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- It can take into account irrelevant terms like lattice effects, non linear bands, Umklapp
- It is based on convergent expansion (Fermions)
- actual limitations; $T = 0$, weak coupling
- I will focus on open questions which should admit an analytical understanding

Weyl semimetals with quasi-periodic disorder

- The effect of weak random disorder in Weyl semimetals is subject to debate; (Altland Bagrets PRL(2015) /Nandkishore, Huse, Sondhi, PRB(2014)). Disorder is perturbatively irrelevant but non perturbative effects can change the behavior (rare region)

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- Lattice model $H_0 = \int \frac{dk}{(2\pi)^3}\, \hat{a}_k^\dagger h(k) \hat{a}_k$ with $h(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ \beta^*(k) & -\alpha(k) \end{pmatrix}$ where $k \in (0, 2\pi]^3$, $\alpha(k) = 2 + \zeta - \cos k_1 - \cos k_2 - \cos k_3$ and $\beta(k) = t_1 \sin k_1 - i t_2 \sin k_2$.

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- $\zeta \in [0, 1)$, in which case $\hat h(k)$ is singular at $k = \pm p_F$, with $p_F = (0, 0, \arccos \zeta)$ (Weyl points).

Dirac Fermions with quasi-periodic disorder

- In the vicinity of $\pm p_F$, $k = q \pm p_F$, Dirac fermions $\hat{H}^0(q \pm p_F) = t_1 \sigma_1 q_1 + t_2 \sigma_2 q_2 \pm \sin p_F \sigma_3 q_3 + O(q^2)$. Lattice realization of Dirac fermions with smaller light velocity.

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- Many body interaction and quasiperiodic disorder

$$H = H_0 + \varepsilon \sum_x \phi_x (a^+_{x,1} a^-_{x,1} - a^+_{x,2} a^-_{x,2}) + \lambda \sum_{x,y} v(x-y) \rho_x \rho_y$$

where $v(x-y)$ is a short range potential, $\rho_x = a^+_{x,1} a^-_{x,1} + a^+_{x,2} a^-_{x,2}$

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- Quasi-periodic disorder, ω_i irrational (rational=periodic)

$$\phi_x = \sum_n \hat{\phi}_n e^{i2\pi(\omega_1 n_1, x_1 + \omega_2 n_2 x_2 + \omega_3 n_3 x_3)}$$

with $n \in \mathbb{Z}^3$, $\hat{\phi}_n = \hat{\phi}_{-n}$ and $|\hat{\phi}_n| \leq C e^{-\xi(|n_1|+|n_2|+|n_3|)}$.

- Is the Weyl phase stable?

Related problems in 1d (Aubry-Andre)

- Without interaction 1d Aubry-Andre' (in 3d Burgain 2002) $-\varepsilon\psi(x+1) - \varepsilon\psi(x-1) + u\cos(2\pi(\omega x + \theta))\psi(x) = E\psi(x)$. Small divisors $1/(E_n - E_m)$ with $E_n = \cos(k - 2\pi\omega n)$ producing $n!$.

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- Early results in the interacting Aubry-Andre' model $T=0$ Mastropietro (CMP 99 proof of irrelevance of weak disorder, PRB15persistence of gaps; PRL15+CMP16 strong disorder) Vidal, Mouhanna, Giamarchi, PRL99 (evidence of relevance with Fibonacci))

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- Recent investigations Iyer, Oganesyan, Refael, Huse, PRB (2013), Varma, Žnidarič PRB19, Cookmeyer, Motruk, Moore PRB (2020)...

Weyl semimetal and qp disorder

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- Strong interaction may produce different behavior Witczak-Krempa, Knap, Abanin (PRL2014), Maciejko, Nandkishore PRB14...

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$$q_{i,1} - q_{2i,} + 2\omega_i n_i \pi + 2l_i \pi + (\varepsilon_1 - \varepsilon_2) p_F = 0$$

The factor $2\omega_i n_i \pi$ is the momentum exchanged with the quasiperiodic disorder while the factor $2l_i\pi$ is exchanged with the lattice

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- In the q-periodic case are relevant or irrelevant?

Renormalization group analysis

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where by $|.|_T$ is $|2\pi\omega n|_T = \inf_p |2\pi\omega n - 2\pi p|$

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- Diophantine numbers have full measure
- Multiscale decomposition $g(x,y) = \sum_{h=-\infty}^{0} g^h(x,y)$, with $\hat{g}^h(k)$, $|q| \sim \gamma^h$, $k = q + \varepsilon p_F$. $\int P(d\psi)e^V = \int P(d\psi^{\leq -1}) \int P(d\psi)e^V$...

