considered for G_{xx})
- Wigner-type needs vector equation even for the density
- Corr. Wigner needs matrix equation. – This is the "true" object!

RECAPITULATION: Three-step strategy.

1. Local density law down to scales $\gg 1/N$

   (Needed in entry-wise form, i.e. control also matrix elements G_{ij} the resolvent $G(z) = (H - z)^{-1}$ and not only TrG)

2. Use local equilibration of Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component

3. Use perturbation theory to remove the tiny Gaussian component.

Step 1 is model dependent and has been the main topic today.

Some short comments on Step 2 and 3.

Step 2: Dyson Brownian Motion

Gaussian convolution matrix interpolates between Wigner and GUE.

$$H = H_0 \;\longrightarrow\; H_t \;\longrightarrow\; H_\infty = \text{GUE}$$

Embed H into an Ornstein-Uhlenbeck matrix flow:

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t \qquad H_t \sim e^{-t/2}H_0 + (1-e^{-t})^{1/2}V.$$

$$\overset{\text{Dyson}}{\Longrightarrow} \qquad \mathrm{d}\lambda_i = \frac{1}{\sqrt{N}}\mathrm{d}B_i + \Big(-\frac{1}{2}\lambda_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\Big)\mathrm{d}t$$

Ev-flow becomes a stochastic dynamics of interacting "particles".

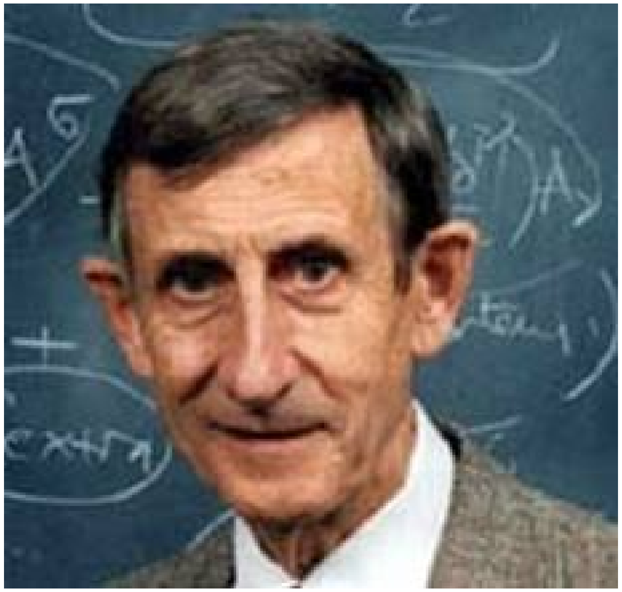
Idea: Equilibrium is GUE/GOE with known local statistics.

Global equilibrium is reached in time $O(1)$ (convexity, Bakry-Emery).

For local statistics, only local equilibrium needs to be achieved which is much faster. The main result proves Dyson's conjecture:
[E-Schlein-Yau] [E-Schlein-Yin-Yau] [Landon-Yau] [E-Schnelli] [Landon-Sosoe-Yau]

"The picture of the gas coming into equilibrium in two well-separated stages, with microscopic and macroscopic time scales, is suggested with the help of physical intuition. A rigorous proof that this picture is accurate would require a much deeper mathematical analysis."

