Relaxation in local many-body Floquet systems

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Quantum transport: disorder, interactions and integrability

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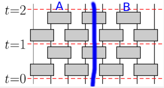
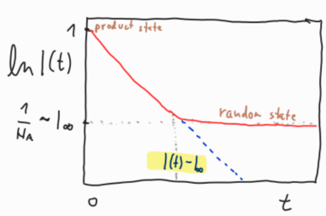
Non-stationary initial condition: **relaxation**

$$|\psi(0)\rangle \xrightarrow{\quad U^t \quad} |\psi(t)\rangle$$

(product state) (random state)



- No conserved quantities: establishment of “typicality”, correlations:
  - OTOC (out-of-time-ordered correlations)
  - purity (entanglement)



Outline

1. Floquet circuits (fixed gates)

2. Purity and OTOC in random circuits (random gates):
   - Markovian average purity (Floquet on average).
   - Purity decay rate not always λ_2.

3. U(d): exact solution for purity $I(t)$

   unitary evolution $\rightarrow$ purity (non-unitary/non-Hermitian)

   - Toeplitz matrix
   - non-Hermitian skin effect (localized eigenvectors)
   - matrix spectrum vs. operator spectrum vs. pseudospectrum

Floquet circuits

(MZ arXiv:2301.06395)

1. product initial state.
2. pick a fixed 2-qubit gate $U_{k,k+1}$:
   $U_{k,k+1} = \exp\left(-\mathrm{i}H\right)$
   $H = \frac{\pi}{4}(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + a_z\sigma_k^z\sigma_{k+1}^z) + 0.82(\sigma_k^x + \sigma_{k+1}^x) + 0.56(\sigma_k^z + \sigma_{k+1}^z)$
3. one-step Floquet propagator: apply all nearest-neighbor $U_{k,k+1}$ in a given order, $U = \prod_k U_{k,k+1}$

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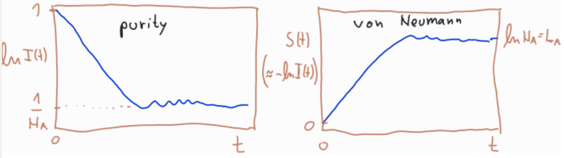
purity: $I(t) = \mathrm{tr}[\rho_A^2(t)]$, say for half-half bipartition

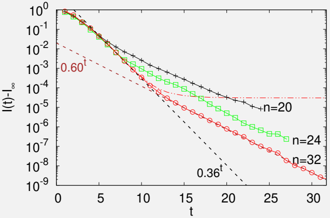
product $|\Psi(0)\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, $|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle$

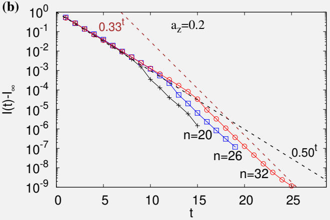
$\rho_A(t) = \mathrm{tr}_B\, |\Psi(t)\rangle\langle\Psi(t)|$, eigenvalues μ_j of $\rho_A(t)$

$I(t) = \sum_j' \mu_j^2$ $\quad$ $S(t) = \sum_j - \mu_j \ln \mu_j$

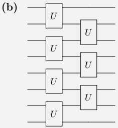


...cont. Floquet circuits

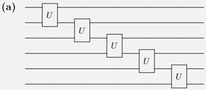


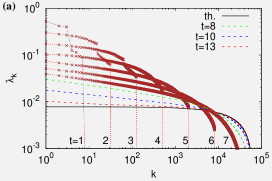


BW circuit with OBC, $a_z = 0.5$



S circuit with OBC, $a_z = 0.2$





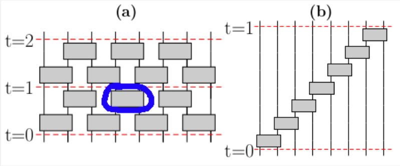
Spectrum of $\rho_A(t)$:

- non-full rank until $t_c \propto n$
- several separated λ_j and non-random bulk (unlike e.g. PT in Scardicchio et al. PRA'10; Vergassola et al. J. Stat. Phys. '11)

Random circuits

Protocol:

1. Pick a pair of sites (n.n.).
2. Apply a given 2-qubit gate, e.g.:
   (a) random U(4)
   (b) **fixed 2-qubit + 1-qubit random U(2)**
3. Repeat in a certain order (e.g. BW, S,...)



