Anomalous transport in random conformal field theory

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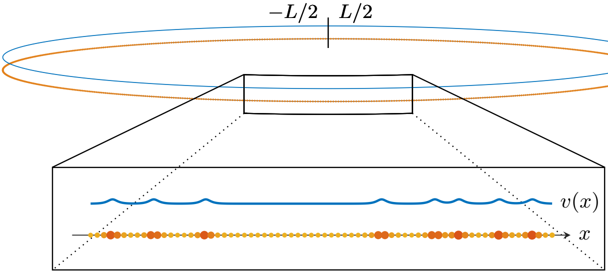
Inhomogeneous CFT

$$H = \int_{-L/2}^{L/2} \mathrm{d}x\, v(x) \big[T_+(x) + T_-(x) \big]$$



E.g.: Effective description of generalized quantum spin chain

$$H_{XXZ} = -\sum_j J_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \Delta S_j^z S_{j+1}^z \right) - \sum_j h_j S_j^z$$

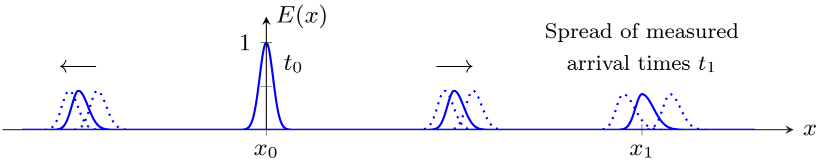
Random CFT

$$v(x) = \frac{v}{1 - \xi(x)} > 0$$

with Gaussian random function $\xi(x)$ specified by

$$\mathbb{E}[\xi(x)] = 0 \qquad \Gamma(x - y) = \mathbb{E}[\xi(x)\xi(y)]$$

Exact analytical results showing diffusion on top of ballistic motion:



P.M., PhD thesis (2018); Langmann, P.M., Phys. Rev. Lett. 122 (2019)

Numerical demonstration of this diffusive effect in random integrable spin chains using generalized hydrodynamics

Agrawal, Gopalakrishnan, Vasseur, Phys. Rev. B 99 (2019)

Outline

- Introduction
- Main tools
- Applications
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Minkowskian conformal field theory

Spacetime: $\mathbb{R}^+ \times S^1$ with S^1 the circle of length L

Conformal group $\cong \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$ with $\mathrm{Diff}_+(S^1)$ the group of orientation-preserving diffeomorphisms of the circle

Right- and left-moving components of the energy-momentum tensor

$$
\begin{aligned}
&\big[T_\pm(x), T_\pm(y)\big] = \mp 2\mathrm{i}\delta'(x-y)T_\pm(y) \pm \mathrm{i}\delta(x-y)T'_\pm(y) \pm \frac{c}{24\pi}\mathrm{i}\delta'''(x-y) \\
&\big[T_\pm(x), T_\mp(y)\big] = 0
\end{aligned}
$$

in light-cone coordinates $x^\pm = x \pm vt$

Recall: $T_\pm = T_\pm(x^\mp)$ with $T_+ = T_{--}$, $T_- = T_{++}$, and $T_{+-} = 0 = T_{-+}$

E.g.: Schottenloher, *A Mathematical Introduction to Conformal Field Theory* (2008)

Observables and conformal transformations

Primary fields

$$\Phi(x^-, x^+) \to f'(x^-)^{\Delta_\Phi^+} f'(x^+)^{\Delta_\Phi^-} \Phi(f(x^-), f(x^+))$$

Energy-momentum tensor

$$T_\pm(x^\mp) \to f'(x^\mp)^2 T_\pm(f(x^\mp)) - \frac{c}{24\pi}\{f(x^\mp), x^\mp\}$$

$f \in \mathrm{Diff}_+(S^1)$ where $\{f(x), x\} = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$

E.g.: Francesco, Mathieu, Sénéchal, *Conformal Field Theory* (1997)

Examples

Non-interacting fermions

$$T_{\pm}(x) = \frac{1}{2}\big[{:}\psi^{+}_{\pm}(x)(\mp \mathrm{i}\partial_x)\psi^{-}_{\pm}(x){:} + \mathsf{h.c.}\big] - \frac{\pi}{12L^2}$$

$$\big\{\psi^{-}_{r}(x), \psi^{+}_{r'}(y)\big\} = \delta_{r,r'}\delta(x-y) \qquad \big\{\psi^{\pm}_{r}(x), \psi^{\pm}_{r'}(y)\big\} = 0$$