Freeman Dyson, 1962

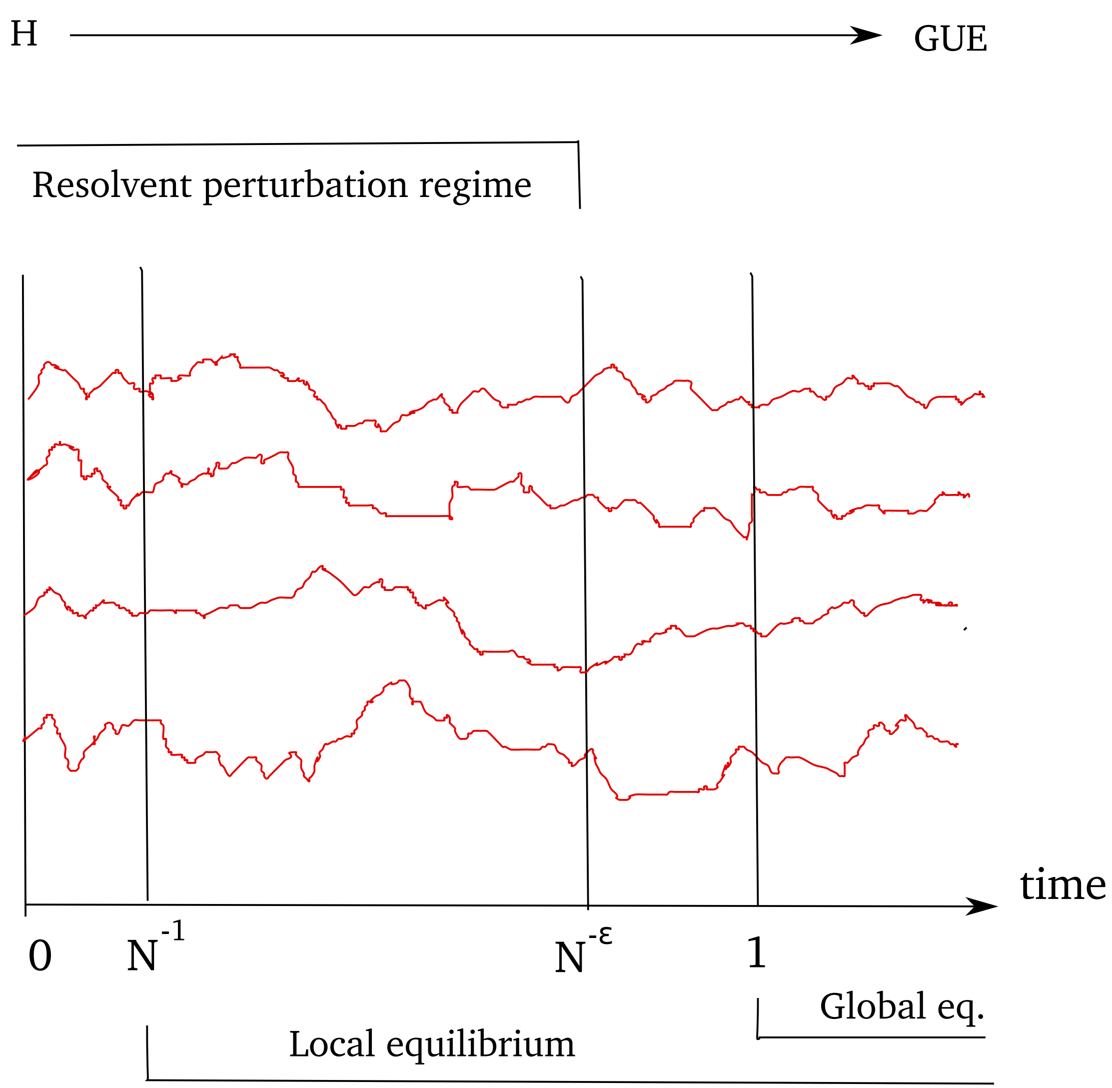


on the approach to equilibrium
of Dyson Brownian Motion

Global equilibrium is reached in time scale of $O(1)$.
Local equilibrium was believed to be reached in $O(N^{-1})$.

Thm. Local law on scale $\eta \gg N^{-1}$ implies local eq. in time $O(N^{-1})$.

Time scales for DBM



Step 3: Resolvent perturbation with four moment matching

Green Function Comparison Thm. Suppose H and $\widehat{H}$ have matching first four moments and entrywise local law is known for both. Then expectations of resolvents with $\mathrm{Im}\, z$ slightly below $1/N$ are close:

$$\Big|\mathbb{E}\Phi\Big[\mathrm{Tr}G(E+iN^{-1-\varepsilon})\Big]-\mathbb{E}\Phi\Big[\mathrm{Tr}\widehat{G}(E+iN^{-1-\varepsilon})\Big]\Big| \ll \|\Phi\|$$

where Φ is a smooth fn, possibly with several G's in the argument.

This can detect local statistics of individual eigenvalues.

Proof. Replace the entries of H to $\widehat{H}$ one by one by res. expansion:

$$G(h_{12}) = G_0 + G_0 h_{12} G_0 + G_0 h_{12} G_0 h_{12} G_0 ... \qquad G_0 := G(h_{12} := 0)$$

$$G(\widehat{h}_{12}) = G_0 + G_0 \widehat{h}_{12} G_0 + G_0 \widehat{h}_{12} G_0 \widehat{h}_{12} G_0 ... \qquad G_0 := G(h_{12} := 0)$$

Using G_0 is indep of h_{12}, $\widehat{h}_{12}$, first four terms are the same in expectation. Fifth order term is $N^{-5/2}$, combinatorics N^2 – small! (For general Φ use Taylor) [Similar idea: Tao-Vu]

SUMMARY

- We reviewed local laws for various random matrix ensembles
- We gave a quantitative analysis of the MDE and its stability.
- MDE is the "correct" equation in RM, with applications: Gram matrices, Inhomogeneous circular law, structured models etc.
- For correlated random matrices with slow correlation decay in both symmetry classes we proved
  - Optimal local law in the bulk
  - Wigner-Dyson-Mehta bulk universality
- Very recently the whole theory is extended to the edge