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so that, if $\bar{N} = \max(N_1, N_2, N_3)$ then $\bar{N} \geq C\gamma^{-h/\tau}$

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- Similar to what was done for KAM Lindtedt series (Gallavotti CMP94) but here there are loops (in KAM no loops)

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- With random disorder still open

Fermions in Z^d with strong quasi-periodic disorder

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- Iyer, Oganesyan, Refael, Huse, PRB (2013) (MBL, all states localized, conductivity zero for any T, debated...)
- $H = -\frac{\varepsilon}{2}\sum_{\vec{x}}\sum_{i=1}^{d}(a^+_{\vec{x}+\vec{e}_i}a^-_{\vec{x}} + a^+_{\vec{x}}a^-_{\vec{x}+\vec{e}_i})+$

$$u\sum_{\vec{x}}(\phi_{\vec{x}}-\mu)a^+_{\vec{x}}a^-_{\vec{x}} + U\sum_{\vec{x}}\sum_{i=1}^{d}a^+_{\vec{x}}a^-_{\vec{x}}a^+_{\vec{x}+\vec{e}_i}a^-_{\vec{x}+\vec{e}_i}$$

$\phi_{\vec{x}} = \cos 2\pi(\vec{\omega}\vec{x})$, ω irrational, $\mu = cos\alpha$

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- Iyer, Oganesyan, Refael, Huse, PRB (2013) (MBL, all states localized, conductivity zero for any T, debated...)
- $H = -\frac{\varepsilon}{2}\sum_{\vec{x}}\sum_{i=1}^{d}(a^+_{\vec{x}+\vec{e}_i}a^-_{\vec{x}} + a^+_{\vec{x}}a^-_{\vec{x}+\vec{e}_i}) +$

$$u\sum_{\vec{x}}(\phi_{\vec{x}} - \mu)a^+_{\vec{x}}a^-_{\vec{x}} + U\sum_{\vec{x}}\sum_{i=1}^{d} a^+_{\vec{x}}a^-_{\vec{x}}a^+_{\vec{x}+\vec{e}_i}a^-_{\vec{x}+\vec{e}_i}$$

  $\phi_{\vec{x}} = \cos 2\pi(\vec{\omega}\vec{x})$, ω irrational, $\mu = cos\alpha$
- $T = 0$ If $\hat{H}_{ii}(\mathbf{p},\vec{y}) = \sum_{\vec{x}\in\Lambda}\int dx_0\, e^{i\mathbf{px}} < j^i_{\vec{x},x_0 0}; j^i_{\vec{y},0} >$ the Drude weight is $D^i_{\vec{y}} = \lim_{p_0\to 0}\lim_{\vec{p}\to 0}[\hat{H}_{ii}(\mathbf{p},\vec{y}) + < \tau^i_{\vec{y}} >]$ with $\tau^i_{\vec{y}} = -\frac{\varepsilon}{2}(a^+_{\vec{y}+\vec{e}_i}a^-_{\vec{y}} + a^+_{\vec{y}}a^-_{\vec{y}+\vec{e}_i})$.

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- $u = 1$ considering ε, U small

Fermions in Z^d with strong quasi-periodic disorder

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- Theorem(M PRL 16, CMP 2016, PRB 17, JSP 2021)*Assuming Diophantine conditions and for* ε, U *small enough the* $T = 0$ *2-point function*

$$|S(x;y)| \leq C \log \Delta \frac{e^{-|\log \varepsilon||\vec{x}-\vec{y}|}}{1 + (\Delta|x_0 - y_0|)^N}$$

with $\Delta = (1 + min(|x|, |y|)^{-\tau}$ *and the Drude weight is vanishing* $D^i_{\vec{y}} = 0$

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- Other disorder much more difficult $\cos \omega_1 x_1 + \cos \omega_2 x_2 + \cos \omega_3 x_3$ (even without interaction, Bourgain)

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- Open problem in the random case (even at $T = 0$) or quasi periodic $T \neq 0$....

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- Experiments found instead a universal result Geim, Nosovelov et al (2008) (of course with error bars...)

Graphene

- Important to take into account lattice effect. Hubbard model on the honeycomb lattice is $H = H_0 + U\sum_{\vec{x}\in\Lambda_A\cup\Lambda_B}\left(n_{\vec{x},\uparrow}-\frac{1}{2}\right)\left(n_{\vec{x},\downarrow}-\frac{1}{2}\right)$ with $H_0 = -t\sum_{\vec{x}\in\Lambda_A, i=1,2,3}\sum_{\sigma=\uparrow\downarrow}\left(a^+_{\vec{x},\sigma}b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma}a^-_{\vec{x},\sigma}\right)$, $\vec{\delta}_1 = (1,0)\ ,\quad \vec{\delta}_2 = \frac{1}{2}(-1,\sqrt{3})\ ,\quad \vec{\delta}_3 = \frac{1}{2}(-1,-\sqrt{3})$, Λ_A periodic triangular lattice.

