Quantum Transport and Universality

From Topological Materials to Quantum Hydrodynamics

Gian Michele Graf
ETH Zurich

PhD School: September 16-20, 2019
@Università degli Studi Roma Tre

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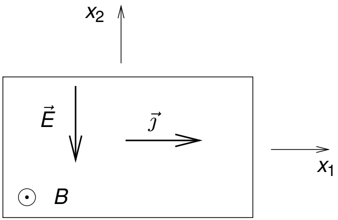
based on joint works with A. Elgart, J. Schenker, M. Porta, J. Shapiro; C. Tauber
and on discussions with Y. Avron, J. Fröhlich

Outline

The phenomenon



B: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Electron gas of
- density n
- Fermi energy μ

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\ , \qquad \underline{\sigma} = \begin{pmatrix} \sigma_\mathrm{D} & \sigma_\mathrm{H} \\ -\sigma_\mathrm{H} & \sigma_\mathrm{D} \end{pmatrix}$$

σ_H: Hall conductance
σ_D: dissipative conductance, ideally $= 0$

The experiment (von Klitzing, 1980)

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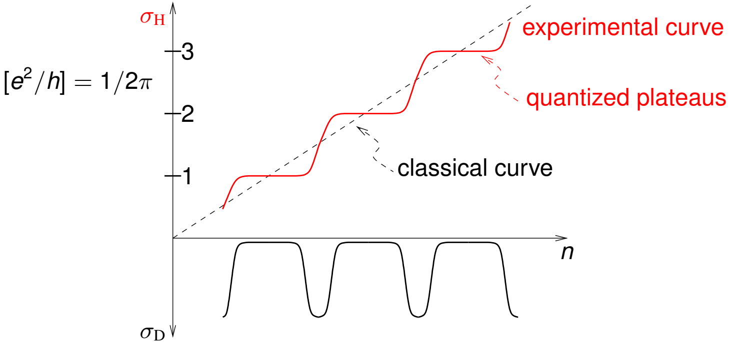

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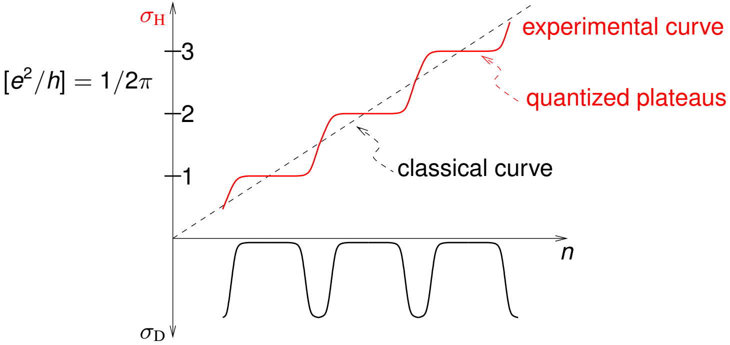

Fractional Quantum Hall effect not discussed

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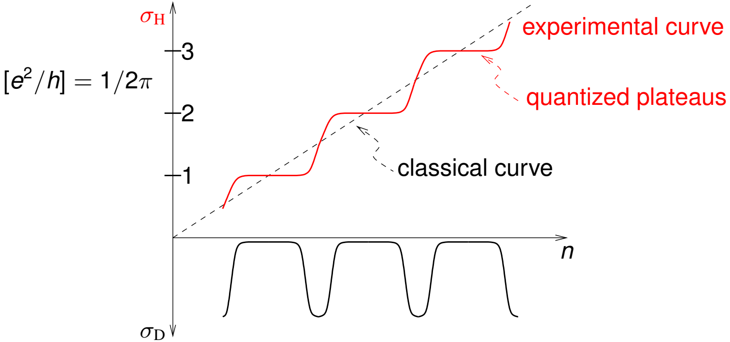

Width of plateaus increases with disorder

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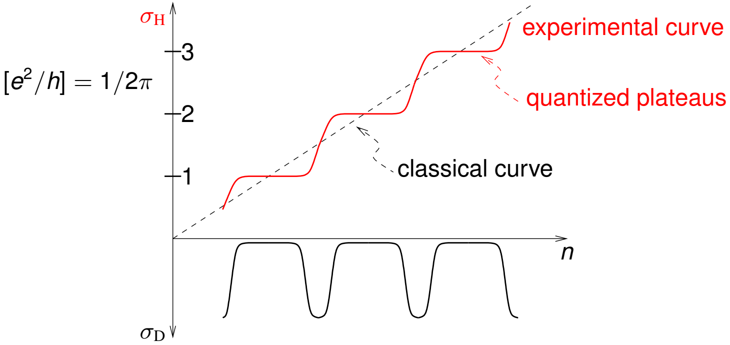
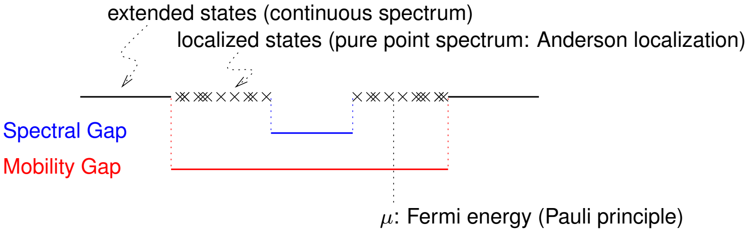


Experiment: $h/e^2 = 25'812.807'4555(59)$ Ohm

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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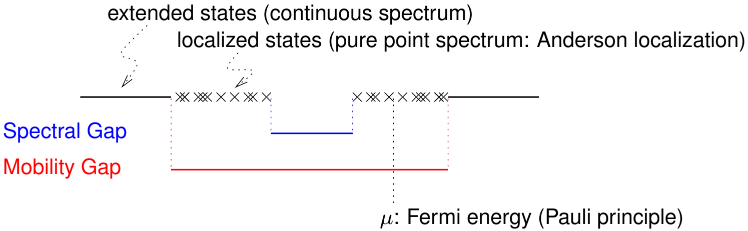

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise

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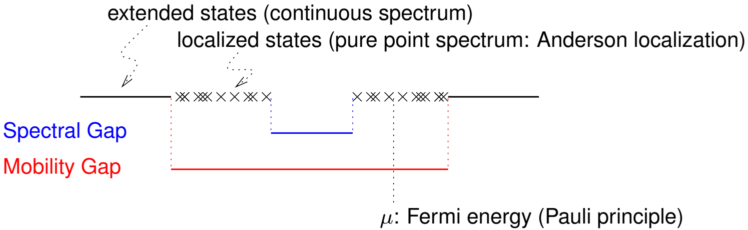

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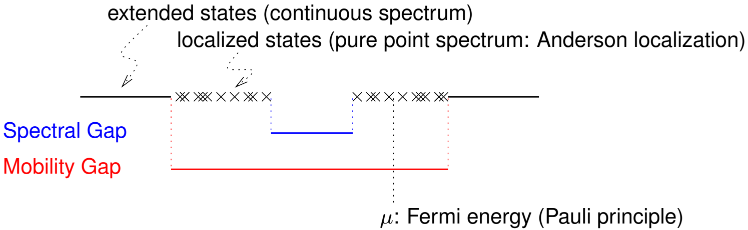

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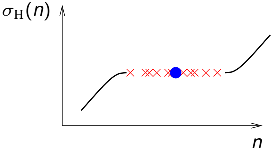
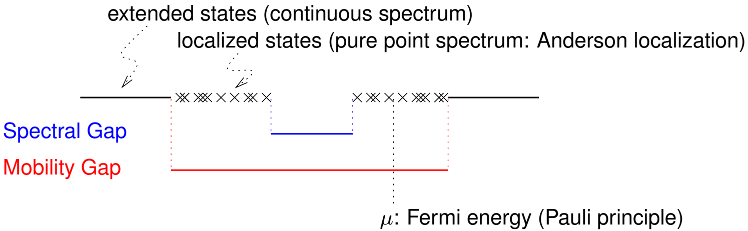


Plateaus arise because of a Mobility Gap only!

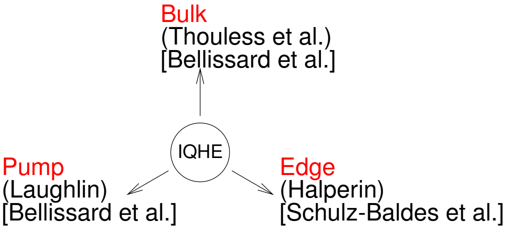
The role of disorder

The spectrum of a single-particle Hamiltonian



- For a periodic (crystalline) medium:
  - Method of choice: Bloch theory and vector bundles (Thouless et al.)
  - Gap is spectral
- For a disordered medium:
  - Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
  - Fermi energy may lie in a mobility gap (better) or just in a spectral gap

Interpretations of IQHE and definitions of σ_H



Interpretations of IQHE and definitions of σ_{H}

Bulk
(Thouless et al.)
[Bellissard et al.]

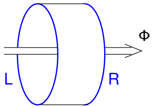
IQHE

Pump
(Laughlin)
[Bellissard et al.]

Edge
(Halperin)
[Schulz-Baldes et al.]

Pump:

$2\pi\sigma_{\mathrm{P}} \equiv$ number n of electrons pumped from L to R upon increasing the magnetic flux Φ by 2π. (Note: $\Phi \rightsquigarrow \Phi + 2\pi$ implies $H \rightsquigarrow UHU^*$.)



Quantization: n is an integer.

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Bulk:

σ_{B} conductivity by Kubo formula: Current density $\vec{\jmath}$ as linear response to an applied (weak) electric field $\vec{E}$ in the bulk.

Quantization: $2\pi\sigma_{\mathrm{B}}$ is a Chern number.

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Bulk
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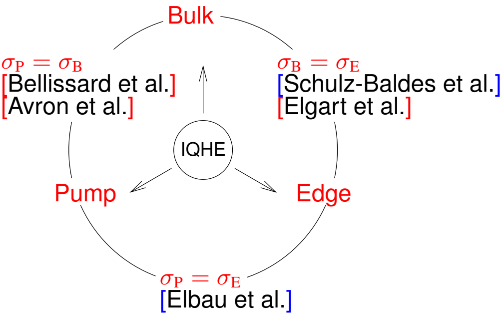
Edge:

σ_{E} conductance: Current carried by edge states per unit voltage, $\sigma_{\mathrm{E}} = dI/d\mu$.

Quantization: $2\pi\sigma_{\mathrm{E}}$ is the number of edge channels.

Equivalences of interpretations

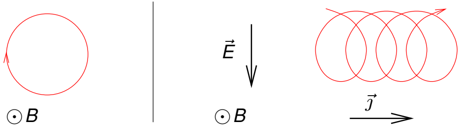


[ ]: spectral gap

[ ]: mobility gap

Bulk vs. Edge

- (Quantum) Hall as a bulk effect



A voltage difference entails an electric field in the bulk

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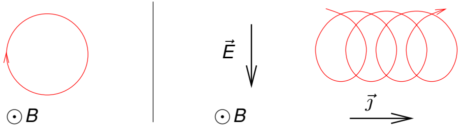

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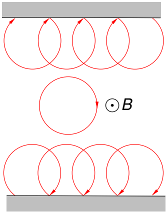

A voltage difference entails different Fermi energies of (chiral) edge states at opposite edges

Heuristic argument for $\sigma_{\mathrm{B}} = \sigma_{\mathrm{E}}$

Bulk: $\vec{\jmath} = -\sigma_{\mathrm{B}} \varepsilon \vec{E}$ with $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (rotation by $\pi/2$)

Edge: $\sigma_{\mathrm{E}} = dI/d\mu$, i.e. $I = \sigma_{\mathrm{E}}(\mu - \varphi)$ with Fermi energy μ and electric potential φ at the edge

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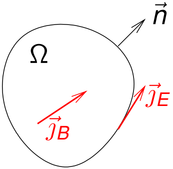

Notation: χ_Ω indicator function of Ω, $\delta_{\partial\Omega}$ delta distribution on $\partial\Omega$, $\vec{n}$ normal vector

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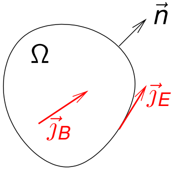

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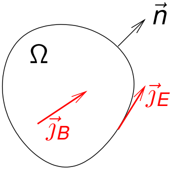

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$$\vec{\jmath}_B = -\chi_\Omega\sigma_{\rm B}\varepsilon\vec{E} = \chi_\Omega\sigma_{\rm B}\varepsilon\vec{\nabla}\varphi \qquad \vec{\jmath}_E = \sigma_{\rm E}(\mu-\varphi)\varepsilon\vec{n}\delta_{\partial\Omega} = -\sigma_{\rm E}(\mu-\varphi)\varepsilon\vec{\nabla}\chi_\Omega$$

$$\mathrm{div}(\varepsilon\vec{v}) = -\,\mathrm{curl}\,\vec{v} \quad (= 0 \text{ for } \vec{v} = \vec{\nabla}\varphi)$$

$$\mathrm{div}\,\vec{\jmath}_B = \sigma_{\rm B}\vec{\nabla}\chi_\Omega\cdot\varepsilon\vec{\nabla}\varphi$$

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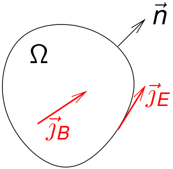

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$$= \chi_\Omega\sigma_{\mathrm{B}}\varepsilon\vec{\nabla}\varphi \qquad = -\sigma_{\mathrm{E}}(\mu - \varphi)\varepsilon\vec{\nabla}\chi_\Omega$$

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$$\mathrm{div}\,\vec{\jmath}_B = \sigma_{\mathrm{B}}\vec{\nabla}\chi_\Omega \cdot \varepsilon\vec{\nabla}\varphi$$

$$\mathrm{div}\,\vec{\jmath}_E = \sigma_{\mathrm{E}}\vec{\nabla}\varphi \cdot \varepsilon\vec{\nabla}\chi_\Omega$$

Thus $\mathrm{div}(\vec{\jmath}_B + \vec{\jmath}_E) = 0$ implies $\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$.

