A (partial) discussion of the Kitaev table

Gian Michele Graf, ETH Zurich

PhD School: September 16-20, 2019
@Università degli Studi Roma Tre

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based on discussions with J. Haag, B. Roos

Outline

The periodic table of topological matter

Symmetry				d							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Notation for symmetries:

- Θ (time-reversal): antiunitary, $H\Theta = \Theta H$, $\Theta^2 = \pm 1$
- Σ (charge-conjugation): antiunitary, $H\Sigma = -\Sigma H$, $\Sigma^2 = \pm 1$
- $\Pi = \Theta\Sigma = \Sigma\Theta$: unitary

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CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

First version: Schnyder et al.; then Kitaev based on Altland-Zirnbauer; based on Bloch theory

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By now: Non-commutative (bulk) index formulae have been found in all cases (Prodan, Schulz-Baldes)

Basic symmetries

V: complex Euclidean vector space

Symmetries: (anti-)unitary maps $U : V \to V$ (names conventional)

1. Time-reversal $\Theta : V \to V$ antiunitary, $\Theta^2 =: \alpha = \pm 1$

If $V = V_+ \oplus V_-$:

2. Particle-hole $\Sigma : V_\pm \to V_\mp$ antiunitary, $\Sigma^2 =: \beta = \pm 1$
3. Chiral symmetry $\Pi : V_\pm \to V_\mp$ unitary

Remarks. 1) Let $U : V \to V$ with $U^2 = \gamma$; consider $\tilde{U} := cU$ with $c \in \mathbb{C}$, $|c| = 1$ to be chosen. Then

- U unitary: $\tilde{U}^2 = c^2\gamma$. W.l.o.g. γ arbitrary.
- U antiunitary: By $U^2 U = UU^2$ we have $\gamma = \bar{\gamma}$, i.e. $\gamma = \pm 1$. Intrinsic γ: $\tilde{U}^2 = |c|^2\gamma = \gamma$

2) For items 2, 3: $\dim V_+ = \dim V_-$.
3) Σ qualifies as Θ, so far.

Combination of symmetries

$V = V_+ \oplus V_-$

1'. $\Theta : V_\pm \to V_\pm$

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1'. $\Theta : V_\pm \to V_\pm$

2. $\Sigma : V_\pm \to V_\mp$

3. $\Pi : V_\pm \to V_\mp$

Let

$$\Pi = \Theta\Sigma\,, \qquad [\Theta, \Sigma] = 0$$

Then any two symmetries imply the third; moreover $\Pi^2 = \alpha\beta = \pm 1$

Remarks. 1) $\Sigma \neq \Theta$ (flip/no flip)

2) $\Pi \neq \Sigma, \Theta$ (unitary/antiunitary)

3) Possible combinations (none, one, three): $1 + 5 + 4 = 10$ symmetry classes

The classification

- Each entry of the table shows a group $G = 0, \mathbb{Z}, \mathbb{Z}_2$ (index group)
- Vector bundles over $\mathbb{T}^d$ (torus) of a given symmetry class ("topological insulators") are assigned an index $I \in G$
- If two of them (with indices I, I') are homotopy equivalent (within the class), then $I = I'$ (strong index).
- However, this is true only if their restrictions to all tori $\mathbb{T}^{d'} \subset \mathbb{T}^d$, ($d' < d$) are homotopy equivalent (weak indices).
- However, also non homotopy equivalent bundles may have $I = I'$, if they are so upon addition of trivial ones (stably homotopic $\rightarrow$ K-theory)

The K-theoretic point of view

Example: Integers $k \in \mathbb{Z}$ may be identified with pairs $(n_+, n_-) \in \mathbb{N}^2$ of naturals, up to equivalence $(n'_+, n'_-) \sim (n''_+, n''_-)$ defined by

$$(n'_+ + \tilde{n}, n'_- + \tilde{n}) = (n''_+ + \tilde{\tilde{n}}, n''_- + \tilde{\tilde{n}})$$

for some $\tilde{n}, \tilde{\tilde{n}} \in \mathbb{N}$.

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Here: Pairs of vector spaces $V = (V_+, V_-)$ instead of $V = V_+ \oplus V_-$.

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Say $V' \sim V''$ if

$$(V'_+ \oplus \tilde{V}, V'_- \oplus \tilde{V}) \cong (V''_+ \oplus \tilde{\tilde{V}}, V''_- \oplus \tilde{\tilde{V}})$$

(homotopy).

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(homotopy). In case of symmetry $U = \Theta, \Sigma, \Pi$, the map $U : V_\pm \to V_{\pm/\mp}$ is augmented to $U \oplus \tilde{U}$ with $\tilde{U} : \tilde{V} \to \tilde{V}$, $\tilde{U}^2 = \pm 1$

Remarks. 1) Dimension redefined (only here):
$\dim V = \dim V_+ - \dim V_-$

2) If Σ, Π are symmetries, $\dim V = 0$.