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- The conductivity is defined via the Euclidean Kubo formula (Wick rotation ok) $\sigma_{ii} = \lim_{p_0\to 0}\lim_{p\to 0} \frac{2}{3\sqrt{3}}\frac{1}{p_0}(K_{ii}(p,p_0) + \Delta)$, Δ Schwinger terms, K_{ii} Fourier transform of current correlation $< J_i J_j >$.

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- Lattice Ward Identities ($p_j = 0$, $j \neq i$)

$$p_0 K_{0i}(p,p_0) + p_i(K_{ii}(p,p_0)+\Delta) = 0$$

and $p_\mu \hat{G}_\mu(\mathbf{k},\mathbf{p}) = \hat{S}(\mathbf{k}+\mathbf{p}) - S(\vec{p})$

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- $K(p)$ is even so if differentiable the conductivity would vanishing but is not
- Mutiscale analysis says

$$\hat{K}_{lm}(\mathbf{p}) = \frac{Z_l Z_m}{Z^2} \langle \hat{j}_{\mathbf{p},l}; \hat{j}_{-\mathbf{p},m} \rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p})$$

where $\langle \cdot \rangle_{0,v_F}$ is the average associated to a non-interacting system with Fermi velocity $v_F(U) = \frac{3}{2}t + dU + ..$ $\quad Z_\mu = \frac{3t}{2} + aU + ..$ and $R_{lm}(\mathbf{p})$ with continous derivative.

- By the lattice WI exact relations $Z_0 = Z$, $Z_1 = Z_2 = v_F Z$

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$$\sigma_{11} = -\frac{2}{3\sqrt{3}} \lim_{\omega \to 0^+} \frac{1}{\omega} \Big[\big(\hat{R}_{11}(\omega, \vec{0}) - \hat{R}_{lm}(0, \vec{0}) \big)$$
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- The first term is differentiable and even hence vanishing, while the first term is identical to the free one so it does not depend from v_F

Universality of the conductivity

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$$\sigma_{lm} = \frac{e^2}{h}\frac{\pi}{2}\delta_{lm} \ .$$
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- Frequency and temperature dependence?

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- There are examples where interaction breaks universality (Avdoshkin ,Kozii, Moore PRB 2020) in chiral photocurrent

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$$H_1 = \lambda \sum v(x,y) S^3_x S^3_y$$

with e.g. $\sum_x |x|^n |v(x,y)| \leq C$. [one can consider more general $H_1 = \sum_{i=1}^3 \sum_x S^i_x S^i_{x+2}$].

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$$H_1 = \lambda \sum v(x,y) S^3_x S^3_y$$

with e.g. $\sum_x |x|^n |v(x,y)| \leq C$. [one can consider more general $H_1 = \sum_{i=1}^3 \sum_x S^i_x S^i_{x+2}$].

- Equivalent to fermions $\varepsilon(k) = Jcosk + h$; if $\cos p_F = -h/J$. $\cos(k \pm p_F) - \cos_F = \pm(\sin p_F)k + r(k)$ with $|r| \leq Ck^2$.

Non-integrable spin chains

- The universality properties in 1d are more subtle as the interaction is marginal (but strategy similar).
- $H = H_0 + H_1$

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- In Luttinger model $\pm vk$ (quadratic terms neglected, continuum limit) at $T = 0$ $D = Kv/\pi$, $\kappa = K\pi/v$, v velocity and $\eta = (K + K^{-1} - 2)/2$ critical exponent.

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- Integrabilty conjecture Zotos (1977) in $d = 1$ at $T > 0$ the Drude weight is vanishing for non solvable models and not vanishing for solvable ($\Delta_\beta > 0$ in XXZ by Mazur bounds Ilievski and Prosen 2013).

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- **Theorem** (Bonetto-M EPL, JSP(2018)) Same relations hold $|\lambda|, |J_3| \leq \varepsilon_0$ up to critical point
- At $T = 0$ no effect if integrability, even far from the linear region where irrelevant terms dominate!

