Violation of bulk-edge correspondence in a hydrodynamic model

Gian Michele Graf
ETH Zurich

PhD School: September 16-20, 2019
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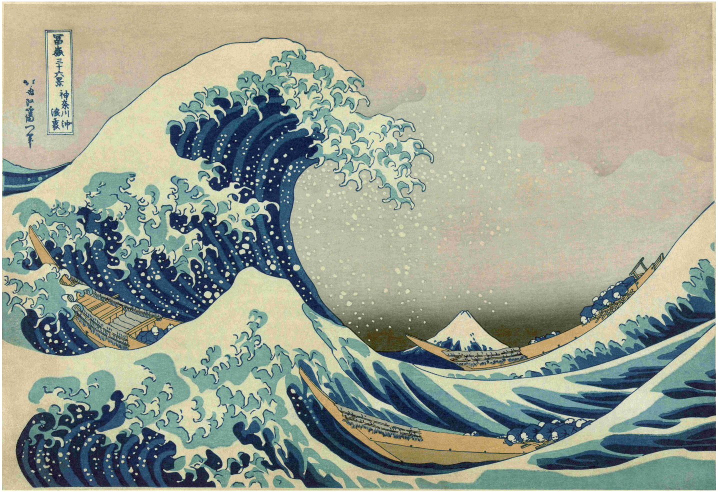
based on joint work with Hansueli Jud, Clément Tauber

Outline

The Great Wave off Kanagawa



(by K. Hokusai, ∼1831)

A hydrodynamic model

Topology by compactification

The Hatsugai relation

Violation

What goes wrong?

The model (take it or leave it)

- The Earth is rotating.

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Incompressible, shallow water equations (preliminary):

$$\frac{\partial \eta}{\partial t} = -h\underline{\nabla} \cdot \underline{v}$$
$$\frac{\partial \underline{v}}{\partial t} = -g\underline{\nabla}\eta - f\underline{v}^{\perp}$$

- fields (dynamic): velocity $\underline{v} = \underline{v}(x, y)$, height above average $\eta = \eta(x, y)$
- parameters: gravity g, average depth h, angular velocity $f/2$

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Steps: (a) Linearization, (b) $(2 + 1)$-split, and (c) dimensional reduction

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(c) Replace $\underline{v}$ by its average over $0 \leq z \leq h$:

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A convenient extension

Momentum equations (in dimension 2):

$$\rho \frac{D\underline{v}}{Dt} = \underline{b} + \underline{\nabla} \cdot \underline{\underline{\sigma}}$$

body forces $\vec{b}$, stress tensor $\underline{\underline{\sigma}}$.

To $\sigma_{ij} = -p\delta_{ij}$ (Euler) add either ($v_{i,j} := \partial v_i / \partial x_j$):

- even viscosity (Navier-Stokes)

$$\underline{\underline{\sigma}} = -\eta \begin{pmatrix} 2v_{1,1} & v_{1,2}+v_{2,1} \\ v_{1,2}+v_{2,1} & 2v_{2,2} \end{pmatrix}, \qquad \underline{\nabla} \cdot \underline{\underline{\sigma}} = \eta \Delta \underline{v}$$

- odd viscosity (Avron)

$$\underline{\underline{\sigma}} = -\eta \begin{pmatrix} -(v_{1,2}+v_{2,1}) & v_{1,1}-v_{2,2} \\ v_{1,1}-v_{2,2} & v_{1,2}+v_{2,1} \end{pmatrix}, \qquad \underline{\nabla} \cdot \underline{\underline{\sigma}} = -\eta \Delta \underline{v}^{\perp}$$

The model (final form)

Equations of motion

$$\frac{\partial \eta}{\partial t} = -h\underline{\nabla} \cdot \underline{v}$$
$$\frac{\partial \underline{v}}{\partial t} = -g\underline{\nabla}\eta - f\underline{v}^{\perp} - \nu\Delta\underline{v}^{\perp}$$

with $\nu = \eta/\rho$.