Topological insulators: definition stated

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  For independent electrons: spectral gap at Fermi energy μ

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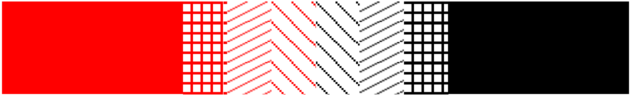
- Integer QHE: $2\pi\sigma_{\mathrm{H}} \in \mathbb{Z}$ tells classes apart
- Refinement: The Hamiltonians enjoy a symmetry which is preserved under deformations. (Classification trivially more restrictive, yet potentially richer: Hamiltonians along deformation may not enjoy symmetry even if endpoints do. Thus finer classes.)

Bulk-edge correspondence

Recall: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and respecting symmetries

Bulk-edge correspondence

Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.

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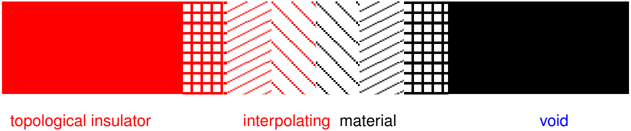

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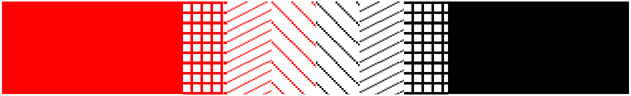

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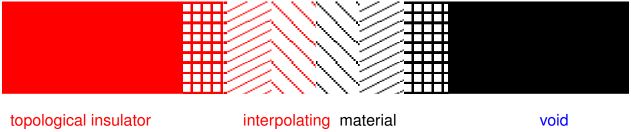

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
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- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence

In a nutshell: Termination of bulk of a topological insulator implies edge states

- Topological insulators are insulating in the bulk, but conducting on the surface

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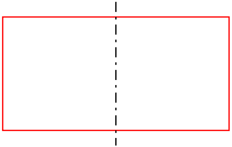

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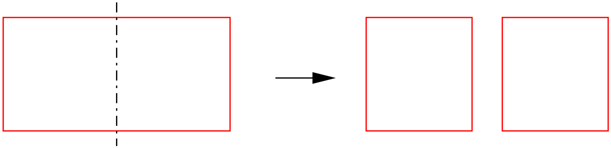

The periodic table of topological matter

Symmetry Class	Θ	Σ	Π	d = 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Notation for symmetries:

- Θ (time-reversal): antiunitary, $H\Theta = \Theta H$, $\Theta^2 = \pm 1$
- Σ (charge-conjugation): antiunitary, $H\Sigma = -\Sigma H$, $\Sigma^2 = \pm 1$
- $\Pi = \Theta\Sigma = \Sigma\Theta$: unitary

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BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

First version: Schnyder et al.; then Kitaev based on Altland-Zirnbauer; based on Bloch theory

The periodic table of topological matter

Symmetry				d							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

By now: Non-commutative (bulk) index formulae have been found in all cases (Prodan, Schulz-Baldes)

Special cases to be considered

Symmetry				*d*							
Class	Θ	Σ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

...and one more

Various approaches to the QHE

- Landau Hamiltonians (not discussed)
- Periodic Hamiltonians (Thouless et al.)
- The role of disorder and non-commutative geometry
- Effective field theories (important, but not discussed; Fröhlich et al.)

Broad mathematical setting

Definitions of σ_{H} and their equivalences should

• be based on a microscopic model (Schrödinger operator), as opposed to an effective theory (conformal or topological field theory).

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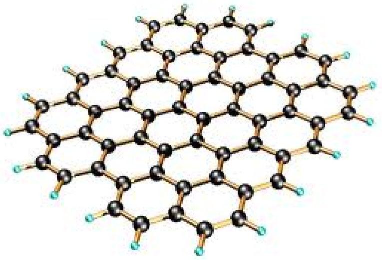
Plane: lattice $\Gamma \ni x = (x_1, x_2)$, e.g. $\Gamma = \mathbb{Z}^2$

Single-particle Hamiltonian H_B: operator on $\ell^2(\Gamma)$ with $H_B(x', x)$ of short range in $|x - x'|$ (tight binding model).



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• apply to infinite systems (thermodynamic limit)

• preferably, be compatible with disorder: Fermi energy μ lies in a Mobility Gap (as opposed to a Spectral Gap).

Mobility gap, technically speaking

Hamiltonian H_B on $\ell^2(\mathbb{Z}^d)$
$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,

Assumption. Fermi projection has strong off-diagonal decay:

$$\sup_{x'} \mathrm{e}^{-\varepsilon|x'|} \sum_x \mathrm{e}^{\nu|x-x'|} |P_\mu(x,x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$)

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- Trivially true for H_B a multiplication operator in position space
- Trivially false for H_B a function of momentum ($P_\mu(x,0) \sim |x|^{-d}$)
- Proven in (virtually) all cases where localization is known.

Mobility gap and dynamical localization (DL)

DL of a random Schrödinger operator H_ω, $(\omega \in \Omega)$ in an interval Δ means (or could equivalently mean) that for some $\nu > 0$ (Notation: $K(x,x') = \langle x|K|x'\rangle$)

$$\mathbb{E}\Big(\sup_{g \in B_1(\Delta)} |\langle x|g(H_\omega)|x'\rangle| \Big) \le C \mathrm{e}^{-2\nu|x-x'|}$$

where

$$B_1(\Delta) = \{ g : \mathbb{R} \to \mathbb{C} \mid |g(\lambda)| \le 1, g \text{ constant on } \lambda \gtrless \Delta \}$$

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Let $g(\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda} E_\Delta(\lambda)$ ($\in B_1(\Delta)$) for $t \in \mathbb{R}$. By DL

$$\mathbb{E}\Big(\sup_{t \in \mathbb{R}} |\langle x|\mathrm{e}^{-\mathrm{i}tH_\omega} E_\Delta(H_\omega)|x'\rangle|\Big) \leq C\mathrm{e}^{-2\nu|x-x'|}$$

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- explains name "DL"
- implies spectral localization

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In particular (drop $\mathbb{E}$, $\sum_{x'}$)

$$\mathrm{e}^{-\varepsilon|x'|} \sum_x |\langle x|P_{\mu,\omega}|x'\rangle| \mathrm{e}^{\nu|x-x'|} \leq C_\omega < +\infty$$

Aside: Rate of change in QM

State space $\mathcal{H}$ state ψ, observable $X = X^*$. Expectation value is

$$(\psi, X\psi)$$

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Because evolution is $\psi \mapsto \mathrm{e}^{-\mathrm{i}Ht}\psi$, so

$$\frac{d}{dt}(\mathrm{e}^{-\mathrm{i}Ht}\psi, X\mathrm{e}^{-\mathrm{i}Ht}\psi)\big|_{t=0} = (\psi, \mathrm{i}[H, X]\psi)$$

Aside: Poor man's second quantization for fermions

Single particle Hilbert space $\mathcal{H} \in \psi$

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Many particle state *S* has single-particle marginal ("density matrix") ρ: operator on $\mathcal{H}$

$$\rho = \rho^* , \qquad 0 \leq \rho \leq 1$$

Meaning: ρ tells expected occupation of any single-particle state $\psi \in \mathcal{H}$, $((\psi, \psi) = 1)$ in the state *S* as

$$(\psi, \rho\psi) = \mathrm{tr}(P\rho) \qquad (\in [0, 1])$$

with $P = \psi(\psi, \cdot)$ the projection onto ψ.

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Expectation value in S:

$$\sum_i x_i \,\mathrm{tr}(P_i\rho) = \mathrm{tr}(X\rho)$$

Aside: Gauge transformations

(Units $e = \hbar = c = 1$)

Electromagnetic (e.m.) fields $\vec{E} = \vec{E}(\vec{x}, t)$, $\vec{B} = \vec{B}(\vec{x}, t)$ expressed in terms of e.m. potentials $\varphi = \varphi(\vec{x}, t)$, $\vec{A} = \vec{A}(\vec{x}, t)$

$$\vec{E} = -\vec{\nabla}\varphi - \partial\vec{A}/\partial t\,, \quad \vec{B} = \mathrm{curl}\,\vec{A}$$

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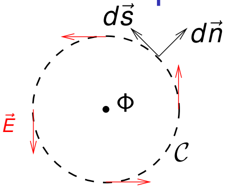
Time-independent gauge transformations are realized as unitaries $U : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \psi \mapsto \mathrm{e}^{\mathrm{i}\chi}\psi$

$$H \mapsto UHU^* = \mathrm{e}^{\mathrm{i}\chi} H \mathrm{e}^{-\mathrm{i}\chi} = H'$$

(by $\mathrm{e}^{\mathrm{i}\chi}(\vec{p} - \vec{A})\mathrm{e}^{-\mathrm{i}\chi} = \vec{p} - \vec{A}'$)

IQHE as a pump: Flux insertion



Flux increase from 0 to Φ
Charge Q traversing $\mathcal{C}$ inwards

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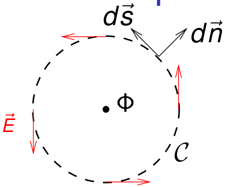

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$$\frac{dQ}{dt} = -\oint_{\mathcal{C}} \vec{\jmath} \cdot d\vec{n} = -\sigma_{\mathrm{H}} \oint_{\mathcal{C}} \vec{E} \cdot d\vec{s} = \sigma_{\mathrm{H}} \frac{d\Phi}{dt}$$

$$Q = \sigma_{\mathrm{H}} \Phi$$

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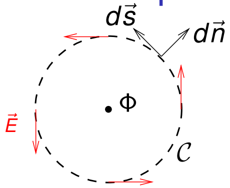

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Flux Φ generated by a gauge potential $\vec{A}$:

$$\oint_{\mathcal{C}} \vec{A}\cdot d\vec{s} = \Phi, \ \text{e.g. } \vec{A} = \vec{\nabla}\Big(\frac{\Phi}{2\pi}\arg\vec{x}\Big) \equiv \vec{\nabla}\chi$$

If $\chi(\vec{x})$ were single-valued:

$$\begin{tikzcd}
\text{gauge} & \vec{A}=0 \arrow[d] & \text{equiv. to} & \vec{A}=\vec{\nabla}\chi \arrow[d] \\
\text{Hamiltonian} & H_B & & UH_BU^*
\end{tikzcd}$$

with $U = \mathrm{e}^{\mathrm{i}\chi}$, unitary. For $\Phi = 2\pi$, U is single-valued, though $\chi(\vec{x}) = \arg\vec{x}$ is not.

Charge Q according to quantum mechanics

Fermi energy μ: all single-particle eigenstates of H_B with eigenvalues (energies) $\leq \mu$ are occupied

Fermi projection (FP) of H_B ($\Phi = 0$): $P_\mu = E_{(-\infty,\mu)}(H_B)$

FP of UH_BU^* ($\Phi = 2\pi$): $UP_\mu U^*$

Evolution of FP as flux $\Phi(t)$ increases from 0 to 2π: $\tilde{U}P_\mu\tilde{U}^*$ with propagator $\tilde{U}$

Tentatively, the charge Q is

$$2\pi\sigma_{\mathrm{P}} = \text{“}\dim \tilde{U}P_\mu\tilde{U}^* - \dim UP_\mu U^*\text{”} = \infty - \infty$$

($\dim P = \dim \operatorname{Ran} P$). The (non existent) expression counts difference in number of electrons: After pumping to $\Phi = 2\pi$, resp. in equilibrium at $\Phi = 2\pi$.

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Rightly interpreted, it is an integer. Hence

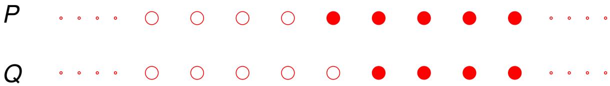
$$2\pi\sigma_{\mathrm{P}} = \text{``}\dim P_\mu - \dim UP_\mu U^*\text{''}$$

since $\tilde{U}$ is connected to 1 (unlike U)

The index of a pair of projections

Orthogonal projections P, Q on a Hilbert space $\mathcal{H}$.

Example (Hilbert's hotel): $\mathcal{H} = \ell^2(\mathbb{Z})$, projections P, Q defined by filled dots $n \in \mathbb{Z}$.



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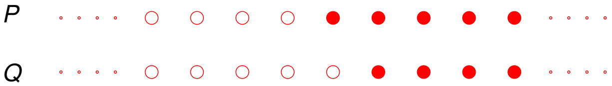

Generalizations of $\dim P - \dim Q$:

$$\operatorname{tr}(P - Q)$$

since $\operatorname{tr} P = \dim P$

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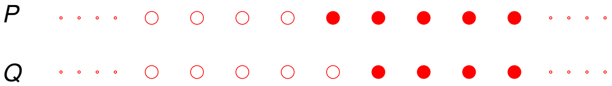

Generalizations of $\dim P - \dim Q$:

$$\operatorname{tr}(P - Q)$$

since $\operatorname{tr} P = \dim P$. More generally:

Definition. The Index of a pair of projections is

$$\begin{aligned}\operatorname{Ind}(P, Q) = \dim\{\psi \in \mathcal{H} \mid P\psi = \psi, Q\psi = 0\} + \\ - \dim\{\psi \in \mathcal{H} \mid Q\psi = \psi, P\psi = 0\}\end{aligned}$$

(if dimensions finite)

Remarks. (i) In the example, both generalizations $= 1$. (ii) In the IQHE only the index is well-defined

Properties of the Index

- Additivity: $\mathrm{Ind}(P,Q) = \mathrm{Ind}(P,R) + \mathrm{Ind}(R,Q)$
- Stability: $\|P-Q\| < 1 \;\Rightarrow\; \mathrm{Ind}(P,Q) = 0$
-
$$\mathrm{Ind}(P,Q) = \mathrm{tr}(P-Q)^{2n+1}$$

  if $P - Q \in \mathcal{J}_{2n+1}$ (trace ideals).