3) Notions extended to vector bundles

The derivation of the table

One more column $d = 0$: Vector bundles over a point $\equiv$ vector spaces

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(Same as claimed for $d = 8$).

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(Same as claimed for $d = 8$). We'll derive it.

Normal forms of lone symmetries

Let

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad \omega = \mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det\varepsilon = \det\omega = 1$$

We'll see: Depending on classes, $N := \dim V = n, 2n, 4n$, $(n = 1, 2, \ldots)$.

We'll construct adapted orthonormal bases $\underline{V} = (v_j)_{j=1}^{k} = (v_1, \ldots, v_k)$ of invariant subspaces of V of dimension $k = 1, 2, 4$; generated by arbitrary v_1, $(\|v_1\| = 1)$.

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Left action: $U\underline{V} = (Uv_1, \ldots, Uv_k)$ ($U : V \to V$ map)

Right action: $\underline{V}M = (\sum_i v_i M_{ij})_{j=1}^{k}$ (M: matrix of order k)

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$$\Theta \underline{V} = \underline{V}$$

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- Θ ($\alpha = +1$) has $k = 1$: v_1 such that $\Theta v_1 = v_1$, i.e.

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$$\Theta \underline{V} = \underline{V}$$

- Θ ($\alpha = -1$) has $k = 2$: $v_2 := \Theta v_1$, $\Theta v_2 = -v_1$; then

$$\Theta \underline{V} = (\Theta v_1, \Theta v_2) = (v_2, -v_1) = \underline{V}\varepsilon$$

Normal forms of lone symmetries (cont.)

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- Π ($\gamma = -1$ w.l.o.g.) has $k = 2$: $v_1 \in V_+$, $v_2 := -\mathrm{i}\Pi v_1 \in V_-$; then

$$\Pi \underline{V} = (\mathrm{i}v_2, \mathrm{i}v_1) = \underline{V}\omega$$

Normal forms of combined symmetries Θ, Σ

- $\alpha = +1, \beta = \pm 1$ has $k = 2$:

$$\Theta \underline{V} = \underline{V}, \qquad \Sigma \underline{V} = \underline{V} \begin{cases} \omega & (\beta = +1) \\ \varepsilon & (\beta = -1) \end{cases}$$

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- $\alpha = -1, \beta = \pm 1$ has $k = 4$:

$$\Theta \underline{V} = \underline{V} \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)\,, \qquad \Sigma \underline{V} = V \begin{pmatrix} 0 & \beta 1_2 \\ 1_2 & 0 \end{pmatrix}$$

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Let $K : \mathbb{C}^N \to \mathbb{C}^N$ be the standard complex conjugation

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Claim: Lone symmetries $U = \Theta, \Sigma$ ($U^2 = \gamma = \pm 1$; $\gamma = \alpha$ or β, hence $2 + 2$ cases) on V can be brought to the following form on $\mathbb{C}^N$

- $\gamma = +1$ ($N = n, 2n$): $U = K$
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- $\gamma = -1$ ($N = 2n$): $U = \varepsilon K$

Remarks. 1) Note that Σ qualifies as Θ (see earlier remark)
2) For $U = \Sigma$ the split $V = V_+ \oplus V_-$ is compatibly realized as

$$V_\pm = \{(v, \pm \mathrm{i} v) \mid v \in \mathbb{C}^n\} \subset \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^N\,, \qquad \text{(or flipped)}$$

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Remarks. 1) Note that Σ qualifies as Θ (see earlier remark)
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$$V_\pm = \{(v, \pm \mathrm{i} v) \mid v \in \mathbb{C}^n\} \subset \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^N, \qquad \text{(or flipped)}$$

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4) This foreshadows: The lone symmetry Θ will not contribute to G (to be checked).

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- Classification: Right cosets $\mathcal{B}/\mathcal{T}$, connected components thereof.

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Remarks. 1) The index trivializes for Σ, Π by $\dim V_+ - \dim V_- = 0$. Do new indices appear?

2) The index survives for just Θ (classes AI, AII). Does the group become larger? (Likely not by earlier remark)

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Classes involving Π (AIII, BDI, CI, DIII, CII)

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If further symmetries: $[H, \Theta] = 0$.

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- The group $Sp(2n)$ of complex symplectic matrices U of order $2n$, i.e.,
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(with composite ε), is connected. Same for $Sp(2n) \cap U(2n)$

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$U(n)$ is connected, hence index group $G = 0$.

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- $G = \pi_0(O(n)) = \mathbb{Z}_2$ by $\det U = \pm 1$

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- So $U\varepsilon U^T = \varepsilon$, i.e.
- $U \in Sp(2n) \cap U(2n)$, which is a connected group.
- Hence $G = 0$

Conclusion

Symmetry				d
Class	Θ	Σ	Π	0
A	0	0	0	$\mathbb{Z}$
AIII	0	0	1	0
AI	1	0	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$
DIII	-1	1	1	0
AII	-1	0	0	$\mathbb{Z}$
CII	-1	-1	1	0
C	0	-1	0	0
CI	1	-1	1	0