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with $\nu = \eta/\rho$. After rescaling ($gh = 1$)

$$\frac{\partial \eta}{\partial t} = -\underline{\nabla} \cdot \underline{v}$$
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In Hamiltonian form ($\underline{v} =: (u, v)$, $p_x := -\mathrm{i}\partial/\partial x$)

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi$$

$$\psi = \begin{pmatrix} \eta \\ u \\ v \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & p_x & p_y \\ p_x & 0 & \mathrm{i}(f - \nu\underline{p}^2) \\ p_y & -\mathrm{i}(f - \nu\underline{p}^2) & 0 \end{pmatrix} = H^*$$

The model as a spin 1 bundle

By translation invariance (momentum $\underline{k} \in \mathbb{R}^2$), H reduces to fibers

$$H = \begin{pmatrix} 0 & k_x & k_y \\ k_x & 0 & \mathrm{i}(f - \nu \underline{k}^2) \\ k_y & -\mathrm{i}(f - \nu \underline{k}^2) & 0 \end{pmatrix}$$

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where $\vec{S}$ is an irreducible spin 1 representation

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}$$

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Eigenvalues

$$\omega_0(\underline{k}) = 0 \,, \qquad \omega_\pm(\underline{k}) = \pm |\vec{d}(\underline{k})| = \pm (\underline{k}^2 + (f - \nu \underline{k}^2)^2)^{1/2}$$

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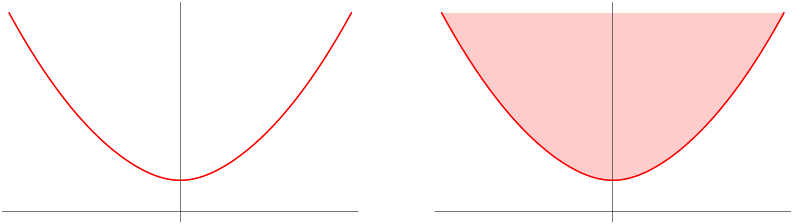

Left: ω_+ as a function of $\underline{k}$
Right: projected along k_y as a function of k_x

Remark: Gap is $f > 0$

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Same as for $\vec{e} \cdot \vec{S}$ with $\vec{e} = \vec{d}/|\vec{d}|$, denoted

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Remarks.

- The compactification of $\mathbb{R}^2$ is S^2.
- $\vec{e}(\underline{k}) \mapsto (0, 0, -\operatorname{sgn} \nu)$ as $\underline{k} \to \infty$ by $\vec{d}(\underline{k}) = (k_x, k_y, f - \nu \underline{k}^2)$
- $\vec{e} : \mathbb{R}^2 \to S^2$ extends to a continuous map $S^2 \to S^2$

Lemma. Let $f\nu > 0$. The line bundle $P_+^{(1)} = |\vec{e}, 1\rangle\langle\vec{e}, 1|$ defined by $\vec{e}(\underline{k})$ on S^2 has Chern number

$$\mathrm{ch}(P_+^{(1)}) = 2$$

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(cf. Souslov et al.; Tauber et al.)

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Proof. If $\vec{S}$ were a spin-$\frac{1}{2}$ representation, then

$$\mathrm{ch}(P_+^{(1/2)}) = \deg(\vec{e}) = +1$$

Now $P_+^{(1)} = P_+^{(1/2)} \otimes P_+^{(1/2)}$, so $\mathrm{ch}(P_+^{(1)}) = 1 + 1$ $\square$

Topological phenomena at interfaces

$f > 0$ (< 0) on northern (southern) hemisphere

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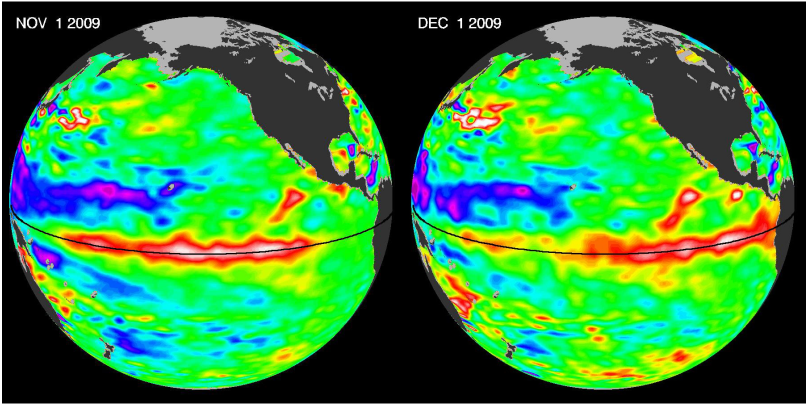
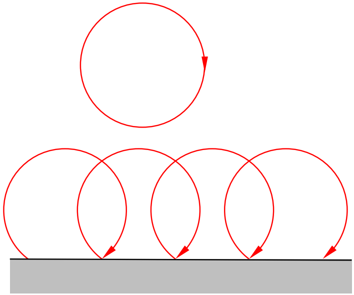


(Source: NASA)

The role of the coast



The figure illustrates the clockwise motion of both a particle in a magnetic field and of a wave in presence of a Coriolis force.