(a) brickwall (BW) (b) staircase (S)

Why:

- In quantum information, generating entanglement:

  Emerson et al., Science '03; Oliveira et al. '07; Žnidarič, PRA '08 [random n.n. obc U(4): $I(t) \asymp (4/5)^t$] . . . .

- Understanding generic many-body systems: e.g. operator dynamics

  Keyserling et al. PRX '18, Nahum et al. PRX '18 [BW obc U(4): $I(t) \asymp (4/5)^{2t}$], Khemani et al. PRX '18,...

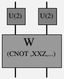
- Classical vs. quantum computation: supremacy Arute et al., Nature '19.

Averaging over $1-$ site Haar

Markovian evolution of average purity $I(t) = \mathrm{tr}[\rho_A^2(t)]$:

Expand many-qubit state: $\rho = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \sigma^{\boldsymbol{\alpha}}$ $(\sigma^{\boldsymbol{\alpha}} = \prod_j \sigma_j^{\alpha_j})$

$$\rho_A^2(t+1) = c_{\boldsymbol{\alpha}}(t) c_{\boldsymbol{\beta}}(t) \mathrm{tr}_B(U\sigma^{\boldsymbol{\alpha}} U^\dagger) \mathrm{tr}_B(U\sigma^{\boldsymbol{\beta}} U^\dagger)$$

$U_{j,k} = V_j V_k W_{j,k} =$ 

average over $V_j, V_k \in \mathrm{Haar\ U(2)}$

Result: $\langle c_{\boldsymbol{\alpha}}^2(t+1)\rangle = M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \langle c_{\boldsymbol{\beta}}^2(t)\rangle$ [Markovian (Oliveira et al. JPA '07)].

Reduction in dimension from $4 \to 2$ (ie., 4×4 for 2-qubits);

for random n.n. protocol M related to integrable spin chains, (MZ PRA'08)

General setting (Kuo, Akhtar, Arovas, and Yi-Zhuang You, PRB '20)

1. Write an abstract state $\Phi(t) = \sum_{\mathbf{s}} I_{\mathbf{s}}(t) |\mathbf{s}\rangle$ $\quad \dim(\mathbf{s}) = 2^n$

   bitstring $\mathbf{s}$ encodes a bipartition, e.g., $\mathbf{s} = 01011$ is a bipartition with sites $A = \{1,3\}$ and $B = \{2,4,5\}$

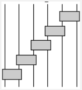
2. Evolution of $\Phi(t)$ is Markovian for any $W_{j,k}$.

$$\boxed{\Phi(t) = M^t \Phi(0)}$$

Example:

XY-gate $W = \exp\left[\mathrm{i}\,\frac{\pi}{4}\left(\sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y\right)\right]$, plus 1-qubit random unitaries

$$M_{i,j} = \frac{1}{9}\begin{pmatrix} 8 & 0 & 0 & 3 \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 3 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}, \qquad M = M_{5,6}M_{4,5}M_{3,4}M_{2,3}M_{1,2} \text{ [not symmetric]},$$



- Works for fixed 2-qubit W + Haar U(2).
- Also for full random U(d) on qudits.

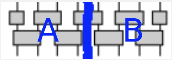
map to statistical model (Nahum, Vijay, Haah PRX '18; Nahum, Zhou PRX'20), dual unitary gates (Bertini et al. PRX '19; Gopalakrishnan et al. PRB '19), Claeys & Lamacraft PRR '20, tripartite (Piroli & Bertini PRB '20) ...

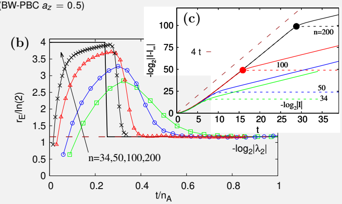
$$I(t) = M^t\, I(0)$$

Standard reasoning:

- M: eigenvalue 1 corresponds to $I_\infty = I(t=\infty)$.
- Decay of $I(t)$ given by $|\lambda_2|$, $I(t) - I_\infty \asymp |\lambda_2|^t$.