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Remarks. (i) $\operatorname{Ind}(P,Q) = \dim P - \dim Q$ (finite-dimensional case)
(ii) $\operatorname{tr}(P-Q)^3 = \operatorname{tr}(P-Q)$ if $P - Q \in \mathcal{J}_1$; because

$$(P-Q) - (P-Q)^3 = [PQ, [Q, P-Q]]$$

$$AB, BA \in \mathcal{J}_1 \;\Rightarrow\; \operatorname{tr}[A,B] = 0$$

(iii) $\operatorname{Ind}(P,Q) = \operatorname{ind}(QP)$ as a map on $\operatorname{ran} P \to \operatorname{ran} Q$

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(iv) If the unitary U has an eigenbasis and $P - UPU^* \in \mathcal{J}_1$, then $\mathrm{tr}(P - UPU^*) = 0$. In fact, by $U\psi_n = u_n\psi_n$

$$(\psi_n, (P - UPU^*)\psi_n) = (1 - |u_n|^2)(\psi_n, P\psi_n) = 0$$

IQHE as a pump: Definition of σ_{P}

Definition.

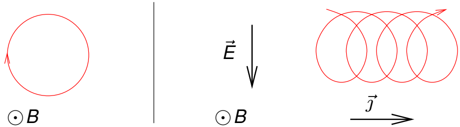
$$\begin{aligned} 2\pi\sigma_{\mathrm{P}} &= \mathrm{Ind}(P_\mu, UP_\mu U^*) && \text{(Bellissard)} \\ &= \mathrm{tr}(P_\mu - UP_\mu U^*)^3 && \text{(Avron et al.)} \end{aligned}$$

where $U = \mathrm{e}^{\mathrm{i}\arg\vec{x}} = z/|z|$.

Remarks. (i) Is a (stable) integer, whenever defined.
(ii) $P_\mu - UP_\mu U^* \notin \mathcal{J}_1$.

IQHE as a Bulk effect

Example: Cyclotron orbit drifting under a electric field $\vec{E}$

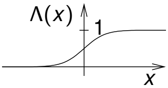


General: Hamiltonian H_B in the plane. Kubo formula (linear response to $\vec{E}$)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \big]$$

where

$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,
$\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches



IQHE as a Bulk effect (remarks)

Kubo formula (Bellissard et al., Avron et al.)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

extends the formula for the periodic case (Thouless et al., Avron)

$$\sigma_{\mathrm{B}} = -\frac{\mathrm{i}}{(2\pi)^2} \int_{\mathbb{T}} d^2k\, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$

Remarks.

$$2\pi\sigma_{\mathrm{B}} = \mathrm{ch}(E)$$

the Chern number of the vector bundle E over $\mathbb{T}$ and fiber range $P(k)$ (see later)

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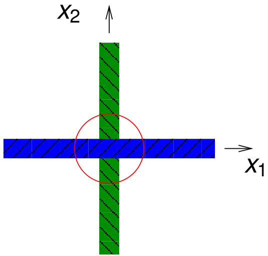
Alternative treatment of disorder (Thouless): Large, but finite system (square); $(k_1, k_2) \rightsquigarrow (\varphi_1, \varphi_2)$ phase slips in boundary conditions

IQHE as a Bulk effect (remarks)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr} P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches. Supports of $\vec{\nabla}\Lambda_i$:



Recall Kubo: $j_1 = -\sigma_{\mathrm{B}} E_2$

Remarks. (i) Λ_1, Λ_2: where from? Current operator across $x_1 = 0$: $\mathrm{i}[H_B, \Lambda_1]$; field $\vec{E} = -\vec{\nabla}\Lambda_2$

(ii) The trace is well-defined. Roughly: An operator has a well-defined trace if it acts non-trivially on finitely many states only. Here the intersection contains only finitely many sites.

Theorem: Quantization and equivalence

Definition. Ergodic operators H_ω, ($\omega \in \Omega$: probability space): actions of (magnetic) $\mathbb{Z}^2$-translations on Ω and on $\ell^2(\mathbb{Z}^2)$ compatible.

Theorem [Index=2π Kubo] (Bellissard, van Elst, Schulz-Baldes)
If μ lies in a Mobility Gap, then $\sigma_{\rm D}(\mu) = 0$ and $2\pi\sigma_{\rm P}(\mu) = 2\pi\sigma_{\rm B}(\mu)$ is an integer and constant.

Proof by non-commutative geometry.

Theorem and proof reformulated

Theorem [Index=2π Kubo] (Avron, Seiler, Simon)
If μ lies in a Mobility Gap, then $2\pi\sigma_{\mathrm{P}} = 2\pi\sigma_{\mathrm{B}}$, i.e.

$$\mathrm{tr}(P_\mu - UP_\mu U^*)^3 = 2\pi\mathrm{i}\ \mathrm{tr}\, P_\mu[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]$$

Remark. No ergodic setting.

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Explicitely,

$$2\mathrm{i} \sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)S(x,y,z) =$$

$$-2\pi\mathrm{i} \sum_{x,y,z\in\mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)[(\Lambda_1(y)-\Lambda_1(x))(\Lambda_2(z)-\Lambda_2(y)) - (1\leftrightarrow 2)]$$

where

$$\begin{aligned} S(x,y,z) &= -\frac{\mathrm{i}}{2}\left(1-\frac{U(x)}{U(y)}\right)\left(1-\frac{U(y)}{U(z)}\right)\left(1-\frac{U(z)}{U(x)}\right) \\ &= \sin\angle(x,0,y) + \sin\angle(y,0,z) + \sin\angle(z,0,x) \end{aligned}$$

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$$= \sin\angle(x,0,y) + \sin\angle(y,0,z) + \sin\angle(z,0,x)$$

Remark. Mobility gap: Substantial contribution only when x, y, z all near 0.

Sketch of proof

• Flux and cross are centered at the origin $p = 0$. Take instead $p \in \mathbb{R}^2$ arbitrary: neither side changes. For $w = x, y, z$ replace

$$\Lambda_i(w) \rightsquigarrow \Lambda_i(w - p), \qquad U(w) \rightsquigarrow U(w - p)$$

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$$L^{-2} \int_{p \in C_L} d^2 p$$

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(by mobility gap) for L large

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- $(p, y, x \in \mathbb{R})$

$$\int dp (\Lambda(y - p) - \Lambda(x - p)) = y - x$$

because $= f(y - x)$, $f(0) = 0$ and $f'(y - x) = \int dp \, \Lambda'(y - p) = 1$.

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- On r.h.s. use

$$\int dp_1 dp_2 (\Lambda(y_1-p_1)-\Lambda(x_1-p_1))(\Lambda(z_2-p_2)-\Lambda(y_2-p_2)) - (1\leftrightarrow 2)$$
$$= (y_1-x_1)(z_2-y_2) - (1\leftrightarrow 2) = 2\,\mathrm{Area}(x,y,z)$$

Sketch of proof (continued)

The claim

$$2\mathrm{i} \sum_{x,y,z \in \mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)S(x,y,z) =$$

$$-2\pi\mathrm{i} \sum_{x,y,z \in \mathbb{Z}^2} P_\mu(x,y)P_\mu(y,z)P_\mu(z,x)[(\Lambda_1(y)-\Lambda_1(x))(\Lambda_2(z)-\Lambda_2(y)) - (1 \leftrightarrow 2)]$$

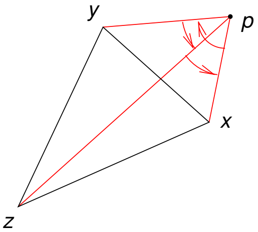
reduces by the above to

$$\int d^2p(\sin\angle(x,p,y) + \sin\angle(y,p,z) + \sin\angle(z,p,x)) = 2\pi \,\mathrm{Area}(x,y,z)$$

Sketch of proof (continued)

$$\int d^2p(\sin\angle(x,p,y) + \sin\angle(y,p,z) + \sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

(Connes' triangle formula)



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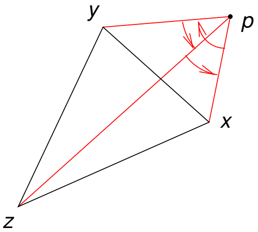

Proof: Observation (Colin de Verdière)

• Drop sin: obvious.

Sketch of proof (continued)

$$\int d^2p(\sin\angle(x,p,y)+\sin\angle(y,p,z)+\sin\angle(z,p,x)) = 2\pi\,\mathrm{Area}(x,y,z)$$

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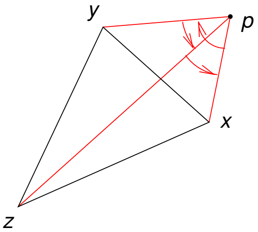


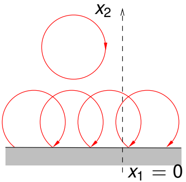
Proof: Observation (Colin de Verdière)

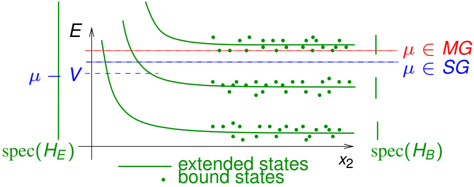
- Drop $\sin$: obvious.
- Let f be odd with $f(t) - t = O(t^3)$, $(t \to 0)$; e.g. $f = \sin$. Then

$$\int d^2p(f(\angle(x,p,y)) - \angle(x,p,y)) = 0$$

by (i) integrand $0(|p|^{-3})$, $(p \to \infty)$ and (ii) reflection symmetry.

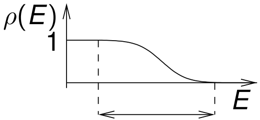
IQHE as an edge effect





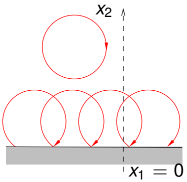
Hamiltonian H_E on the upper half-plane: restriction of H_B through boundary conditions at $x_2 = 0$.

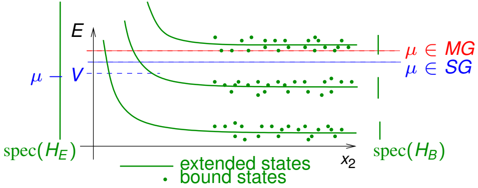
State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth



supp $\rho' \subset$ Spectral Gap for H_B (not for H_E)

IQHE as an edge effect





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State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth

Current operator across $x_1 = 0$: $\mathrm{i}[H_E, \Lambda_1]$

$$I = \mathrm{i}\ \mathrm{tr}(\rho(H_E + V) - \rho(H_E))[H_E, \Lambda_1]$$

As $V \to 0$: $I/V \to \sigma_{\mathrm{E}}$

$$\sigma_{\mathrm{E}} = \mathrm{i}\ \mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

Equality of conductances

Theorem (Schulz-Baldes, Kellendonk, Richter). Ergodic setting. If the Fermi energy μ lies in a Spectral Gap of H_B, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}.$$

In particular, σ_{E} does not depend on ρ', nor on boundary conditions.

What about the case of a Mobility Gap?

Is

$$\sigma_{\mathrm{E}} = -\mathrm{i}\,\mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

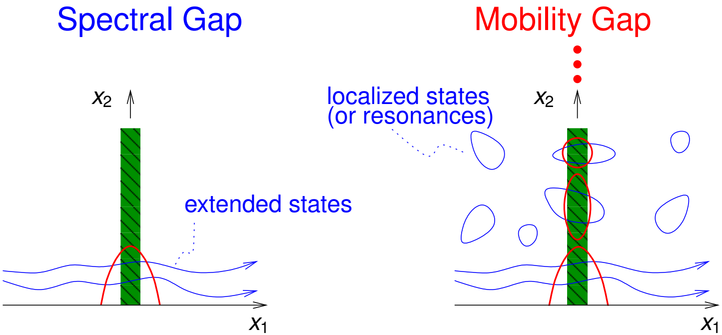
well-defined? (Here, switches Λ_i ($i = 1, 2$) with flipped orientations)

What about the case of a Mobility Gap?

Is

$$\sigma_{\mathrm{E}} = -\mathrm{i}\,\mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

well-defined?

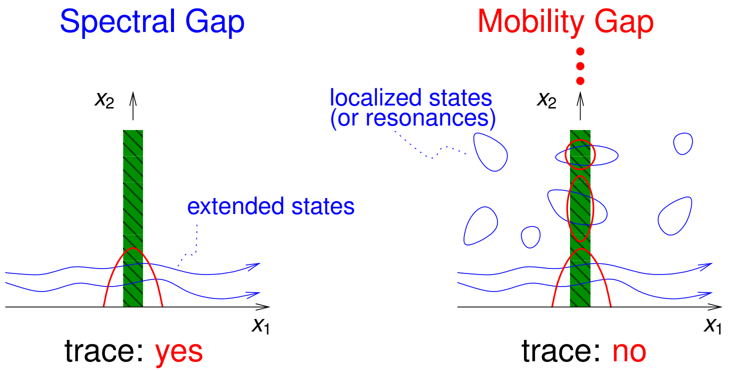


trace: yes

trace: no

$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

What about the case of a Mobility Gap?



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Guiding principle: Localized states should not contribute to the edge current

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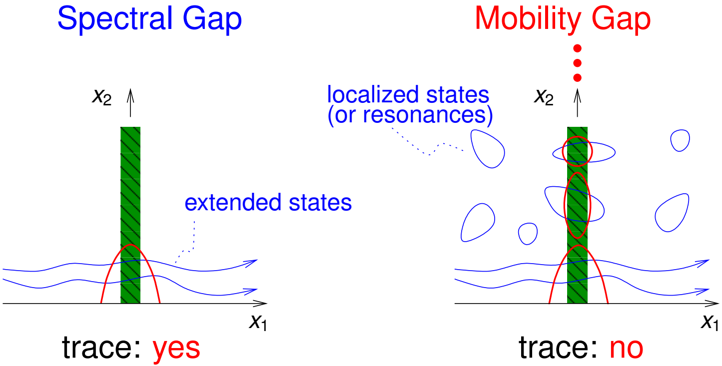


$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

Analogy: Electrodynamics of continuous media

$$\vec{\jmath} = \vec{\jmath}_F + \vec{\jmath}_M \equiv \text{free} + \text{molecular currents} \qquad \vec{\jmath}_M = \text{curl}\, \vec{M}$$

Localized states should not contribute to the (free) edge current

Equality of conductances

For a so amended definition of σ_{E}:

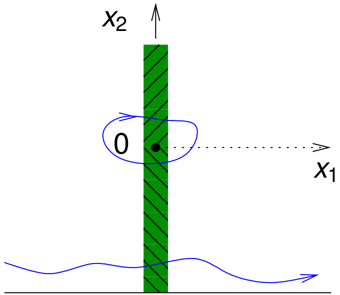
Theorem (Elgart, G., Schenker). If $\operatorname{supp} \rho'$ lies in a Mobility Gap, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$$

In particular σ_{E} does not depend on ρ', nor on boundary conditions.