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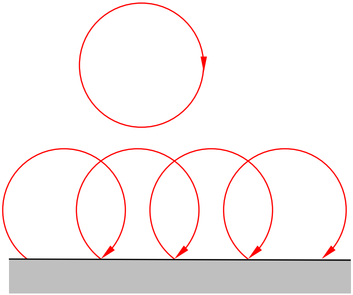

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Boundary waves are gapless (Halperin 1982, Kelvin 1879).

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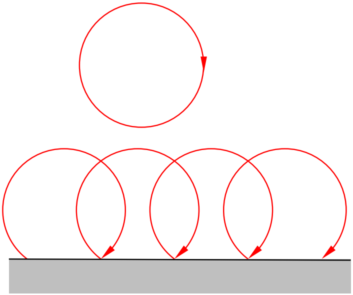

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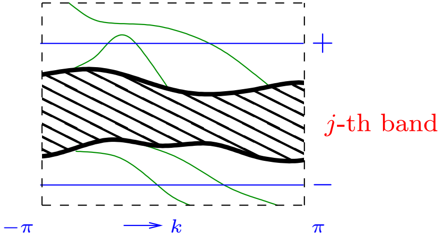
Halperin's work led to the far reaching bulk-edge correspondence.

The Hatsugai relation and bulk-edge correspondence

A (projected) band separated from the rest of the bulk spectrum; edge states (aka evanescent states, bound states).



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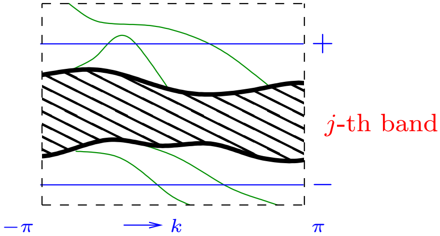

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Alternatively: merging with the band from above/below

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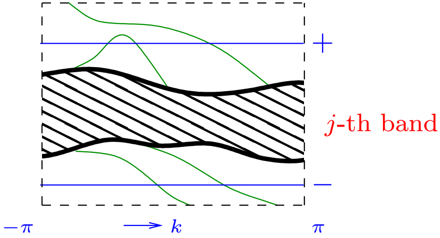

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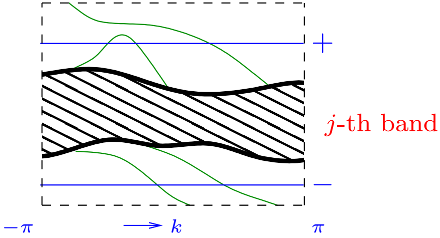

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- Edge index: $\mathcal{N}^\sharp := n_j^+$ for uppermost occupied band j
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- Bulk-edge correspondence: $\mathcal{N} = \mathcal{N}^\sharp$

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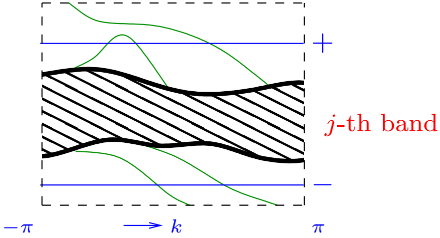

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- Bulk-edge correspondence: $\mathcal{N} = \mathcal{N}^\sharp$
- Proof: Telescoping sum.

Bulk-edge correspondence?

Sea restricted to upper half-space $y > 0$.
Boundary condition at $y = 0$ (parametrized by real parameter a):

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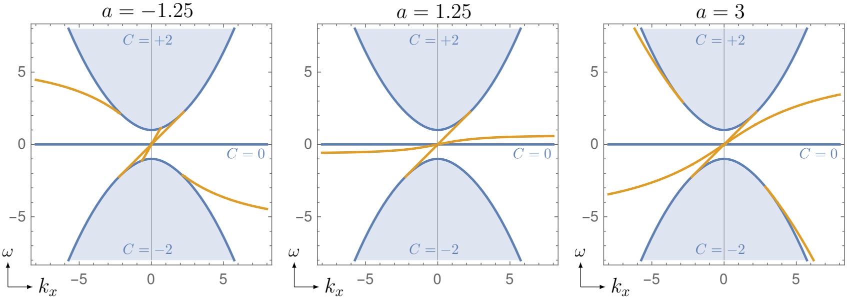
Remark. Merging with the band from below, but boundary is negatively oriented.