Local Luttinger model (renormalized)

$$T_{\pm}(x) = \pi\,{:}\widetilde{\rho}_{\pm}(x)^2{:} - \frac{\pi}{12L^2}$$

$$\widetilde{\rho}_{\pm}(x) = \frac{1+K}{2\sqrt{K}}\rho_{\pm}(x) + \frac{1-K}{2\sqrt{K}}\rho_{\mp}(x) \qquad \rho_{\pm}(x) = {:}\psi^{+}_{\pm}(x)\psi^{-}_{\pm}(x){:}$$

Voit, Rep. Prog. Phys. 58 (1995)

Schulz, Cuniberti, Pieri, Fermi liquids and Luttinger liquids, p. 9 in *Field Theories for Low-Dim. . . .* (2000)

Outline

Projective unitary representations of diffeomorphisms

Proj. unitary reps. $U_{\pm}(f)$ of $f \in \widetilde{\mathrm{Diff}}_{+}(S^1)$ given by

$$U_{\pm}(f) = I \mp \mathrm{i}\varepsilon \int_{-L/2}^{L/2} \mathrm{d}x \, \zeta(x) T_{\pm}(x) + o(\varepsilon)$$

for infinitesimal $f(x) = x + \varepsilon\zeta(x)$ with $\zeta(x+L) = \zeta(x)$

Meaning of projective:

$$U_{\pm}(f_1)U_{\pm}(f_2) = \mathrm{e}^{\pm \mathrm{i}cB(f_1,f_2)/24\pi} U_{\pm}(f_1 \circ f_2)$$

E.g.: Khesin, Wendt, *The Geometry of Infinite-Dimensional Groups* (2009)

Gawędzki, Langmann, P.M., J. Stat. Phys. 172 (2018)

Virasoro-Bott group and Virasoro algebra

Bott cocycle

$$B(f_1, f_2) = \frac{1}{2} \int_{-L/2}^{L/2} \mathrm{d}x \, [\log f_2'(x)]' \, \log[f_1'(f_2(x))]$$

Virasoro-Bott group: Central extension of $\mathrm{Diff}_+(S^1)$ given by $B(f_1, f_2)$

Corresponding Lie algebra: The Virasoro algebra

$$\begin{aligned} \left[L_n^\pm, L_m^\pm\right] &= (n-m)L_{n+m}^\pm + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ \left[L_n^\pm, L_m^\mp\right] &= 0 \end{aligned}$$

and

$$T_\pm(x) = \frac{2\pi}{L^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pm \frac{2\pi \mathrm{i} n x}{L}} \left(L_n^\pm - \frac{c}{24}\delta_{n,0} \right)$$

E.g.: Khesin, Wendt, *The Geometry of Infinite-Dimensional Groups* (2009)

Gawędzki, Langmann, P.M., J. Stat. Phys. 172 (2018)

Adjoint action

Using the Bott cocycle:

$$U_\pm(f)T_\pm(x)U_\pm(f)^{-1} = f'(x)^2 T_\pm(f(x)) - \frac{c}{24\pi}\{f(x), x\}$$
$$U_\pm(f)T_\mp(x)U_\pm(f)^{-1} = T_\mp(x)$$

Given a smooth L-periodic function $v(x) > 0$, define

$$f(x) = \int_0^x \mathrm{d}x' \, \frac{v_0}{v(x')} \qquad \frac{1}{v_0} = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x' \, \frac{1}{v(x')}$$

Then $f \in \widetilde{\mathrm{Diff}}_+(S^1)$ and $U(f) = U_+(f)U_-(f)$ gives

$$U(f)HU(f)^{-1} = \int_{-L/2}^{L/2} \mathrm{d}x \, v_0 \big[T_+(x) + T_-(x)\big] + \text{c-number}$$

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Non-equilibrium dynamics

Focus on heat transport in inhomogeneous CFT:

$$\text{Only need reps. of } \mathrm{Diff}_+(S^1)$$

Can also do both heat and charge transport:

$$\text{Need reps. of } \mathrm{Map}(S^1, G) \rtimes \mathrm{Diff}_+(S^1)$$

Simplest example: $G = \mathsf{U}(1)$ as for the local Luttinger model

Time evolution from smooth-profile states

Non-equilibrium initial states defined by

$$G = \int_{-L/2}^{L/2} \mathrm{d}x \, \beta(x) v(x) [T_+(x) + T_-(x)]$$

with smooth inverse-temperature profile $\beta(x)$

Recipe to compute

$$\langle \mathcal{O}_1(x_1; t_1) \dots \mathcal{O}_n(x_n; t_n) \rangle_{\mathsf{neq}} = \frac{\mathrm{Tr}\big[\mathrm{e}^{-G} \mathcal{O}_1(x_1; t_1) \dots \mathcal{O}_n(x_n; t_n)\big]}{\mathrm{Tr}\big[\mathrm{e}^{-G}\big]}$$

for $\mathcal{O}_j(x; t) = \mathrm{e}^{\mathrm{i}Ht} \mathcal{O}_j(x) \mathrm{e}^{-\mathrm{i}Ht}$