$\lambda_2 = \frac{1}{9}(2 - \cos(\pi a_z))^2$ (BW-PBC $a_z = 0.5$)





large n required

many-body effect

Rate is not always determined by λ_2! (J. Bensa & MZ, PRX '21)

related math: Lindbladian gap doesn't always give nonequilibrium relaxation time (Mori & Shirai PRL '20); (Ueda et al. PRL '21); (Clerk et al. arXiv '22)

long transient boundary modes in Lindbladian decay (Flynn, Cobanera & Viola PRL '21)

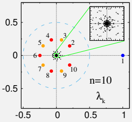
How,...?

- Non-Hermitian: $M = \sum_j \lambda_j |R_j\rangle\langle L_j|$, $|L_j\rangle$ not orthogonal.
- $|I(t)\rangle = \sum_j c_j \lambda_j^t |R_j\rangle$, $c_j = \langle L_j | I(0)\rangle$.

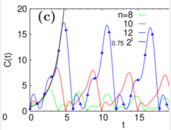
c_j can be exponentially large in n!

$\Longrightarrow$ λ_2 may not matter for $t \leq n/2$.

phantom
S OBC
$a_z = 0$



$$\sim \Big(\sum_j{}' c_j e^{i\phi_j t} \sim 2^t\Big) \overbrace{\left|\tfrac{1}{4}\right|}^{\lambda_2}{}^{t} \sim \underbrace{\left(\tfrac{1}{2}\right)}_{\text{phantom eigenvalue}}{}^{t}$$

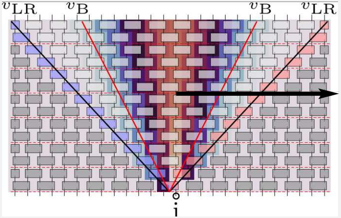


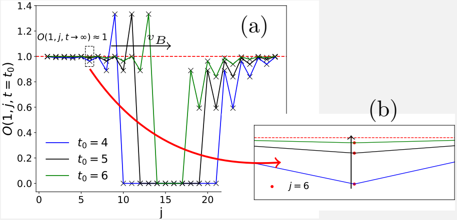
Also in OTOC (J. Bensa & MZ, *PRR '22*)

Out-of-time-ordered correlation (OTOC):

$$O^{\beta}(1,j,t) = \frac{1}{2^{n+1}}\mathrm{tr}\left|\left[\sigma_1^{\alpha}(t),\sigma_j^{\beta}\right]\right|^2 = 1 - \frac{1}{2^n}\mathrm{tr}(\sigma_1^{\alpha}(t)\sigma_j^{\beta}\sigma_1^{\alpha}(t)\sigma_j^{\beta})$$

spreading of operators, $\sigma_1^{\alpha}(t) = U^{\dagger}\sigma_1^{\alpha}U$, correlations



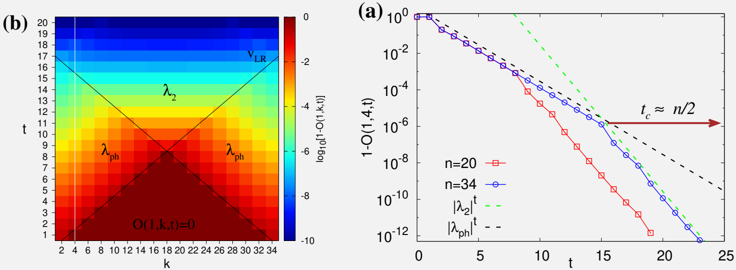


At long times $O^{\beta}(1,j,t\to\infty)\to 1$.

The question:

relaxation towards $O(t=\infty)=1$, i.e., $O(t)-O(\infty)$?

BW PBC XXZ $a_z = 0.2$:



thermalization with phantom $\lambda_{\mathrm{ph}} = 0.39$ ($|\lambda_2|_{\mathrm{S-PBC}} = \frac{1}{3}(2-\cos(a_z\pi))$; also Bertini& Piroli PRB '20)

instead of "correct" $|\lambda_2|_{\mathrm{BW-PBC}} = 0.16$ ($= |\lambda_2|^2_{\mathrm{S-PBC}}$).