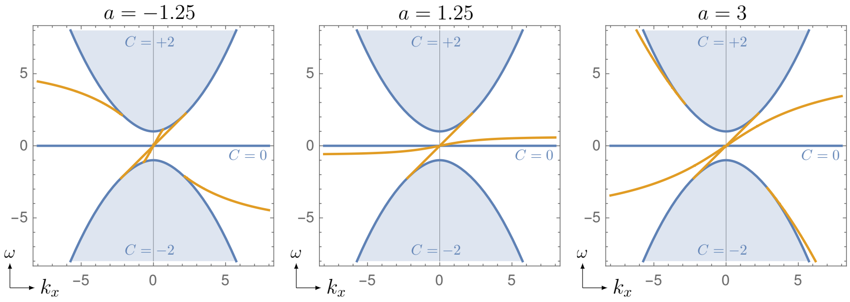
Bulk-edge correspondence?

Spectra of H_a



- Kelvin waves are seen in all cases
- Bulk-edge correspondence is violated!
- There are edge states never merging with a band
- There are edge states “merging at infinity”

Bulk-edge correspondence?

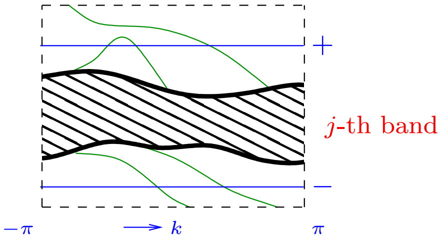


Theorem. (Violation of correspondence) As a function of the boundary parameter a, the edge index takes the values

$$\mathcal{N}^\sharp = \begin{cases} 2 & (a < -\sqrt{2}) \\ 3 & (-\sqrt{2} < a < 0) \\ 1 & (0 < a < \sqrt{2}) \\ 2 & (a > \sqrt{2}) \end{cases}$$

Recall: The bulk index is $\mathcal{N} = 2$.

Back to the Hatsugai relation



$$\mathrm{ch}(P) = n^{+} - n^{-}$$

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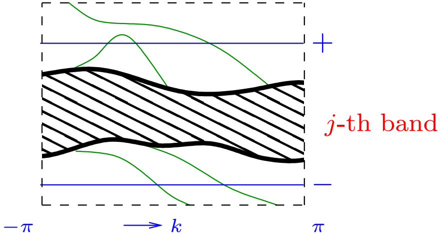

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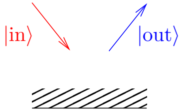
Relation to scattering from inside the bulk:

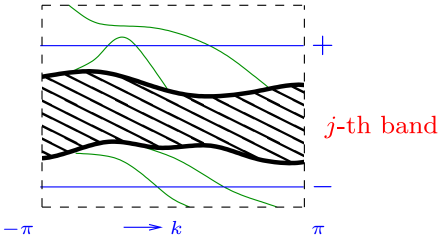


defines scattering map

$$S : |\mathrm{in}\rangle \mapsto |\mathrm{out}\rangle$$

and scattering phase $S(k, E) = \langle \mathrm{in} | \mathrm{out} \rangle$ (k: longitudinal momentum)

Back to the Hatsugai relation



$$\mathrm{ch}(P) = n^{+} - n^{-}$$

Relation can be split in two (Porta, G.):

$$\mathrm{ch}(P) = \mathcal{N}(S^{+}) - \mathcal{N}(S^{-})$$
$$\mathcal{N}(S^{\pm}) = n^{\pm} \quad \text{(Levinson theorem)}$$

where

- $S^{\pm} = S^{\pm}(k) = S(k, E^{\pm}(k) \mp 0)$, $(k \in S^1)$
- $\mathcal{N}(f)$ winding number of $f : S^1 \to S^1$.

What goes wrong?

Is it?