Definition of σ_{E} in case of a Mobility Gap

Replace H_E to H_a ($a > 0$) as follows



edge: $x_2 = -a$
(eventually: $-a \to -\infty$)

- Current across the portion of $x_1 = 0$:

$$-\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) \quad \text{(exists!)}$$

- Current across the portion :

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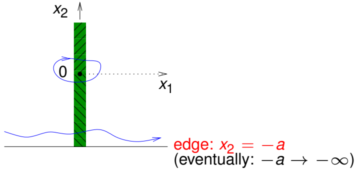

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- Current across the portion : In the limit $a \to \infty$ pretend that

$$\rho'(H_a) \rightsquigarrow \rho'(H_B) = \sum_\lambda \rho'(\lambda)\psi_\lambda(\psi_\lambda, \cdot)$$

(sum over eigenvalues λ of H_B: $H_B\psi_\lambda = \lambda\psi_\lambda$)

$$(\psi_\lambda, [H_B, \Lambda_1](1 - \Lambda_2)\psi_\lambda) = -(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

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(sum over eigenvalues λ of H_B: $H_B\psi_\lambda = \lambda\psi_\lambda$)

$$(\psi_\lambda, [H_B, \Lambda_1](1 - \Lambda_2)\psi_\lambda) = -(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

- Together:

$$\sigma_{\mathrm{E}} = \lim_{a\to\infty} -\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) + \\ + \mathrm{i}\sum_\lambda \rho'(\lambda)(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

Magnetization

Question? What is the term

$$\mathrm{i}(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda) \ ?$$

Magnetization

Question? What is the term

$$\mathrm{i}(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda) \text{ ?}$$

Or better after hermitization of $\mathrm{i}[H_B, \Lambda_1]\Lambda_2$, i.e.

$$\frac{\mathrm{i}}{2}([H_B, \Lambda_1]\Lambda_2 - \Lambda_2[\Lambda_1, H_B]) = \frac{\mathrm{i}}{2}[H_B, \Lambda_1\Lambda_2] - \frac{\mathrm{i}}{2}(\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)$$

where we get

$$-\frac{\mathrm{i}}{2}(\psi_\lambda, (\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)\psi_\lambda) \text{ ?}$$

Magnetization

Question? What is the term

$$-\frac{\mathrm{i}}{2}(\psi_\lambda, (\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)\psi_\lambda) \ ?$$

Answer: Replacement $x_i \rightsquigarrow \Lambda_i$, $(i = 1, 2)$ signifies extensive $\rightsquigarrow$ intensive. Thus

$$m = \frac{1}{2}\vec{x} \wedge \dot{\vec{x}} \rightsquigarrow M = \frac{1}{2}(\Lambda_1 \dot{\Lambda}_2 - \Lambda_2 \dot{\Lambda}_1)$$

signifies "magnetic moment $\rightsquigarrow$ magnetization". So, by $\dot{\Lambda}_i = \mathrm{i}[H_B, \Lambda_i]$,

$$M = \frac{\mathrm{i}}{2}(\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)$$

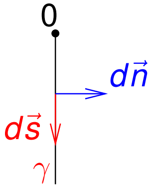
$\therefore$

$$-\frac{\mathrm{i}}{2}(\psi_\lambda, (\Lambda_1 H_B \Lambda_2 - \Lambda_2 H_B \Lambda_1)\psi_\lambda) = -(\psi_\lambda, M\psi_\lambda)$$

Magnetization (alternate)

Magnetization current: $\vec{\jmath}_M = \operatorname{curl} M = -\varepsilon\vec{\nabla} M$

- Classically: Magnetization is current across Dirac string γ ($d\vec{n} = \varepsilon d\vec{s}$)



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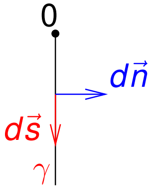

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Then hermitize

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Note: A state $\mathbb{T} \ni k \mapsto \psi_k \in \mathfrak{h}$ is a section of the (trivial) vector bundle $\mathbb{T} \times \mathfrak{h}$

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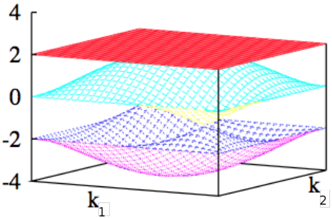

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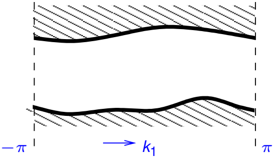

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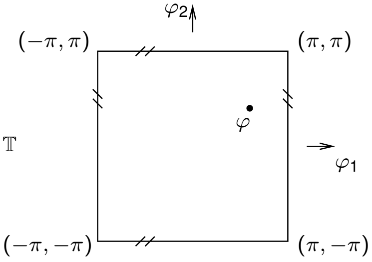
Note: It is a subbundle of $\mathbb{T} \times \mathfrak{h}$, possibly not trivial.

Bundles $(E, \mathbb{T})$ on the 2-torus



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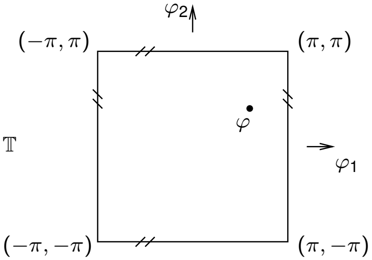

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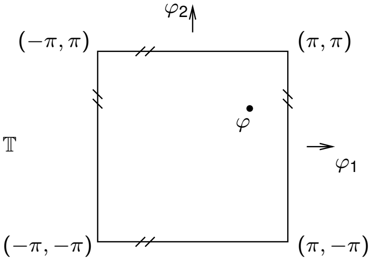

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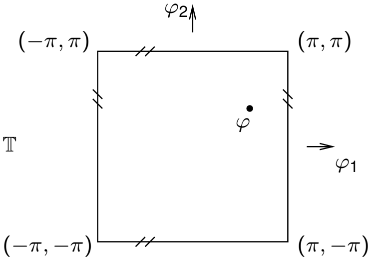

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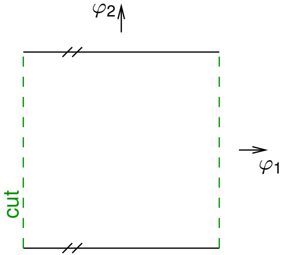
Classification by a Chern number



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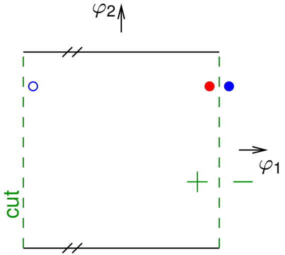

Lemma. On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$

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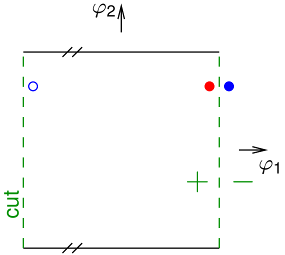

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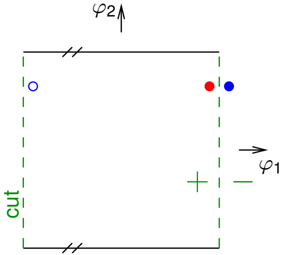

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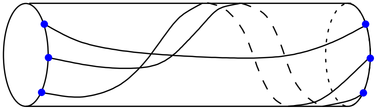

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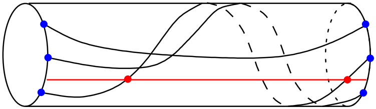

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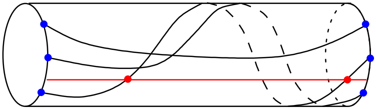

winding number=
signed number of crossings of fiducial line
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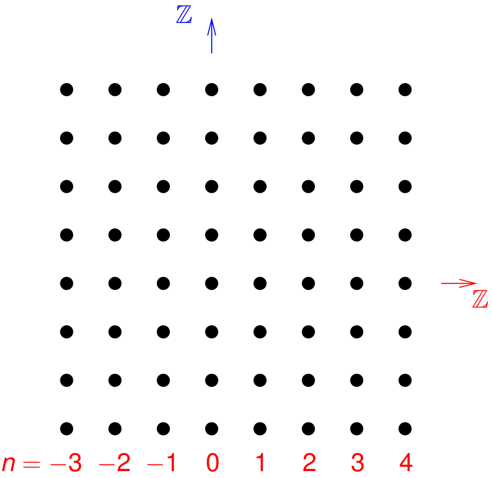
Remark.

$$\mathrm{ch}(E) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} d^2k\, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

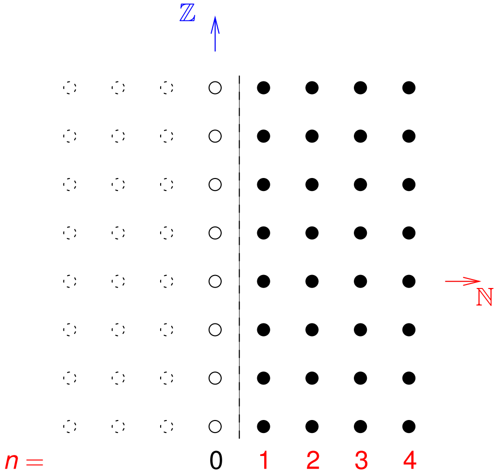
From plane (bulk) to half-plane (edge)

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



From plane (bulk) to half-plane (edge)

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



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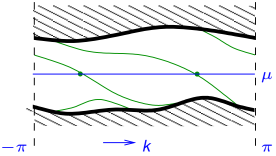
- $H^\sharp(k)$ acting on $L^2(\mathcal{C}^\sharp)$ has continuous and (possibly) discrete spectrum



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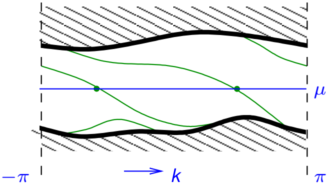

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$\mathcal{N}^\sharp$ = signed number of eigenvalue crossings of Fermi energy

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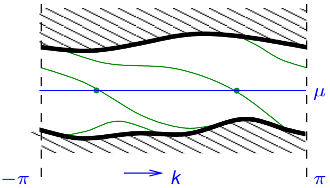

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(cf. Hatsugai)

Goal

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A result to be recalled: Levinson's theorem

Two-body Hamiltonian

$$H = p^2 + V$$

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Levinson's theorem. The scattering phase $S(E) = \mathrm{e}^{2\mathrm{i}\delta(E)}$ at threshold $E = 0$ equals the number N of bound states,

$$\arg S\big|_{E=0+} = 2\pi N$$

(normalization: $\delta(E) \to 0$, ($E \to +\infty$))

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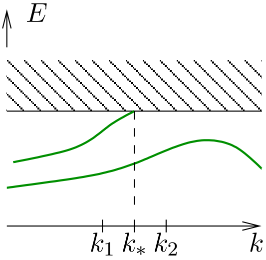
Idea: Scattering states and bound states are related by analytic continuation ... just as bulk and edge states are (Hatsugai, 1993).

Levinson's theorem (relative version)

Suppose $H(k)$ depends on some parameter $k \in \mathbb{R}$



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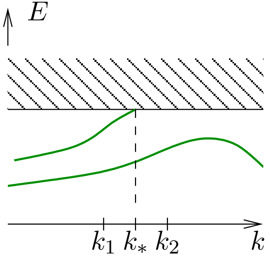

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$$\lim_{E \to 0} \arg S(k, E)\Big|_{k_1}^{k_2} = \mp 2\pi$$

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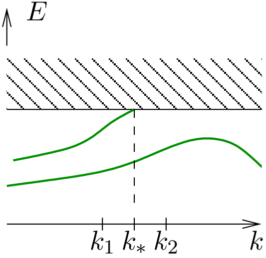

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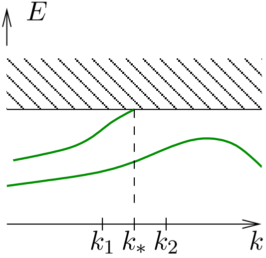

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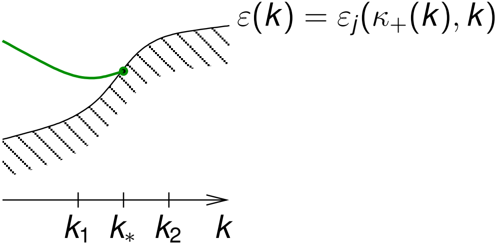
- Normalization of phase forgone

Proof of Levinson's theorem (sketch)



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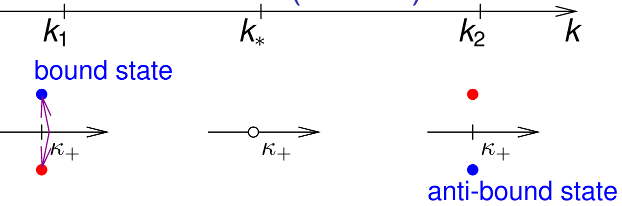
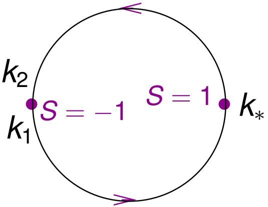


$S(\kappa) \equiv S(\varepsilon(\kappa))$
in κ plane
• pole
• zero

$S(\kappa_+ + \delta)$
in S plane
as a function of k
($\delta > 0$ small)



$$|\mathrm{out}\rangle = S|\mathrm{in}\rangle$$

Bound state: $|\mathrm{out}\rangle$ in absence of $|\mathrm{in}\rangle \equiv |\kappa\rangle$ with $\mathrm{Im}\,\kappa < 0$
Thus: Pole of $S(\kappa)$.