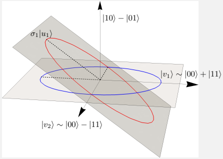
Solution for $U(d)$ random gates

(MZ arXiv:2205.01321)

Simple 2-site transfer matrix:

$$M_{j,k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & \alpha \\ \alpha & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \frac{d}{d^2+1}$$

$$M_{j,k} = |r_1\rangle\langle l_1| + |r_2\rangle\langle l_2| = \sigma_1 |u_1\rangle\langle v_1| + |u_2\rangle\langle v_2|$$



Staircase configuration with OBC

- Markovian process simplifies:

recursion among only n contig. bipartitions (instead of 2^n)

$$I_k(t) = \alpha^k + \sum_{r=2}^{k+1} \alpha^{k+2-r} I_r(t-1)$$

- in matrix notation: $\mathbf{I} := (I_2, I_3, \ldots, I_{n-1})$

$$\mathbf{I}(t+1) = \mathbf{a} + T\mathbf{I}(t)$$

$$T = \begin{pmatrix} \alpha^2 & \alpha & 0 & \cdots & 0 \\ \alpha^3 & \alpha^2 & \alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \ddots & \ddots & \alpha \\ \alpha^{n-1} & \alpha^{n-2} & \cdots & \cdots & \alpha^2 \end{pmatrix}$$

- left and right eigenvectors localized at the edge
  non-Hermitian skin effect
- $\langle L_j | R_j \rangle \sim \exp(-\kappa n)$; almost co-linear basis

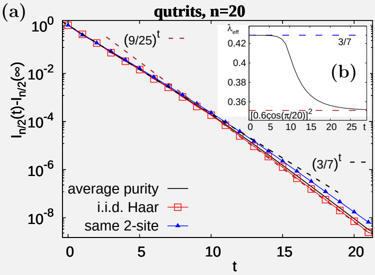
..cont.

- for $t < t_c \propto n$: (from recursion)

$$I(t) = \lambda_{\mathrm{ph}}^t, \qquad \lambda_{\mathrm{ph}} = \frac{d}{d(d-1)+1}$$

- for $t > t_c$: (from exact spectrum)

$$I(t) \sim \lambda_2^t, \qquad \lambda_2 = \frac{4d^2}{(d^2+1)^2}$$

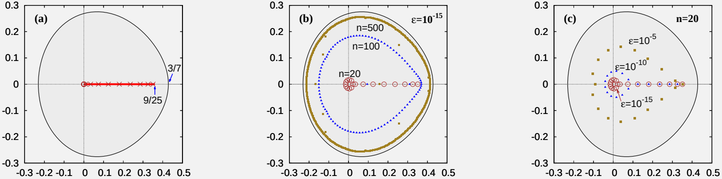


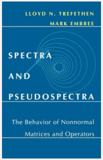
Why?

- Many almost co-linear $|L_j\rangle$; expan. coeff. grow with n.
- What/where is λ_{ph}?

$$T = \begin{pmatrix} \alpha^2 & \alpha & 0 & \cdots & 0 \\ \alpha^3 & \alpha^2 & \alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \ddots & \ddots & \alpha \\ \alpha^{n-1} & \alpha^{n-2} & \cdots & \cdots & \alpha^2 \end{pmatrix}$$

- **Spectrum completely discontinuous in TDL**
  - finite matrix T ($\lambda_j = 4\alpha^2 \cos^2 \frac{\pi j}{n}$) vs. operator T_∞





  - spectrum of T_∞ given by the symbol $a(z) = \frac{\alpha}{z} + \frac{\alpha^2}{1-\alpha z}$.
- **Pseudospectrum $T + \varepsilon E$ is the right thing**
  - $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathrm{spec}(T + \varepsilon E)$

Being exact is wrong.

Spectrum of exact finite T gives an incorrect rate λ_2

Being (slightly) wrong is correct.

Spectrum of noisy T gives the correct λ_{ph}

BW configuration with OBC

- T is symmetric tridiagonal, the same λ_j as for S
- no phantoms, $I(t) \asymp (2\alpha)^{2t} = \left(\frac{2d}{d^2+1}\right)^{2t}$

(known before, Haah et al. PRX'18; Sondhi et al. PRX'18; You et al. PRB '20)

Summary: purity *arXiv:2101.05579*; OTOC *arXiv:2112.07281*; U(d) *arXiv:2205.01321*; Floquet *arXiv:2301.06395*