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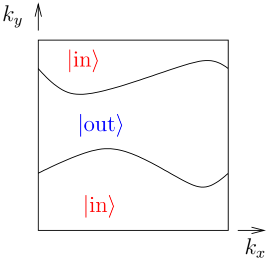
Pictures of torus (Brillouin zone; k_x, k_y longitudinal/transversal momentum)



Regions of $|\mathrm{out}\rangle$, $|\mathrm{in}\rangle$ states

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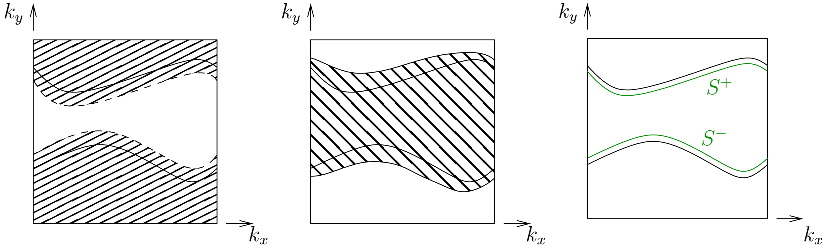

Left: Region admitting (extended) section of states $|\mathrm{in}\rangle$
Middle: Region admitting (extended) section of states $|\mathrm{out}\rangle$
Right: The scattering phases $S^{\pm}(k)$ as transition functions

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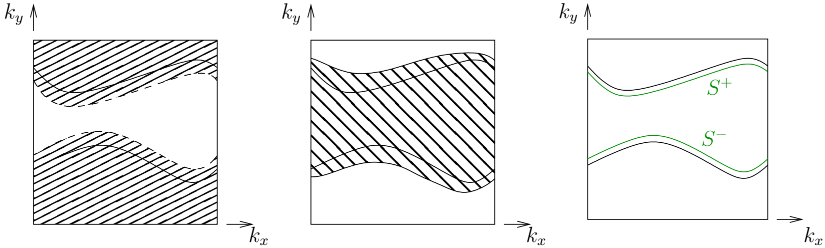

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That still holds for waves: On the compactified sphere (instead of torus) one hemisphere contains incoming states, one outgoing.

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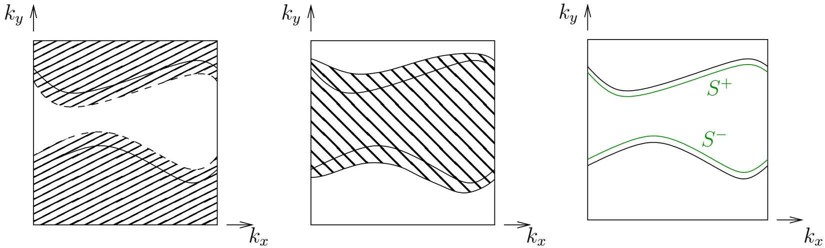

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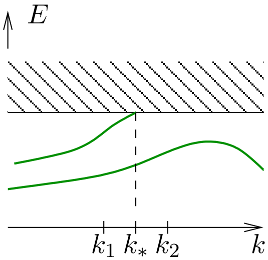
More precisely: Suppose $H(k)$ depends on some parameter $k \in \mathbb{R}$



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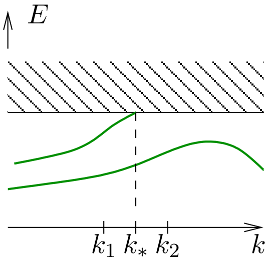

The scattering phase jumps when a bound state reaches threshold

$$\lim_{E \to 0} \arg S(k, E) \Big|_{k_1}^{k_2} = \mp 2\pi$$

The Levinson scenario

$$\lim_{E \to 0} \arg S(k_x, E)\Big|_{k_1}^{k_2} = \mp 2\pi$$

Structure of scattering phase

$$S(k_x, E) = -\frac{g(k_x, \tilde{k}_y)}{g(k_x, k_y)}$$

where

- $\tilde{k}_y$ and k_y are the incoming/outgoing momenta with $E(k_x, k_y) = E(k_x, \tilde{k}_y) = E$

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Bound states of $H(k_x)$ correspond to poles of $S(k_x, E)$ with $\operatorname{Im} k_y < 0$ ("bound out-state without in state"); i.e. to $g(k_x, k_y) = 0$

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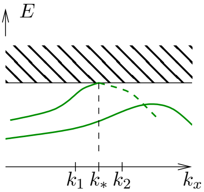
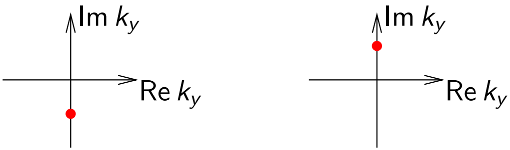


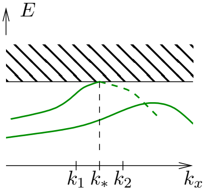
Bound states of $H(k_x)$ correspond to complex zeros k_y of $g(k_x, k_y)$



$(k_x < k_*)$ $(k_x > k_*)$

Fact 1: As k_x crosses zero, a bound state disappears.