$$S(\kappa) \approx \frac{\kappa - \overline{\kappa_0}}{\kappa - \kappa_0}, \qquad (\kappa \to \kappa_+)$$

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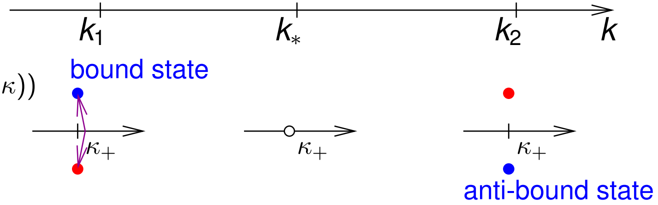

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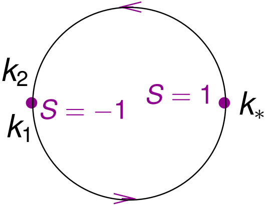

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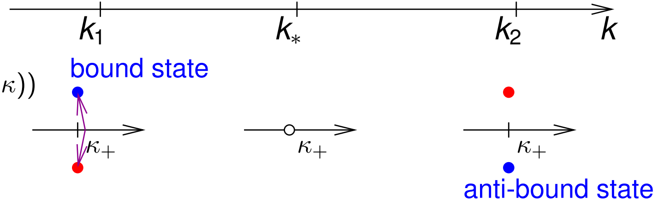

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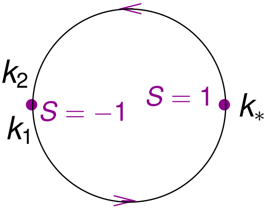


$$\lim_{\delta\to 0} \arg S_+\big(\underbrace{\kappa_+(k) + \delta}_{\sim\,\varepsilon(k) - \delta'}\big)\Big|_{k_1}^{k_2} = 2\pi$$

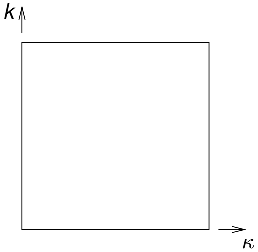
The Great Wall of China and its Towers

The Great Wall of China and its Towers

bulk (top view)

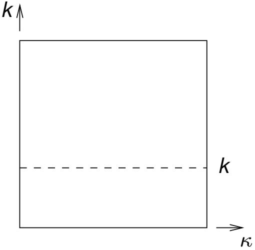
edge (side view)

The Bloch landscape

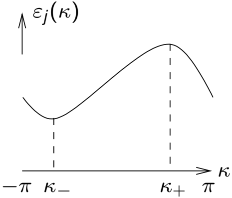


Brillouin zone $\ni (\kappa, k)$
Energy band $\varepsilon_j(\kappa, k)$

The Bloch landscape



Minimum $\kappa_-(k)$ and maximum $\kappa_+(k)$ of energy band $\varepsilon_j(\kappa, k)$ in κ at fixed k



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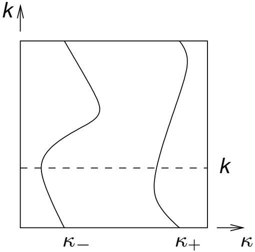

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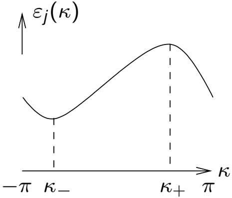

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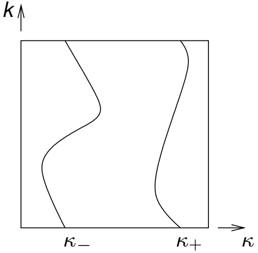

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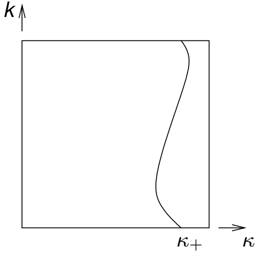

Maxima $\kappa_+(k)$

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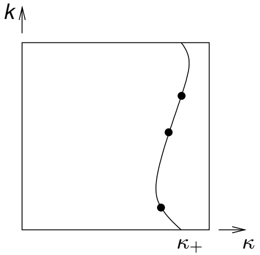
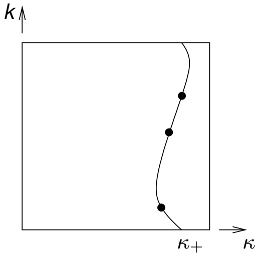
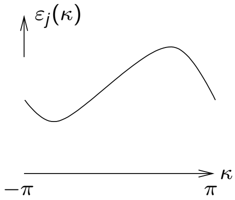


Maxima $\kappa_+(k)$ with semi-bound states (to be explained)

The Bloch landscape

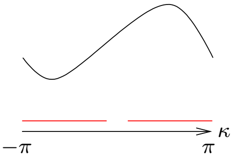


Duality via scattering



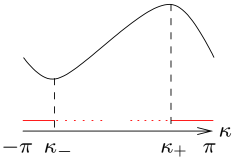
At fixed k: Energy band $\varepsilon_j(\kappa, k)$ and the line bundle P_j of Bloch states

Duality via scattering



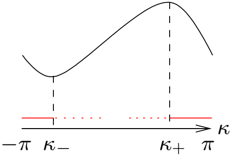
Line indicates choice of a section $|\kappa\rangle$ of Bloch states (from the given band). No global section in $\kappa \in \mathbb{R}/2\pi\mathbb{Z}$ is possible, as a rule.

Duality via scattering

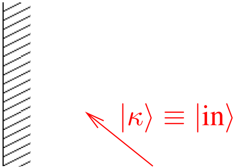


States $|\kappa\rangle$ above the solid line are left movers ($\varepsilon_j'(\kappa) < 0$)

Duality via scattering



They are incoming asymptotic (bulk) states for scattering at edge (from inside)



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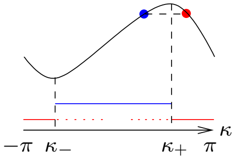
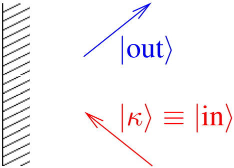


Scattering determines section $|\mathrm{out}\rangle$ of right movers above line



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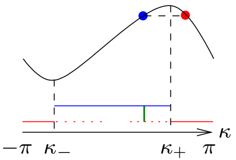
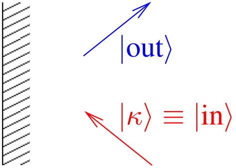


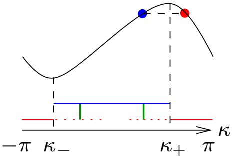


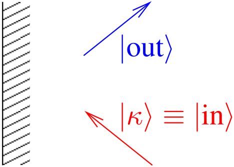
Scattering matrix

$$|\mathrm{out}\rangle = S_+|\kappa\rangle$$

as relative phase between two sections of the same fiber (near κ_+)

Duality via scattering





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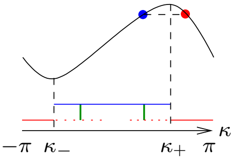
Likewise S_- near κ_-.

Duality via scattering

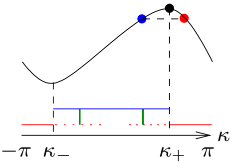


Chern number computed by sewing

$$\mathrm{ch}(P_j) = \mathcal{N}(S_+) - \mathcal{N}(S_-)$$

with $\mathcal{N}(S_\pm)$ the winding of $S_\pm = S_\pm(k)$ as $k = -\pi \dots \pi$.

Duality via scattering



As $\kappa \to \kappa_+$, whence

$$|\mathrm{in}\rangle = |\kappa\rangle \to |\kappa_+\rangle \qquad |\mathrm{out}\rangle = S_+|\kappa\rangle \to |\kappa_+\rangle \text{ (up to phase)}$$

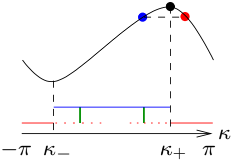
their limiting span is that of

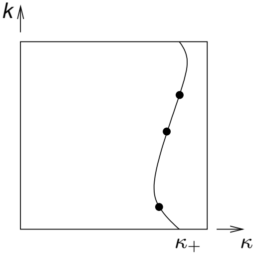
$$|\kappa_+\rangle, \quad \left.\frac{d|\kappa\rangle}{d\kappa}\right|_{\kappa_+}$$

(bounded, resp. unbounded in space). The span contains the limiting scattering state $|\psi\rangle \propto |\mathrm{in}\rangle + |\mathrm{out}\rangle$.

If (exceptionally) $|\psi\rangle \propto |\kappa_+\rangle$ then $|\psi\rangle$ is a semi-bound state.

Duality via scattering





As a function of k, semi-bound states occur exceptionally.

Aside: Generalized Bloch solutions

$$\left.\frac{d|\kappa\rangle}{d\kappa}\right|_{\kappa_+}$$

is an eigensolution unbounded in space $\mathbb{Z} \ni n$.

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is an eigensolution unbounded in space $\mathbb{Z} \ni n$. In fact, let $\psi(\kappa, n) = \langle n|\kappa\rangle$ be a Bloch solution (p period):

$$(H - \varepsilon(\kappa))\psi(\kappa, n) = 0, \quad \psi(\kappa, n + mp) = \mathrm{e}^{\mathrm{i}\kappa m}\psi(\kappa, n)$$

Then

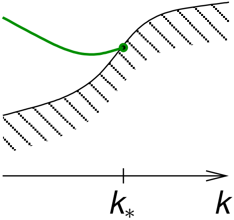
$$(H - \varepsilon(\kappa_+))\frac{d\psi}{d\kappa}(\kappa_+, n) = 0, \quad \left(\frac{d\varepsilon(\kappa)}{d\kappa}\Big|_{\kappa_+} = 0\right)$$

$$\frac{d}{d\kappa}\psi(\kappa, n + mp) = \mathrm{e}^{\mathrm{i}\kappa m}\frac{d}{d\kappa}\psi(\kappa, n) + \mathrm{i}m\mathrm{e}^{\mathrm{i}\kappa m}\psi(\kappa, n)$$

(unbounded in m)

Levinson's theorem (relative version)

Spectrum of edge Hamiltonian



$$\varepsilon(k) = \varepsilon_j(\kappa_+(k), k)$$

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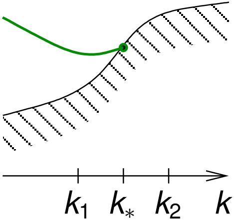
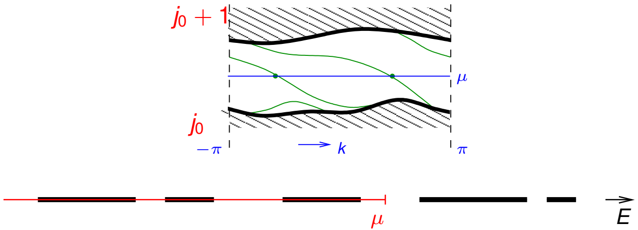


$$\varepsilon(k) = \varepsilon_j(\kappa_+(k), k)$$

$$\lim_{\delta \to 0} \arg S_+(\varepsilon(k) - \delta)\Big|_{k_1}^{k_2} = \pm 2\pi$$

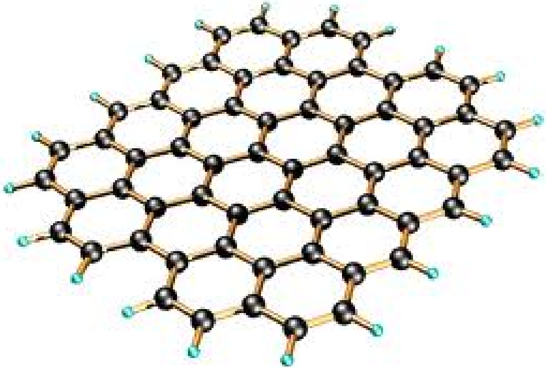
Proof of duality



$$\mathcal{N}^\sharp = \mathcal{N}(S_+^{(j_0)}) \quad (= \mathcal{N}(S_-^{(j_0+1)}))$$
$$= \sum_{j=0}^{j_0} \mathcal{N}(S_+^{(j)}) - \mathcal{N}(S_-^{(j)})$$
$$= \sum_{j=0}^{j_0} \mathrm{ch}(P_j)$$

$(\mathcal{N}(S_-^{(0)}) = 0)$

An application: Quantum Hall in graphene

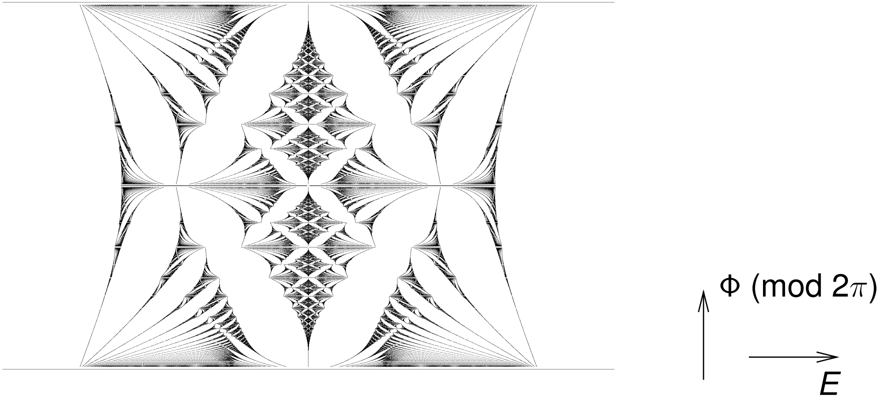


An application: Quantum Hall in graphene

Hamiltonian: Nearest neighbor hopping with flux Φ per plaquette.