Ward Identities

- Ward Identities (conservation of current)

$$-ip_0 < \hat\rho_{\mathbf{p}} \hat\psi^+_{\mathbf{k}} \hat\psi^-_{\mathbf{k}+\mathbf{p}} > +(1-e^{i\mathbf{p}}) < \hat j_{\mathbf{p}} \hat\psi^+_{\mathbf{k}} \hat\psi^-_{\mathbf{k}+\mathbf{p}} >=$$
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- $D^E = \lim_{p_0\to 0}\lim_{p\to 0} D^E(\mathbf{p})$; by the WI $\lim_{p\to 0}\lim_{p_0\to 0} D^E(\mathbf{p}) = 0$; hence a finite Drude weight is possible only of $D(\mathbf{p})$ is non continuous (in FT $1/x^2$). Relation between regularity and transport properties

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- If $< .. >_R$ are the correlations of the reference model and $<>$ of lattice one

$$< j_{\mathbf{x}} j_{\mathbf{y}} >=< j_{\mathbf{x}} j_{\mathbf{y}} >_R + G(\mathbf{x}, \mathbf{y})$$

with $|G(\mathbf{x}, \mathbf{y})| \leq \frac{C}{1+|\mathbf{x}-\mathbf{y}|^3}$ (bound by convergence); in Fourier transform we have decomposed in a continuous and non continuous part.

QFT model

- The reference model verifies an extra chiral symmetry which is not true in the lattice model, that is $\psi^\pm_\omega \to e^{\pm i\alpha_\omega}\psi^\pm_\omega$. If $D_\omega = -ip_0 + \omega v_s k$, $\tau = \frac{\lambda_\infty}{4\pi v_s}$

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- By the WI for the reference model (Z wave finction renormalization, $Z^{(2)}$ vertex renormalization)

$$< \hat{j}_{\mathbf{p}} \hat{j}_{-\mathbf{p}} > = \frac{1}{\pi v Z^2} \frac{(Z^{(2)})^2}{1-\tau^2} [\frac{D_-}{D_+} + \frac{D_+}{D_-} + 2\tau] + G(p)$$

$G(p)$ continuous

Reference model

- $< j\psi\psi >$ and $< j\psi\psi >_R$ coincides up to subleading terms for small $\mathbf{p}$; similarly $< j\psi\psi >$ and $< j\psi\psi >_R$. The lattice WI and the QFT one must coincide, so that

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. LL relation hold for solvable and non solvable models; no effect of solvability at $T=0$.

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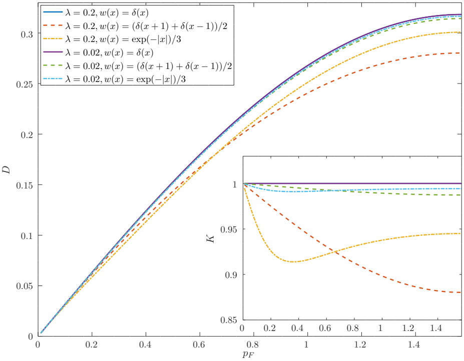
- At criticality one gets the non interacting values $K \to 1 \quad D/D_0 \to 1$ as $r \to 0$. μ_c is shifted by the interaction. $O(\lambda r)$r convergent series.

Spinless case



D and K as function of density (or magnetic field), both in Heisenberg or non solvable cases. D/D_0 and K tend to 1: Features found in the solvable case (Bethe ansatz) persists up to the critical point.

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- One cannot take the $r\to 0$ limit; however for λ small one can see that K does not tend to the non interacting value 1 but $D/D_0\to 1$ for small r.

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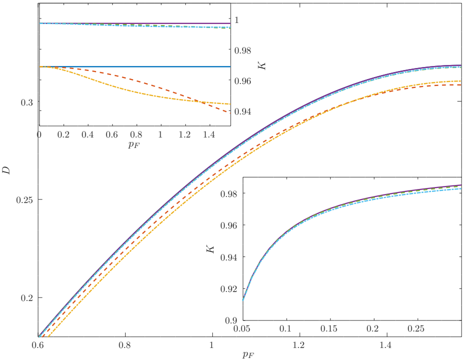

Contrary to the spinless case, we cannot get $p_F = 0$. K show the tendency to a strongly interacting fixed point while D is close to the non interacting value. Cfr the behavior of the Hubbard model by Bethe ansatz (Fig 13 14 in Schultz 93). Again no difference in solvable or non solvable

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- Universality or not with long range forces (graphene, Weyl semimetals etc), temperature and frequency dependence (why small correction in graphene) etc?

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- Drude weight at finite T in 1d with integrable or non integrable interactions in 1d (integrability conjecture true?).
- Analytical results with random disorder even in the ground state in Weyl?
- Localization in the ground state with strong random disorder and interaction at $T = 0$?
- The same with quasi periodic but any state not only GS?
- Universality or not with long range forces (graphene, Weyl semimetals etc), temperature and frequency dependence (why small correction in graphene) etc?
- ...