The Levinson scenario



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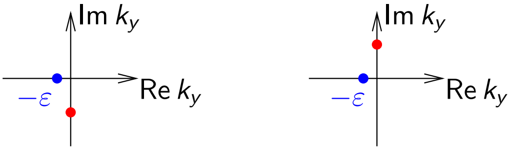

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Fact 2: As k_x crosses zero, $\arg g(k_x, k_y = -\varepsilon)$ changes by $-\pi$ (and $\arg g(k_x, \varepsilon)$ by π), hence S winds by -2π.

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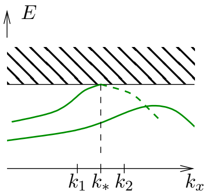

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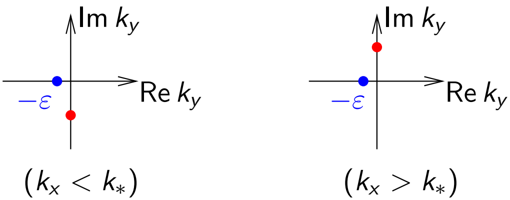

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As for waves, this is the relevant scenario for (almost) all critical, finite momenta k_x.

Waves at infinite momentum

A convenient, orientation preserving change of coordinates on compactified momentum space S^2 is

$$\lambda_x = \frac{k_x}{k_x^2 + k_y^2}, \quad \lambda_y = -\frac{k_y}{k_x^2 + k_y^2}$$

The map $\underline{k} \mapsto \underline{\lambda}$ maps $\infty \to 0$. (Antipodal map in stereographic coordinates.)

Not the Levinson scenario

$\lambda_x = 0$ is always critical (regardless of whether an edge state merges there).

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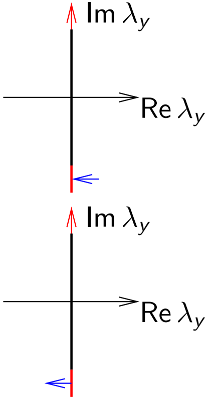
Structure of $g(\lambda_x, \lambda_y)$ for λ_x fixed, small: Two sheets joined by slits.



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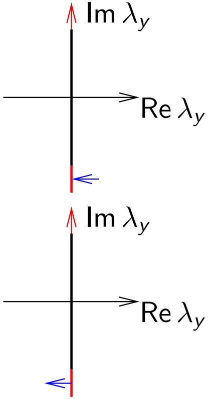

It takes two zeros, both with $\text{Im}\,\lambda_y < 0$, to make a bound state

Not the Levinson scenario: Alternative I

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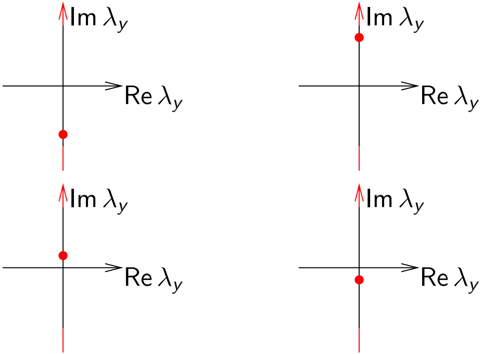

$(\lambda_x < 0)$ $(\lambda_x > 0)$

Fact 1: No bound state is created nor destroyed at transition.

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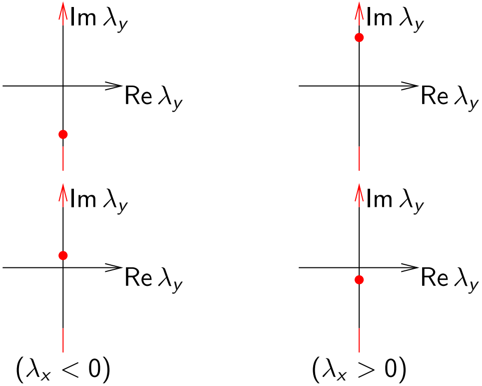

Fact 1: No bound state is created nor destroyed at transition.