Energy density and heat current

The energy density operator

$$\mathcal{E}(x) = v(x)\big[T_+(x) + T_-(x)\big]$$

and the heat current operator

$$\mathcal{J}(x) = v(x)^2\big[T_+(x) - T_-(x)\big]$$

satisfy

$$\partial_t \mathcal{E}(x) + \partial_x \mathcal{J}(x) = 0$$

$$\partial_t \mathcal{J}(x) + v(x)\partial_x\big[v(x)\mathcal{E}(x) + \mathcal{S}(x)\big] = 0$$

with

$$\mathcal{S}(x) = -\frac{c}{12\pi}\left[v(x)v''(x) - \frac{1}{2}v'(x)^2\right]$$

Energy density and heat current – Results

Given smooth L-periodic functions $v(x)$ and $\beta(x)$ defining the time evolution and the initial state as above, then

$$\langle \mathcal{E}(x;t) \rangle_{\mathsf{neq}}^{\infty} = \frac{1}{2v(x)} \left[F(\mathring{x}^-) + F(\mathring{x}^+) \right] - \frac{1}{v(x)} \mathcal{S}(x)$$
$$\langle \mathcal{J}(x;t) \rangle_{\mathsf{neq}}^{\infty} = \frac{1}{2} \left[F(\mathring{x}^-) - F(\mathring{x}^+) \right]$$

in the thermodynamic limit $L \to \infty$ with

$$\mathring{x}^{\pm} = f^{-1}(f(x) \pm v_0 t)$$

and

$$F(x) = \frac{\pi c}{6\beta(x)^2} + \frac{cv(x)^2}{12\pi} \left[\frac{\beta''(x)}{\beta(x)} - \frac{1}{2} \left(\frac{\beta'(x)}{\beta(x)} \right)^2 + \frac{v'(x)}{v(x)} \frac{\beta'(x)}{\beta(x)} \right]$$

Thermal conductivity

Dynamically:

$$\kappa_{\mathsf{th}}(\omega) = \beta^2 \frac{\partial}{\partial(\delta\beta)} \left(\int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int_{\mathbb{R}} \mathrm{d}x\, \partial_t \langle \mathcal{J}(x;t) \rangle_{\mathsf{neq}}^{\infty} \right) \Big|_{\delta\beta=0}$$

for a kink-like initial profile $\beta(x) = \beta + \delta\beta W(x)$ with height $\delta\beta$

or equivalently

Green-Kubo formula:

$$\kappa_{\mathsf{th}}(\omega) = \beta \int_0^{\beta} \mathrm{d}\tau \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}x'\, \partial_{x'} [-W(x')] \big\langle \mathcal{J}(x;t) \mathcal{J}(x';\mathrm{i}\tau) \big\rangle_{\beta}^{c,\infty}$$

with $\langle \cdots \rangle_{\beta} = \langle \cdots \rangle_{\mathsf{neq}} \big|_{\beta(x)=\beta}$

P.M., PhD thesis (2018)

Thermal conductivity – Results

On general grounds

$$\mathrm{Re}\,\kappa_{\mathsf{th}}(\omega) = D_{\mathsf{th}}\pi\delta(\omega) + \mathrm{Re}\,\kappa_{\mathsf{th}}^{\mathsf{reg}}(\omega)$$

Given a smooth $v(x)$, then

$$D_{\mathsf{th}} = \frac{\pi v c}{3\beta} \qquad \mathrm{Re}\,\kappa_{\mathsf{th}}^{\mathsf{reg}}(\omega) = \frac{\pi c}{6\beta}\left[1 + \left(\frac{\omega\beta}{2\pi}\right)^2\right] I(\omega)$$

with

$$I(\omega) = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' \left(1 - \frac{v}{v(x)}\right)\partial_{x'}\left[-W(x')\right]\cos\left(\omega\int_{x'}^{x}\frac{\mathrm{d}\widetilde{x}}{v(\widetilde{x})}\right)$$

where v is arbitrary in the thermodynamic limit

Full counting statistics

"Full counting statistics of energy transfers in inhomogeneous nonequilibrium states of (1+1)D CFT"

Gawędzki, Kozłowski, arXiv:1906.04276 (2019)

Alternative approach

Standard Euclidean CFT in curved spacetime with the metric

$$h = \mathrm{d}x^2 + v(x)^2 \mathrm{d}\tau^2$$

(imaginary time $\tau = it$)