An application: Quantum Hall in graphene

Spectrum in black



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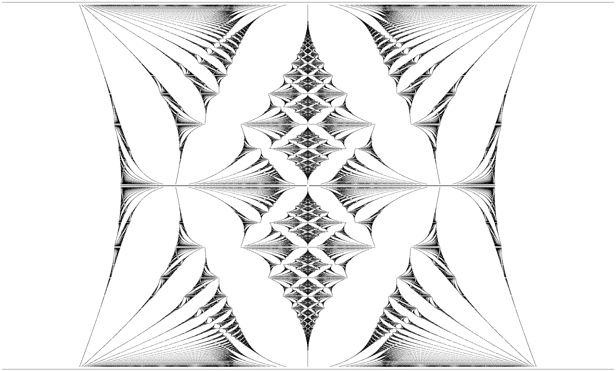

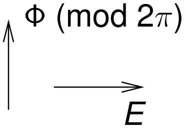

What is the Hall conductance (Chern number) in any white point?

An application: Quantum Hall in graphene

What is the Hall conductance (Chern number) s in any white point?

Bulk approach (Thouless/Avron et al.): If $\Phi = p/q$, (p, q coprime) then

$$r = sp + tq$$

where:

- r number of bands below Fermi energy
- s, t integers

s is so determined only modulo q.

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$\rightarrow$ Edge approach, method by Schulz-Baldes et al.

The colors of graphene

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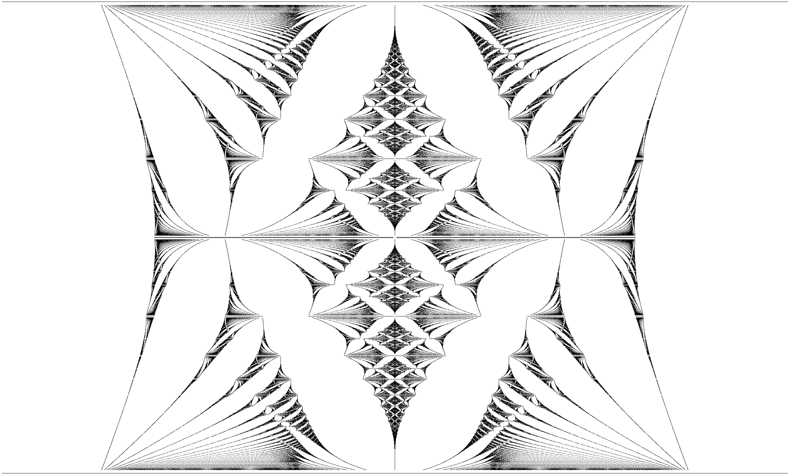

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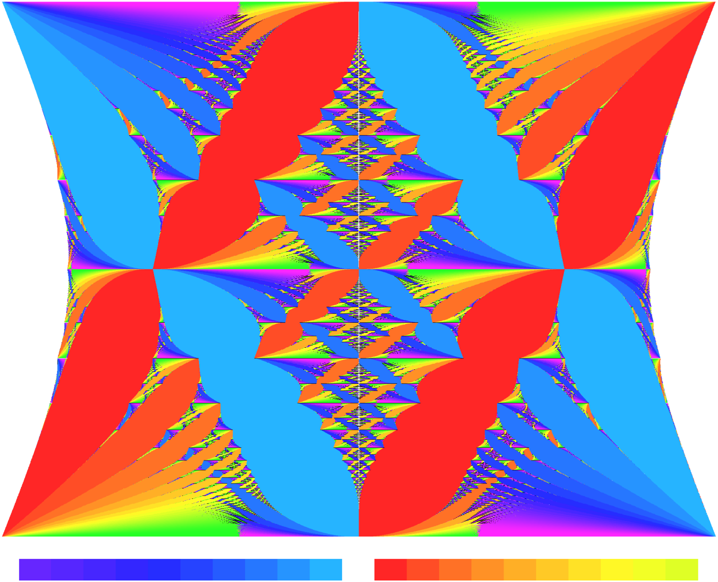
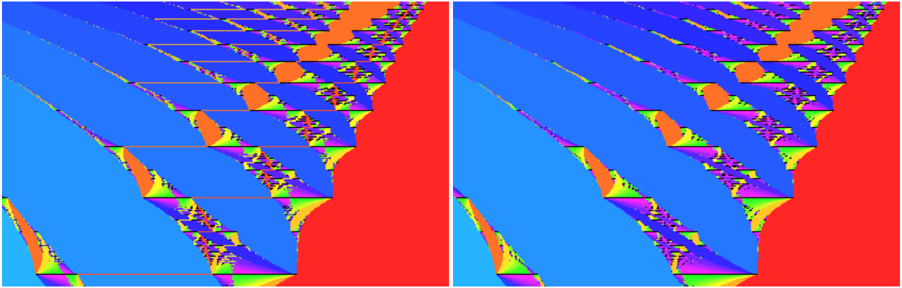


Naive Bulk vs. Edge computation



Left: “Natural” window condition $s \in (-q/2, q/2)$
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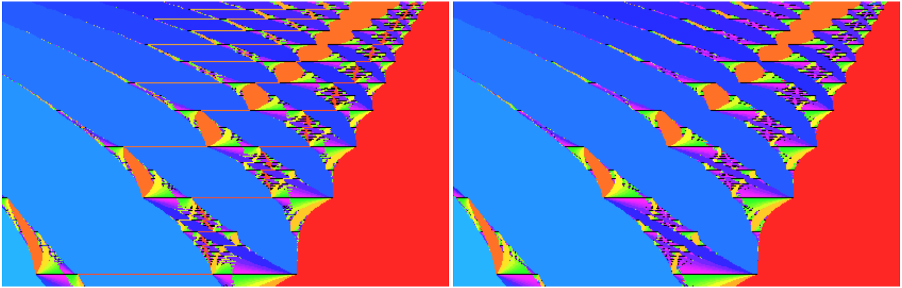

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cf. Avron et al.

Topological insulators: time-reversal invariant case

- Insulator in the Bulk: Excitation gap
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Time-reversal invariance explained

There is a map Θ on $\mathcal{H}$ (time-reversal) such that

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Such insulators are trivial from the Quantum Hall point of view. Yet interesting in their own class.

Classification of time-reversal invariant bundles

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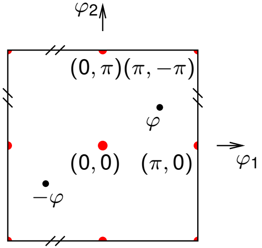
- Kane, Mele; Fu, Kane: Index realized as Pfaffian

Time reversal invariant bundles $(E, \mathbb{T}, \Theta)$

$\mathbb{T}$



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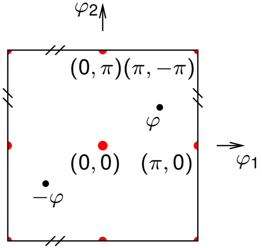

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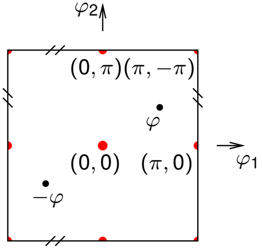

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- Set

$$\mathcal{I}(E) := \prod_{a \in TRIP} \frac{\mathrm{pf}\ W(\varphi_a)}{\sqrt{\det W(\varphi_a)}} = \pm 1$$

(Pfaffian defined for antisymmetric matrices, $\det W = (\mathrm{pf}\ W)^2$)

The Fu-Kane index restated

- Family of matrices $W(\varphi_2)$ with single parameter $0 \leq \varphi_2 \leq \pi$, $\det W(\varphi_2) \neq 0$, antisymmetric at endpoints $\varphi_2 = 0, \pi$

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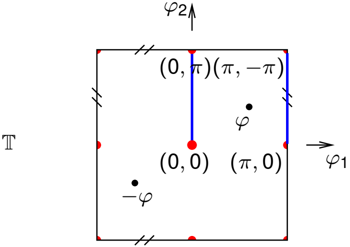

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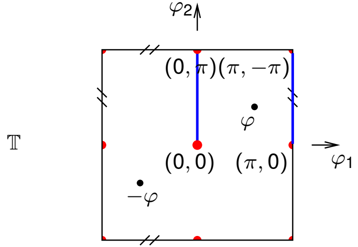

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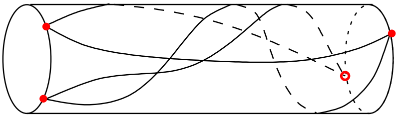
Then

$$\widehat{\mathcal{I}}(E) = \widehat{\mathcal{I}}(W_0)\widehat{\mathcal{I}}(W_\pi)$$

The index of a rueda

Consider a fixed even number of lines moving forward along a (finite) cylinder.

Condition: Lines pair up at the ends



D

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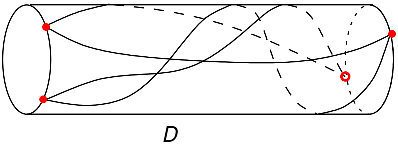

$D = (D(t))_{a \leq t \leq b}$ with $D(t)$ a collection of points on the circle.

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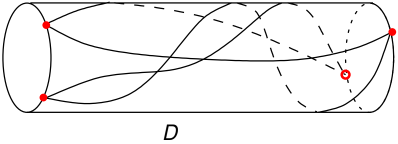

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(Lines can be thought of as world lines of dancers of a rueda)

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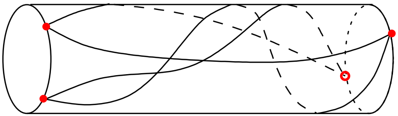

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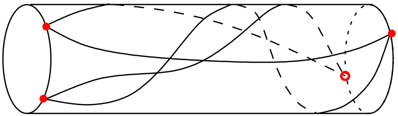

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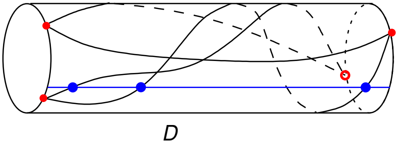

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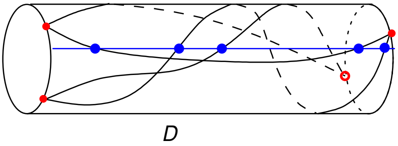

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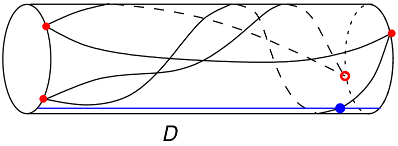

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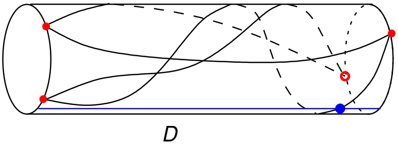

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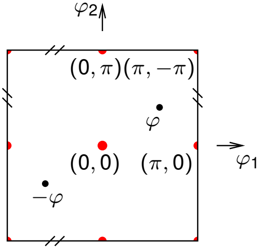
$$\mathcal{I}(D) = \text{parity of number of crossings of fiducial line}$$

Time reversal invariant bundles $(E, \mathbb{T}, \Theta)$

$\mathbb{T}$



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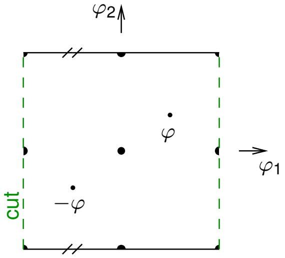
Towards another index

Consider the cut torus:



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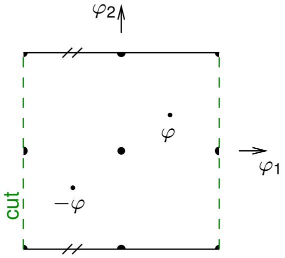

Lemma On the cut torus the frame bundle admits a section $\varphi \mapsto v(\varphi) \in F(E)_\varphi$ which is time-reversal invariant:

$$v(-\varphi) = (\Theta v(\varphi))\varepsilon$$

with ε the block diagonal matrix with blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Idea: At a time reversal invariant point, that means $(N = 2)$

$$v_2 = \Theta v_1 \qquad v_1 = -\Theta v_2$$

Towards another index (cont.)

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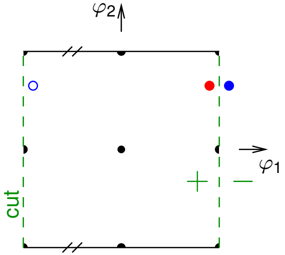

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$$v_+(\varphi_2) = v_-(\varphi_2)T(\varphi_2)\,, \qquad (\varphi_2 \in S^1)$$

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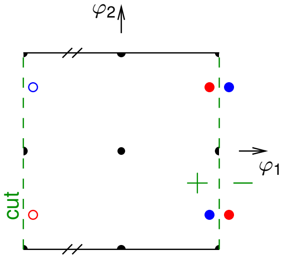

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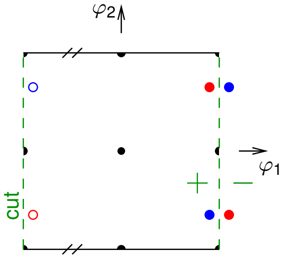

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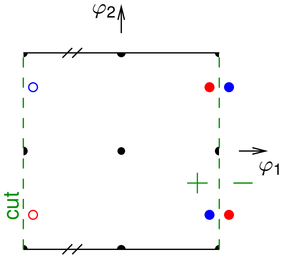

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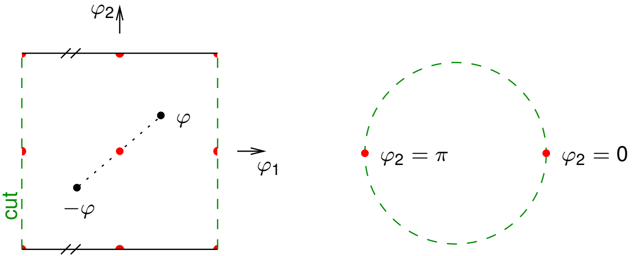
with $\Theta_0 = \varepsilon C$, (C complex conjugation on $\mathbb{C}^N$)

Towards another index (cont.)