Fact 2: There is a jump of $\arg g$ by $\pm\pi$, hence S winds by $\pm 2\pi$

Not the Levinson scenario: Alternative II

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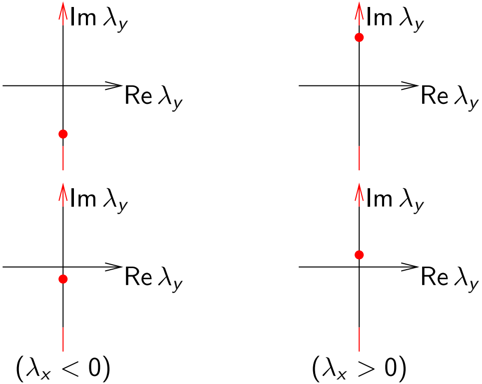

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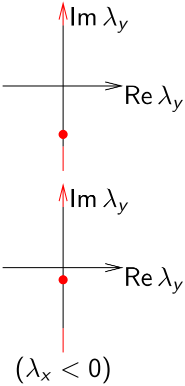


Fact 1: A bound state is destroyed at transition

Fact 2: There is no jump of $\arg g$ and hence S does not wind.

Back to Theorem

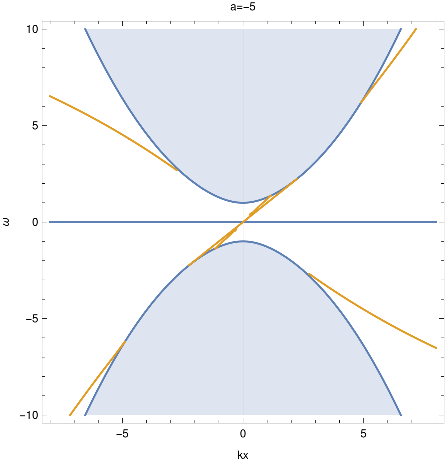
Edge:

$$\mathcal{N}^\sharp = \begin{cases} 2 & (a < -\sqrt{2}) \\ 3 & (-\sqrt{2} < a < 0) \\ 1 & (0 < a < \sqrt{2}) \\ 2 & (a > \sqrt{2}) \end{cases}$$

Bulk:

