Magnetic Weyl Quantization and Semiclassical Limit

Satoshi Okumura

Tohoku University Graduate School of Mathematics

2019. 9. 19.

Table of contants

- What is Semiclassical Limit?
- Motivation

What is Semiclassical Limit?

Semiclassical Limit

- Approximate quantum mechanics using classical mechanics.
- Introducing a semiclassical parameter $\varepsilon > 0$.
- When the limit of $\varepsilon \to 0$ is taken, we expect classical mechanics and quantum mechanics to match.

$$\begin{cases} \dfrac{d}{dt}x = \nabla_\xi h \\ \dfrac{d}{dt}\xi = -\nabla_x h + B\dot{x} \end{cases} \overset{?}{\longleftrightarrow} \begin{cases} i\varepsilon \dfrac{\partial \psi(t)}{\partial t} = \hat{H}^A \psi(t) \\ \psi(0) = \psi_0 \in L^2(\mathbb{R}^d) \end{cases}$$

f : Classical Observable, i.e. real function on $\mathbb{R}^d \times \mathbb{R}^d$

B : Magnetic Field, A : Vector Potential of Magnetic Field,

When $d = 2$, Asatisfies $\mathrm{rot} A = B$.

Motivation

Motivation

- To approximate time evolution generated by Quantum Hamiltonian $\hat{H}^A$ with time evolution generated by Classical Hamiltonian h.

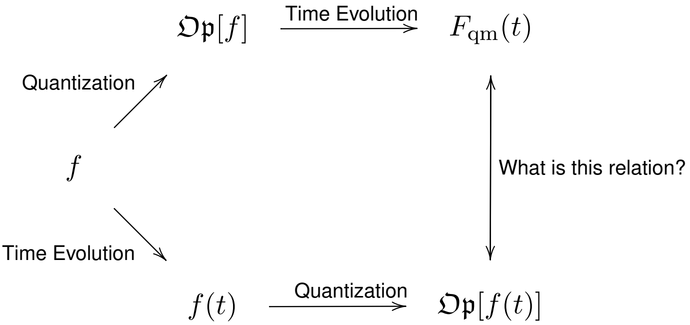

Table of contants

- Definition of Magnetic Weyl Quantization
- Time Evolution in Classical System and Quantum System
- Egorov type Theorem

Magnetic Weyl Quantization(Măntoiu, Purice 2004)

Magnetic Weyl Quantization

We consider integral of f :

$$\mathfrak{Op}^A[f] := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathscr{F}_\sigma[f](Y) W_\varepsilon^A(Y) dY.$$

We call a map $f \mapsto \mathfrak{Op}^A[f]$ as Magnetic Weyl Quantization of f.

$$\sigma((x,\xi),(y,\eta)) := \xi \cdot y - x \cdot \eta, \quad \text{(syplectic form)},$$

$$W_\varepsilon^A(Y) := \exp\left(i\sigma(Y,(Q,P_\varepsilon^A))\right), \quad (Y = (y,\eta) \in \mathbb{R}^d \times \mathbb{R}^d),$$

$$\mathscr{F}_\sigma[f](Y) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i\sigma(Y,Y')} f(Y') dY'.$$

Classical Time Evolution

$$f(t) = f(x(t), \xi(t)),$$

$$\frac{d}{dt}(f(t)) = \sum_{j=1}^{d} \frac{\partial f}{\partial \xi_j}\frac{\partial h}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial h}{\partial \xi_j} + \sum_{j,k=1}^{d} B_{jk}\frac{\partial f}{\partial \xi_j}\frac{\partial h}{\partial \xi_k}$$

$$=: \{h, f(t)\}_B, \quad f(0) = f$$

Quantum time evolution

- $F^A_{\mathrm{qm}}(t)$: Quantum observable at time $t \in \mathbb{R}$.

$$\begin{cases} i\varepsilon \frac{\partial}{\partial t} F^A_{\mathrm{qm}}(t) = [F^A_{\mathrm{qm}}(t), \hat{H}^A], \quad F^A_{\mathrm{qm}}(0) = \mathfrak{Op}^A[f] \\ F^A_{\mathrm{qm}}(t) = e^{\frac{i}{\varepsilon}t\hat{H}^A}\mathfrak{Op}^A[f]e^{-\frac{i}{\varepsilon}t\hat{H}^A}. \end{cases}$$

Semiclassical Limit

Egorov Type Theorem(Lein [6] 2010)

$$F_{\mathrm{cl}}(t) := \mathfrak{Op}^A[f(t)], \quad F_{\mathrm{qm}}(t) := e^{i\frac{t}{\varepsilon}\hat{H}^A}\mathfrak{Op}^A[f]e^{-i\frac{t}{\varepsilon}\hat{H}^A},$$

Then there exists $C_T > 0$ such that,

$$\|F_{\mathrm{cl}}(t) - F_{\mathrm{qm}}(t)\|_{B(L^2(\mathbb{R}^d))} \leq \varepsilon^2 C_T, \quad (^\forall t \in [-T, T])$$

Egorov type theorem(O. 2018)

There exists g_t such that

$$F_{\mathrm{qm}}(t) = \mathfrak{Op}^A[g_t] + O_{B(L^2(\mathbb{R}^d))}(\varepsilon^\infty) \tag{1}$$

$$|\partial_\xi^\alpha \partial_x^\beta (g_t(x,\xi) - f(x(t),\xi(t)))| \leq C_T \varepsilon^2 \tag{2}$$

$$g_t(x,\xi) \asymp \sum_{n=0}^{\infty} \varepsilon^{2n} g_n(x,\xi).$$

Summary

- Magnetic Weyl Quantization is good quantization when magnetic field exists :

$$\mathfrak{Op}^A[f] := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathscr{F}_\sigma[f](Y) W^A_\varepsilon(Y) dY.$$

- The difference in time evolution generated by Classical Hamiltonian h and Quantum Hamiltonian $\mathfrak{Op}^A[h]$ is small.

References

- A.Arai, "Mathematical Structure of Quantum Mecanics I, II", Asakura, 1999 in Japanese.
- A.Arai, "Mathematical Aspects of Quantum Phenomenon", Asakura, 2006 in Japanese.
- Helmut Abels, "*Pseudodifferential and Singular Integral Operators*, " De Gruyter, 2012.
- Maciej Zworski, "*Semiclassical Analysis*", AMS, 2012.
- Marius Măntoiu and Radu Purice, *The magnetic Weyl calculus* Journal of Mathematical Physics 45, 1394 (2004)
- Max Lein, "*Semiclassical Dynamics and Magnetic Weyl Calculus*" PhD thesis Technische Universität München, Germany, 2010

References

- Max Lein , “Two-parameter Asymptotics in Magnetic Weyl Calculus”, Journal of Mathematical Physics 51, p. 123519, 2010.
- M.W.Wong, "*Weyl Transformations*", Springer, 1999.
- D. Robert,"*de l’Approximation Semi-Classique*|" Birkhäuser, 1987
- Viorel Iftimie, Marius Măntoiu and Radu Purice, *"Magnetic Pseudodifferential Operators"*", RIMS,Kyoto Univ, 43, 585-623, 2007.