We have

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)



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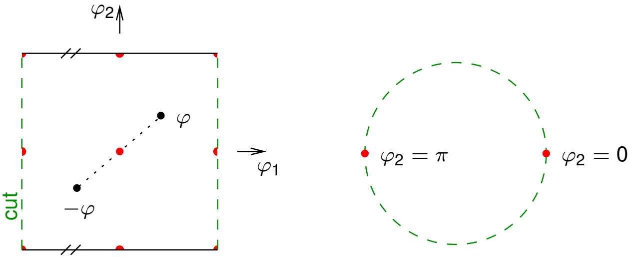

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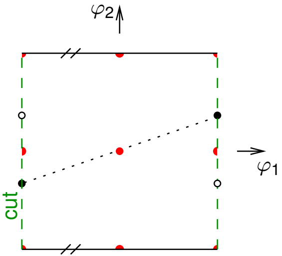

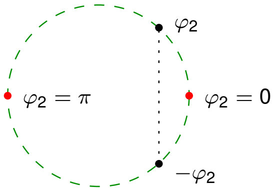

- a (compatible) section of the frame bundle of E
- the transition matrices $T(\varphi_2) \in \mathrm{GL}(N)$ across the cut

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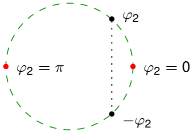
with $\Theta_0 : \mathbb{C}^N \to \mathbb{C}^N$ antilinear, $\Theta_0^2 = -1$

Time-reversal invariant bundles on the torus



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut ($0 \leq \varphi_2 \leq \pi$) matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

$$\Theta_0 T = T^{-1}\Theta_0$$

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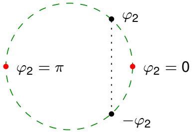

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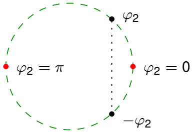
Time-reversal invariant bundles on the torus



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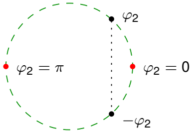

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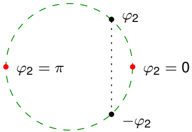

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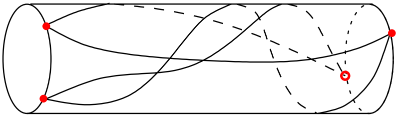

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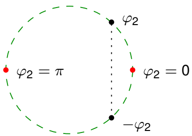

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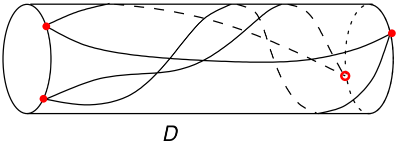

Definition (Index): $\mathcal{I}(E) := \mathcal{I}(T) := \mathcal{I}(D)$

Why are the $\mathbb{Z}_2$ indices equal?

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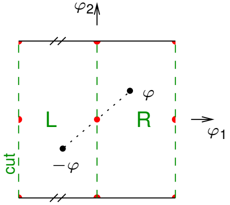
$$v(\varphi) = \begin{cases} u(\varphi)\ , & (\varphi \in L) \\ \Theta u(-\varphi)\varepsilon\ , & (\varphi \in R) \end{cases}$$



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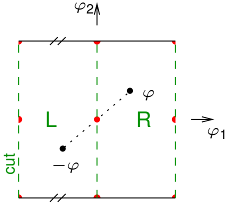

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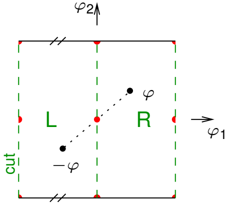

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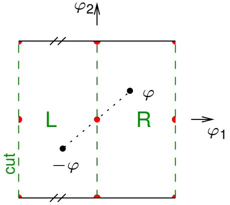

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- $W(\varphi_2) = T(\varphi_2)\varepsilon$. Then $\widehat{\mathcal{I}}(W) = \mathcal{I}(T)$ and hence

$$\widehat{\mathcal{I}}(E) = \mathcal{I}(E)$$

Rueda de casino. Time 0′15″

Rueda de casino. Time 0′25″

Rueda de casino. Time $0'35''$

Rueda de casino. Time $0'44''$

Rueda de casino. Time $0'44.25''$

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Rueda de casino. Time 0′45.25″

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Rueda de casino. Time 0′46″

Rueda de casino. Time $0'47''$

Rueda de casino. Time $0'55''$

Rueda de casino. Time 1′16″

Rueda de casino. Time 3′23″

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor

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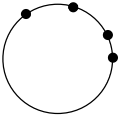
There are dances which can not be deformed into one another.

What is the index that tells the difference?

The index of a Rueda

A snapshot of the dance



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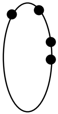
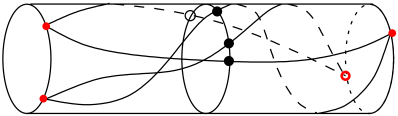


Dance D as a whole



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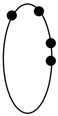

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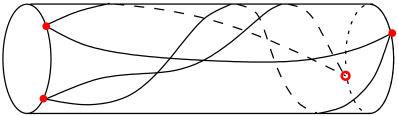

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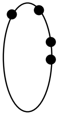

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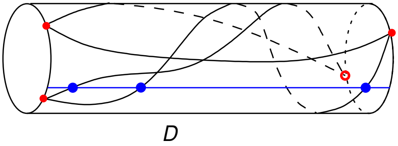

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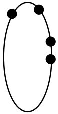

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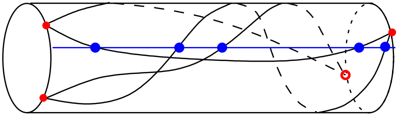

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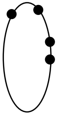

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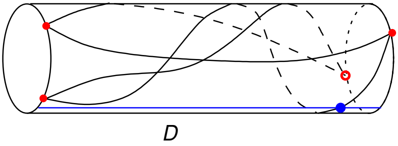

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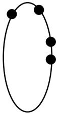

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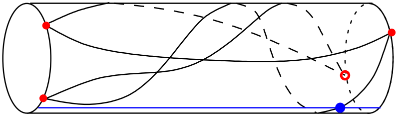

D

$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

The $\mathbb{Z}_2$ index in the non-periodic case

Recall: Index without time-reversal symmetry based on index of pair of projections

$$\begin{aligned}
&\mathrm{Ind}(P, Q) = \\
&\dim\{\psi \in \mathcal{H} \mid P\psi = \psi, Q\psi = 0\} - \dim\{\psi \in \mathcal{H} \mid Q\psi = \psi, P\psi = 0\} \\
&\qquad = \dim\ker(A-1) - \dim\ker(A+1), \qquad A = P - Q
\end{aligned}$$

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With time-reversal symmetry:

$$\mathcal{I} = (-1)^{\dim\ker(A-1)}$$

(cf. Atiyah; Schulz-Baldes; Katsura, Koma)

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In both cases, apply to $P = P_\mu$, $Q = UP_\mu U^*$.

An experiment: Amo et al.

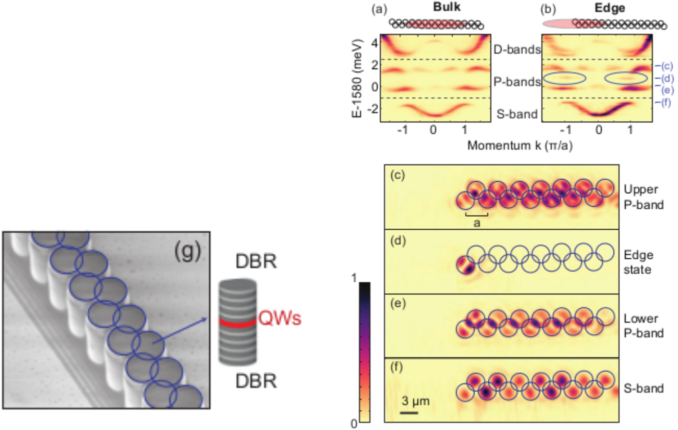

Figure: Zigzag chain of coupled micropillars and lasing modes

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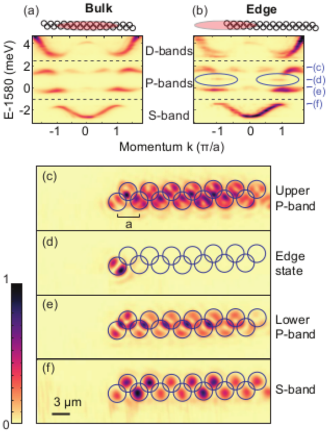


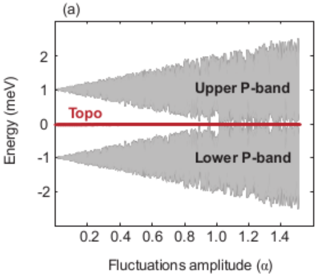
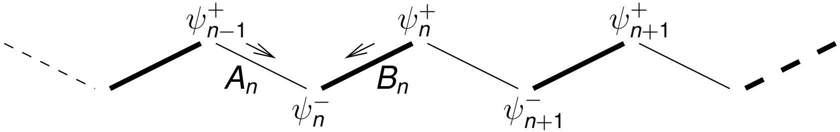


Figure: Lasing modes: bulk and edge

The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping



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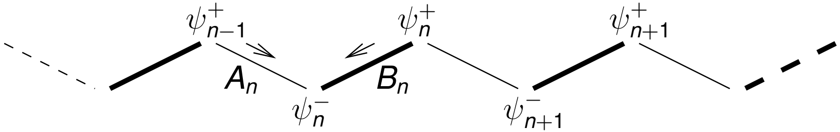

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n\in\mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n\psi^+_{n-1} + B_n\psi^+_n\,, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$H\psi = \lambda\psi \quad \Longrightarrow \quad H(\Pi\psi) = -\lambda(\Pi\psi)$$

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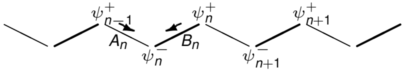


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

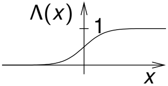
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

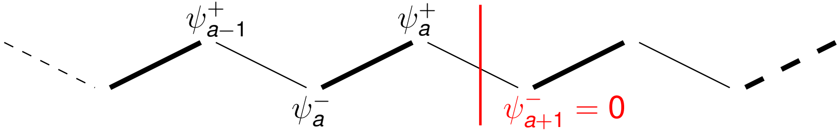
Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2}\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma])$$



with $\Lambda = \Lambda(n)$ a switch function (cf. Prodan et al.)

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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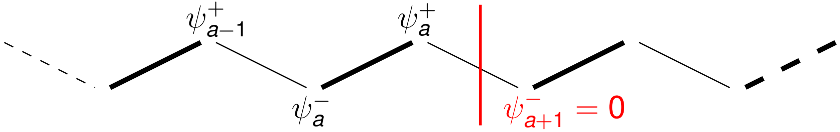

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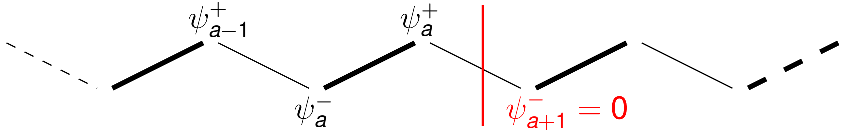

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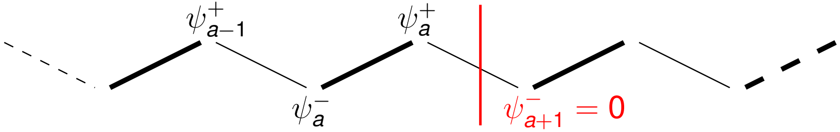

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$$\mathcal{N}^\pm_a := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

Definition. The Edge index is the spectral asymmetry

$$\mathcal{N}^\sharp_a := \mathcal{N}^+_a - \mathcal{N}^-_a$$

and can be shown to be independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\sharp$$

Bulk-edge duality: Remarks

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Remarks.

- Spectral gap case ($0 \notin \sigma_{\mathrm{ess}}(H) \supset \sigma_{\mathrm{ess}}(H_a)$)

$$H_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{N}_a^\sharp := \dim\ker S_a - \dim\ker S_a^* = \operatorname{ind} S_a \qquad \text{(Fredholm index)}$$

Bulk-edge duality by Schulz-Baldes. In mobility gap case, S_a is not Fredholm.

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  Bulk-edge duality by Schulz-Baldes. In mobility gap case, S_a is not Fredholm.
- Supersymmetry: Is realized as $(H_a, \Pi) = (\text{supercharge}, \text{grading})$. Then $\mathcal{N}_a^\sharp$ is Witten index.

Bulk-edge duality: Remarks

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\sharp$$

Remarks.

- Spectral gap case ($0 \notin \sigma_{\mathrm{ess}}(H) \supset \sigma_{\mathrm{ess}}(H_a)$)

$$H_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{N}_a^\sharp := \dim\ker S_a - \dim\ker S_a^* = \mathrm{ind}\, S_a \qquad \text{(Fredholm index)}$$

  Bulk-edge duality by Schulz-Baldes. In mobility gap case, S_a is not Fredholm.
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- Periodic case

$$S = \int_{S^1}^{\oplus} S(k)$$

  Toeplitz index theorem:

$$\mathcal{N}^\sharp = -\mathrm{Wind}(k \mapsto \det S(k))$$

Bulk-edge duality: Lyapunov exponents

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Remark. Consider the dynamical system $A_n \psi_{n-1}^+ + B_n \psi_n^+ = 0$ with Lyaponov exponents

$$\gamma_1 \geq \cdots \geq \gamma_N$$

The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\sharp = \sharp\{i \mid \gamma_i > 0\}$.