- Iterating non-Hermitian matrix M

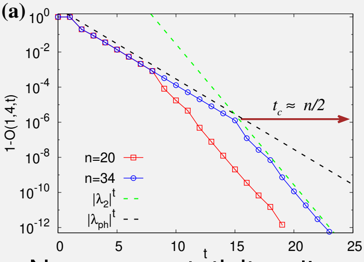
$$I(t) = M^t I(0)$$

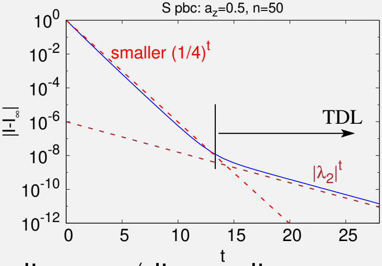
examples: purity, OTOC,...(shows also in von Neumann $S(t)$)

beware of λ_2:

Physically relevant relaxation often not $\sim \lambda_2^t$!

slower: faster:





Non-commutativity: $\lim_{L\to\infty}\lim_{t\to\infty} \neq \lim_{t\to\infty}\lim_{L\to\infty}$

Expansion coefficients can blow-up in a many-body system

- Common: Floquet, random circuits

► Purity:

Gate	Protocol		Rate r_E	Reference
	Configuration	Boundary conditions		
1-dim.(NN):				
U(4)	Random	PBC	$\frac{2}{5} = 0.4$	[24]
U(4)	Random	OBC	$\frac{1}{5} = 0.2$	[24]
U(4)	Permutation	OBC	$\ln\frac{3}{2} \approx 0.40$	[34]
U(4)	BW	PBC	$4\ln\frac{5}{4} \approx 0.89$	
U(4)	BW	OBC	$2\ln\frac{5}{4} \approx 0.45$	[4,5,35]
∞-dim. (all-all):				
U(4)	Random		$\frac{6}{5} = 1.2$	[24,25]
XY, CNOT	Random		$\frac{4}{3} \approx 1.33$	[24]

Gate	Protocol		Rate r_E	$-\ln\lvert\lambda_2\rvert$ (for XY)
	Configuration	Boundary conditions		
1-dim.(NN):				
XXZ	**BW**	**PBC**	$\mathbf{\ln 16 \approx 2.77}$	$\ln 9 \approx 2.20$
XXZ	**BW**	**OBC**	$\mathbf{\ln 4 \approx 1.39}$	$\ln 4 \approx 1.39$
XXZ	S	PBC	$\ln 4 \approx 1.39$	$\ln 3 \approx 1.10$
XXZ	S	OBC	$\ln 2 \approx 0.69$	$\ln 4 \approx 1.39$
CNOT	BW	PBC	≈ 0.591	
CNOT	BW	OBC	≈ 0.284	
CNOT	S	PBC	≈ 0.570	
CNOT	S	OBC	≈ 0.296	

Configuration	Gate	
	XY, $a_z = 0$	XXZ, $a_z \in (0,1)$
BW OBC	λ_2	λ_2 $[a_z \in (0, a_c)]$, faster
BW PBC	Faster	Faster
S OBC	Phantom	Phantom, faster
S PBC	Faster	Faster

► OTOC:

Gate	Protocol		Phantom	True eig.
	config.	b.c.	λ_{ph}	$\lvert\lambda_2\rvert$
	S	OBC	no	$0.86 = \lvert\lambda_2\rvert_{obc}$
W_g	S	PBC	$0.86 = \lvert\lambda_2\rvert_{obc}$	0.74
$\mathbf{a} = (0.5, 0.3, 0.1)$	BW	OBC	no	$0.86 = \lvert\lambda_2\rvert_{obc}$
	BW	PBC	$0.86 = \lvert\lambda_2\rvert_{obc}$	0.73
	S	OBC	$\frac{2}{3} = \lvert\lambda_2\rvert_{Spbc}$	0.45
XXZ	S	PBC	no	$\frac{2}{3} = \lvert\lambda_2\rvert_{Spbc}$
e.g. $\mathbf{a} = (1, 1, 0.5)$	BW	OBC	$\frac{2}{3} = \lvert\lambda_2\rvert_{Spbc}$	0.45
	BW	PBC	$\frac{2}{3} = \lvert\lambda_2\rvert_{Spbc}$	$\frac{4}{9} = \lvert\lambda_2\rvert_{BWpbc}$