$$\mathcal{N} = 2$$

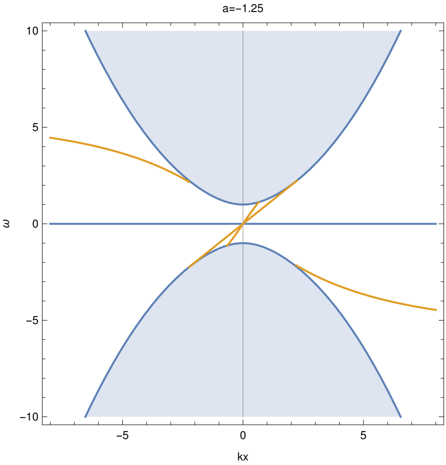
Back to Theorem, case by case



$$\mathcal{N}^\sharp = 2\,, \qquad (a < -\sqrt{2})$$

Alternative II: Edge state merging at infinity; no winding of S there

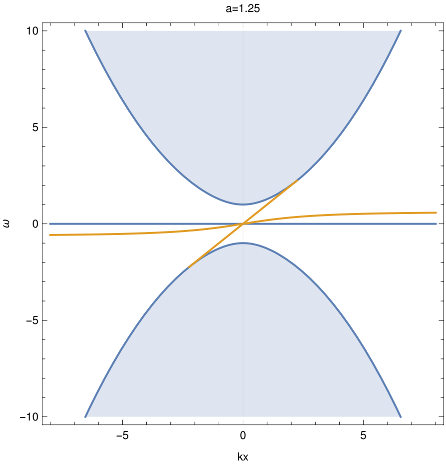
Back to Theorem, case by case



$$\mathcal{N}^\sharp = 3\,, \qquad (-\sqrt{2} < a < 0)$$

Alternative I: No edge state merging at infinity; winding of S by -1

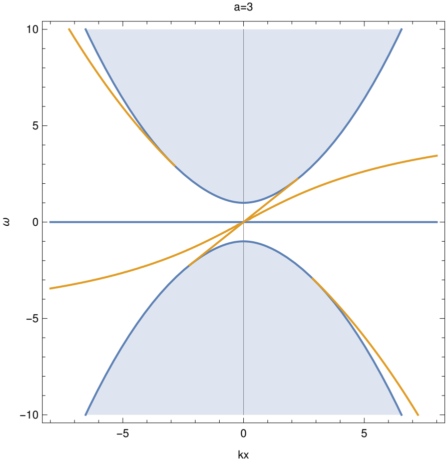
Back to Theorem, case by case



$$\mathcal{N}^\sharp = 1\,, \qquad (0 < a < \sqrt{2})$$

Alternative I: No edge state merging at infinity; winding of S by $+1$

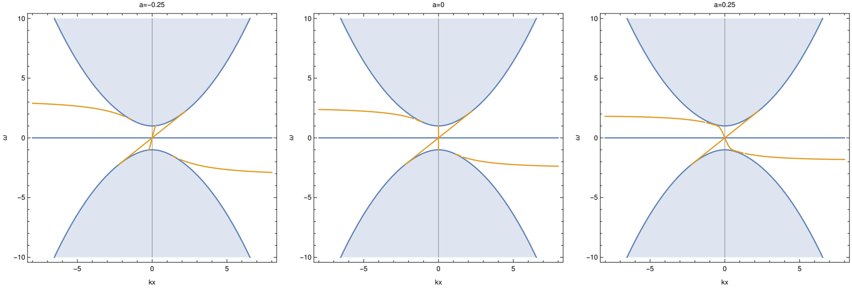
Back to Theorem, case by case



$$\mathcal{N}^\sharp = 2\,, \qquad (a > \sqrt{2})$$

Alternative II: Edge state merging at infinity; no winding of S there

The transition at $a = 0$



$a = -0.25$ $\quad$ $a = 0$ $\quad$ $a = 0.25$

- The transition occurs within Alternative 1.
- Winding of S at infinity changes from -1 to $+1$
- The fibers $H_a(k_x)$ of the edge Hamiltonian are self-adjoint for almost all k_x (as it must)

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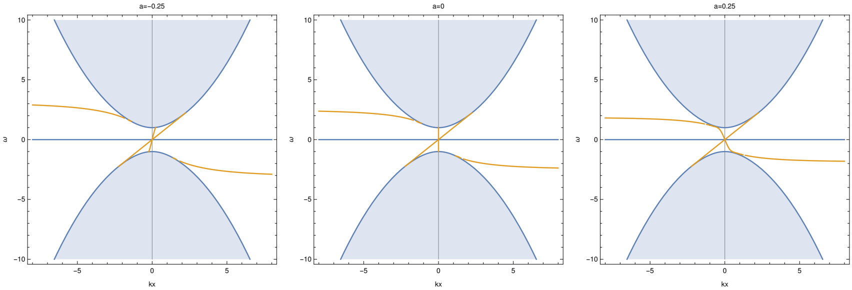

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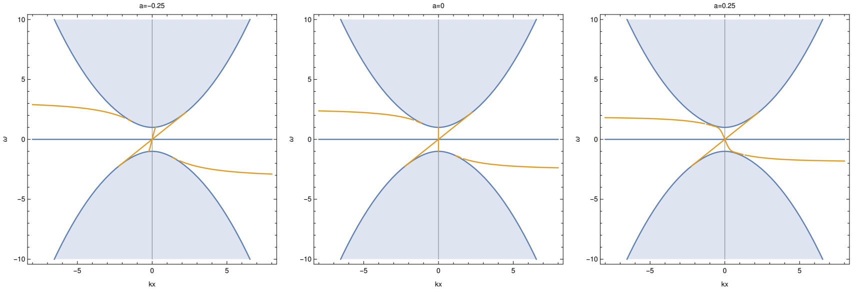

$a = -0.25$ $\qquad$ $a = 0$ $\qquad$ $a = 0.25$

- The transition occurs within Alternative 1.
- Winding of S at infinity changes from -1 to $+1$
- The fibers $H_a(k_x)$ of the edge Hamiltonian are self-adjoint for almost all k_x (as it must), but not for $a = 0$, $k_x = 0$. In fact the boundary condition

$$\mathrm{i} k_x u + a \partial_y v = 0$$

becomes empty.

Summary

- The shallow water model has edge states in presence of Coriolis forces.
- The model is topological if compactified by odd viscosity
- The model violates bulk-boundary correspondence
- Scattering theory (of waves hitting shore) clarifies the cause
- Levinson's theorem does not apply in its usual form