Dubail, Stéphan, Viti, Calabrese, SciPost Phys. 2 (2017)
Dubail, Stéphan, Calabrese, SciPost Phys. 3 (2017)

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Ballistic and anomalous/normal diffusive contributions

Recall: Random CFT with $v(x) = v/[1-\xi(x)]$ and Gaussian random function $\xi(x)$ specified by $\mathbb{E}[\xi(x)] = 0$ and $\Gamma(x-y) = \mathbb{E}[\xi(x)\xi(y)]$

After averaging:

$$D_{\mathsf{th}} = \frac{\pi v c}{3\beta}$$

$$\operatorname{Re}\kappa_{\mathsf{th}}^{\mathsf{reg}}(\omega) = \frac{\pi c}{6\beta}\left[1 + \left(\frac{\omega\beta}{2\pi}\right)^2\right]\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}(\omega/v)^2\Lambda(x)} \cos\left(\frac{\omega x}{v}\right)$$

$$L_{\mathsf{th}} = \lim_{\omega\to 0}\operatorname{Re}\kappa_{\mathsf{th}}^{\mathsf{reg}}(\omega) = \frac{\pi c}{6\beta}\Gamma_0$$

with $\Lambda(x) = \int_0^x \mathrm{d}x_1 \int_0^x \mathrm{d}x_2\, \Gamma(x_1 - x_2)$ and $\Gamma_0 = \int_{\mathbb{R}} \mathrm{d}x\, \Gamma(x)$

Wave propagation in random media

Solving random PDEs for the expectations $E(x;t)$ and $J(x;t)$ of $\mathcal{E}(x;t)$ and $\mathcal{J}(x;t)$ in an arbitrary state with $E(x;0) = e_0(x)$ and $J(x;0) = 0$ gives:

$$\mathbb{E}[E(x;t)] = \int_{\mathbb{R}} \mathrm{d}y \left[G_+^{\mathcal{E}}(x-y;t) + G_-^{\mathcal{E}}(x-y;t) \right] e_0(y)$$

$$\mathbb{E}[J(x;t)] = \int_{\mathbb{R}} \mathrm{d}y \left[G_+^{\mathcal{J}}(x-y;t) + G_-^{\mathcal{J}}(x-y;t) \right] e_0(y)$$

with $G_\pm^{\mathcal{E}}(x;t)$ and $G_\pm^{\mathcal{J}}(x;t)$ expressed in terms of

$$G_\pm(x;t) = \theta(\pm x) \frac{\mathrm{e}^{-(x \mp vt)^2/2\Lambda(x)}}{\sqrt{2\pi\Lambda(x)}}$$

Propagation-diffusion equation

$$\boxed{\left[v^{-1}\partial_t \pm \partial_x - \gamma(x)\partial_t^2 \right] G_\pm(x;t) = 0} \quad (\pm x > 0,\ t > 0)$$

with temporal diffusion coeff. $\gamma(x)$

Boon, Grosfils, Lutsko, Euro. Phys. Lett. 63 (2003)
Langmann, P.M., Phys. Rev. Lett. 122 (2019)

Heat-wave reference frame

Define $\widetilde{G}_\pm(\widetilde{x};\widetilde{t}) = G_\pm(x;t)$ with $\begin{cases} \widetilde{x} = x \mp vt \\ \widetilde{t} = |x|/v \end{cases}$

Diffusion equation

$$\boxed{\left[\partial_{\widetilde{t}} - \alpha_{\mathsf{th}}(\widetilde{t})\partial_{\widetilde{x}}^2\right]\widetilde{G}_\pm(\widetilde{x};\widetilde{t}) = 0} \quad (\widetilde{t} > 0,\ \pm\widetilde{x} > -vt)$$

with thermal diffusivity $\alpha_{\mathsf{th}}(\widetilde{t})$ where

$$\alpha_{\mathsf{th}} = \lim_{\widetilde{t}\to\infty} \alpha_{\mathsf{th}}(\widetilde{t}) = \frac{v}{2}\Gamma_0$$

Relation between L_{th} and α_{th}

Einstein relation

$$L_{\mathsf{th}} = c_V \alpha_{\mathsf{th}}$$

with the volume-specific heat capacity

$$c_V = -\beta^2 \frac{\partial}{\partial \beta} \mathbb{E}\Big[\langle \mathcal{E}(x;t) \rangle_\beta^\infty \Big] = \frac{\pi c}{3 \beta v}$$

where $\langle \cdots \rangle_\beta = \langle \cdots \rangle_{\mathsf{neq}} \big|_{\beta(x)=\beta}$

Thank you for your attention!