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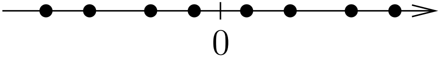
Lyapunov spectrum of the full chain has $2N$ exponents, spectrum is even (Example: $N = 4$)

- at energy $\lambda \neq 0$ (simple spectrum)



  - Spectrum is simple because measure on transfer matrices is irreducible
  - so $\gamma = 0$ is not in the spectrum; localization follows

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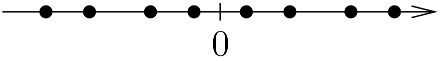

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces

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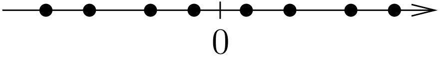
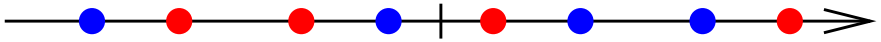
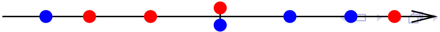


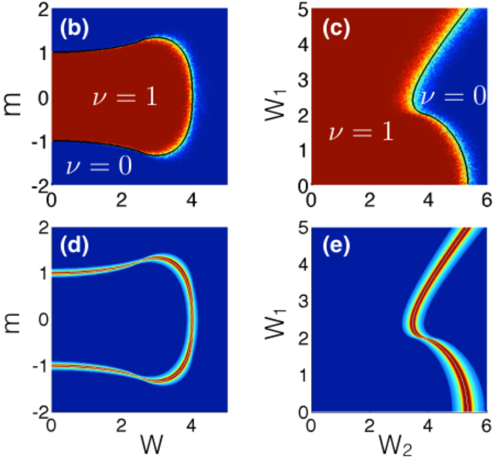
- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$



- at energy $\lambda = 0$ (phase boundary)



Some numerics



Left/right column: two parameterized chiral models ($N = 1$)
upper/lower row: index and Lyapunov exponent (from Prodan et al.)

Proof

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi P_{0,a})$

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi \Lambda) = N\Big(\sum_{n \leq a} \Lambda(n)\Big) \operatorname{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi\Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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$$\begin{aligned}\mathrm{tr}(\Pi\Lambda P_{+,a}) &= \mathrm{tr}(P_{+,a}\Pi\Lambda P_{+,a}) = \mathrm{tr}(\Pi P_{-,a}\Lambda P_{+,a}) \\ &= \mathrm{tr}(\Pi P_{-,a}[\Lambda, P_{+,a}])\end{aligned}$$

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$$\operatorname{tr}(\Pi\Lambda) = 0$$

So,

$$\operatorname{tr}(\Pi\Lambda) = \underbrace{\operatorname{tr}(\Pi\Lambda P_{0,a})}_{\to\mathcal{N}^\sharp} + \underbrace{\operatorname{tr}(\Pi\Lambda P_{+,a}) + \operatorname{tr}(\Pi\Lambda P_{-,a})}_{\to\operatorname{tr}(\Pi P_-[\Lambda,P_+])+\operatorname{tr}(\Pi P_+[\Lambda,P_-])=-\mathcal{N}}$$

q.e.d.

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$H = H(t)$ (bulk) Hamiltonian in the plane with period T

$$H(t + T) = H(t)$$

(disorder allowed, no adiabatic setting)

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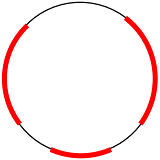
Assumption: Spectrum of $\widehat{U}$ has gaps:



spec $\widehat{U} \subset S^1$

Bulk index

Special case first: $U(t)$ periodic, i.e.

$$\widehat{U} = 1$$

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$$\mathcal{N}_{\mathrm{B}} = \frac{1}{2} \int_0^T dt \, \mathrm{tr}\big(U^* \partial_t U \big[U^*[\Lambda_1, U], U^*[\Lambda_2, U]\big]\big)$$

with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$

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Remark. Extends the formula for the periodic case (Rudner et al.)

$$\mathcal{N}_{\mathrm{B}} = \frac{1}{8\pi^2} \int_0^T dt \int_{\mathbb{T}} d^2k \, \mathrm{tr}(U^* \partial_t U [U^* \partial_1 U, U^* \partial_2 U])$$

with $U = U(t, k)$ acting on the space of states of quasi-momentum $k = (k_1, k_2)$. Map U: 3-torus $\to$ unitary group $\mathcal{U}$; $\pi_3(\mathcal{U}) = \mathbb{Z}$.

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Bulk index $\mathcal{N}_{\mathrm{B}}$ is degree of map.

Edge index

$H_{\mathrm{E}}(t)$ restriction of $H(t)$ to right half-space $x_1 > 0$

$\widehat{U}_{\mathrm{E}}$ corresponding fundamental propagator

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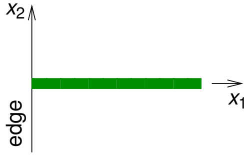
$$\mathcal{N}_{\mathrm{E}} = \mathrm{tr}(\widehat{U}_{\mathrm{E}}^*[\Lambda_2, \widehat{U}_{\mathrm{E}}]) = \mathrm{tr}(\widehat{U}_{\mathrm{E}}^* \Lambda_2 \widehat{U}_{\mathrm{E}} - \Lambda_2)$$

Remarks.

- The trace is well-defined



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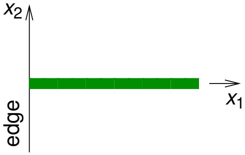

- $\mathcal{N}_{\mathrm{E}}$ is charge that crossed the line $x_2 = 0$ during a period.
- $\mathcal{N}_{\mathrm{E}}$ is independent of Λ_2 and an integer.

General case: Pair of Hamiltonians

$$\widehat{U} \neq 1$$

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$$H(t) = \begin{cases} H_1(t) & (0 < t < T) \\ -H_2(-t) & (-T < t < 0) \end{cases}$$

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Theorem (G., Tauber) $\mathcal{N} = \mathcal{N}_{\mathrm{E}}$

Duality in time and space

Let the interface Hamiltonian $H_{\mathrm{I}}(t)$ be a bulk Hamiltonian with

$$H_{\mathrm{I}}(t) = \begin{cases} H_1(t) \\ H_2(t) \end{cases} \quad \text{on states supported on large } \pm x_1$$

(still assuming $\widehat{U}_1 = \widehat{U}_2 =: \widehat{U}_\bullet$)

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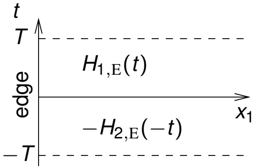
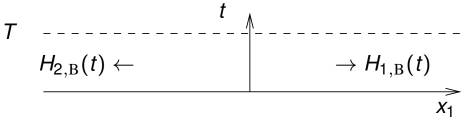
Interface index

$$\mathcal{N}_{\mathrm{I}} = \mathrm{tr}(\widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}[\Lambda_2, \widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}])$$





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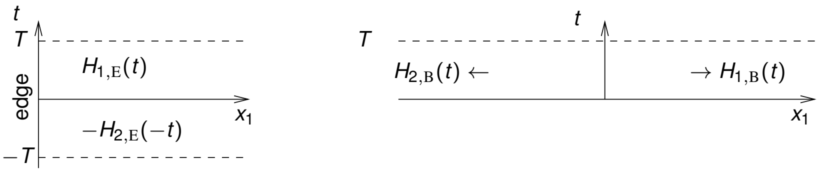
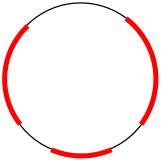


Theorem (G., Tauber) The indices for the two diagrams agree:

$$(\mathcal{N} =)\mathcal{N}_{\mathrm{E}} = \mathcal{N}_{\mathrm{I}}$$

Back to single Hamiltonian

$\widehat{U} \neq 1$



$\text{spec}\, \widehat{U} \subset S^1$

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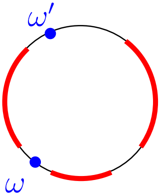

Let $\alpha \in \mathbb{R}$ and $\omega = \mathrm{e}^{\mathrm{i}\alpha}$. For $z \notin \omega\mathbb{R}_+$ (ray) define the branch

$$\log_\alpha z = \log|z| + \mathrm{i}\arg_\alpha z$$

by $\alpha - 2\pi < \arg_\alpha z < \alpha$.

Back to single Hamiltonian

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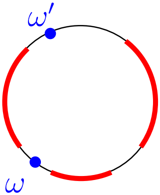

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Comparison Hamiltonian H_α: For $\omega \notin \mathrm{spec}\,\widehat{U}$ set

$$-\mathrm{i}H_\alpha T := \log_\alpha \widehat{U}$$

So,

- $\widehat{U}_\alpha = \widehat{U}$
- $U_{\alpha+2\pi}(t) = U_\alpha(t)\mathrm{e}^{2\pi\mathrm{i}t/T}$
- $\mathcal{N}_{\mathrm{B},\alpha+2\pi} = \mathcal{N}_{\mathrm{B},\alpha} =: \mathcal{N}_\omega$

Back to single Hamiltonian

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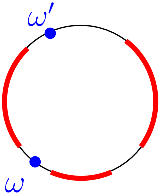

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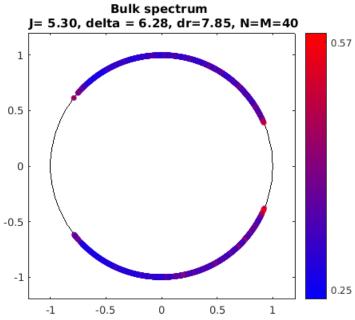
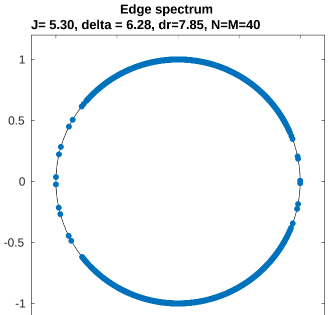
Theorem (Rudner et al.; G., Tauber) For ω, ω' in gaps

$$\mathcal{N}_{\omega'} - \mathcal{N}_\omega = \mathrm{i}\,\mathrm{tr}\,P\big[[P,\Lambda_1],[P,\Lambda_2]\big]$$

where $P = P_{\omega,\omega'}$ is the spectral projection associated with $\mathrm{spec}\,\widehat{U}$ between ω, ω' (counter-clockwise)

Bulk and Edge spectrum



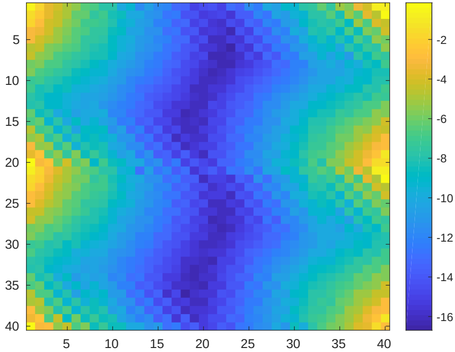


Computing the edge index

Edge index based $\mathcal{N}_{\mathrm{E},\alpha}$ based on the pair (H, H_α) (with $\alpha = \pi$)

$$\mathcal{N}_{\mathrm{E},\alpha} = \operatorname{tr} A \qquad A = \widehat{U}_{\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\mathrm{E}} - \widehat{U}_{\alpha,\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\alpha,\mathrm{E}}$$

The diagonal integral kernel $A(x,x)$ as $\log |A(x,x)|$



Boundary conditions:

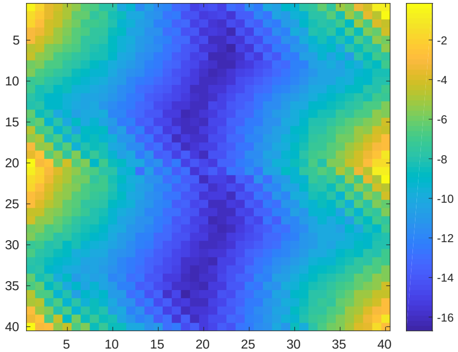
- Vertical edges: Dirichlet
- Horizontal edges: Periodic

Computing the edge index

Edge index based $\mathcal{N}_{\mathrm{E},\alpha}$ based on the pair (H, H_α) (with $\alpha = \pi$)

$$\mathcal{N}_{\mathrm{E},\alpha} = \operatorname{tr} A \qquad A = \widehat{U}_{\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\mathrm{E}} - \widehat{U}_{\alpha,\mathrm{E}}^{*} \Lambda_2 \widehat{U}_{\alpha,\mathrm{E}}$$

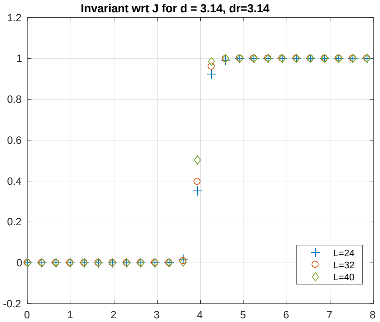
The diagonal integral kernel $A(x,x)$ as $\log |A(x,x)|$

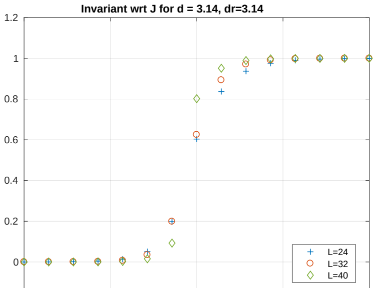


Boundary conditions:

- Vertical edges: Dirichlet
- Horizontal edges: Periodic

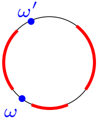
The transition





The anomalous phase

The spectrum of $\widehat{U}$ be fully localized (Rudner et al.): $\widehat{U}\psi_z = z\psi_z$, ($z$: eigenvalues $\in S^1$)



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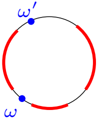
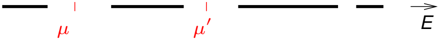


Remark. In the Hamiltonian case (e.g. IQHE)



the index would vanish in all gaps: $\mathcal{N}_\mu = \mathcal{N}_{\mu'} = 0$

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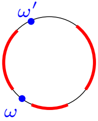

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Theorem (Rudner; Tauber, Shapiro) Let $\widehat{U}_1 = \widehat{U}_2 \equiv \widehat{U}$. Then the index $\mathcal{N}$ for the pair satisfies

$$\mathcal{N} = \mathcal{M}(U_1) - \mathcal{M}(U_2)$$

where

$$\mathcal{M}(U) = \int_0^T \sum_z (\psi_z, U(t)^* M(t) U(t) \psi_z) dt$$

with magnetization $M(t) = (\mathrm{i}/2)(\Lambda_1 H(t) \Lambda_2 - \Lambda_2 H(t) \Lambda_1)$

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If H is time independent, then $\mathcal{M}(U) = 0$. So, for $(H_1(t), H_2(t)) = (H(t), H_\alpha)$ we have $\mathcal{N} = \mathcal{M}(U)$

Summary

Thank you for your attention!