Lecture Course: T. Prosen, Rome 09/19

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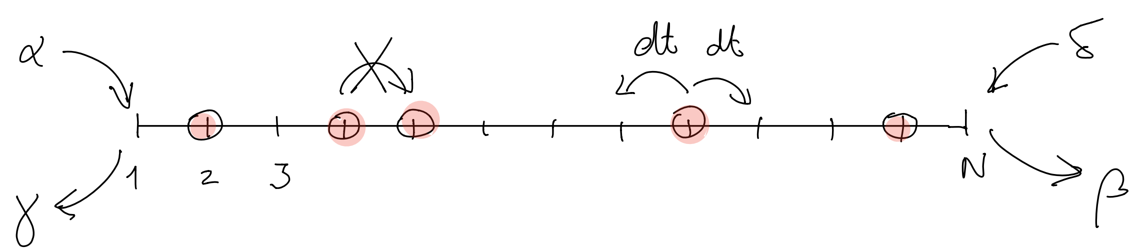
I: Boundary driven systems: Integrability and MPA

a) Stochastic systems: Ex. Symmetric Single Excl. process (SSEP)



Probability state vector

$$\left(p_{s_1 s_2 \dots s_N};\ s_j \in \{0,1\} \right)$$

$$|p\rangle = \sum_{s_1 \dots s_N} p_{s_1 s_2 \dots s_N} |s_1 s_2 \dots s_N\rangle$$

Continuous time Markov process

$$\frac{d}{dt}|p(t)\rangle = \hat{\mathcal{L}}|p(t)\rangle$$

$$\hat{\mathcal{L}} = \sum_{j=1}^{N-1} P_{j,j+1} + \text{boundary terms} \quad ; \quad P_{j,j+1} = \mathbb{1}_{2^{j-1}} \otimes P_{1,2} \otimes \mathbb{1}_{2^{N-j-1}}$$

$$P_{1,2} = \begin{matrix} {}_{s_1 s_2} \backslash {}^{s_1' s_2'} \\ 00 \\ 01 \\ 10 \\ 11 \end{matrix} \begin{pmatrix} 0 & & & \\ & -1 & 1 & \\ & 1 & -1 & \\ & & & 0 \end{pmatrix} = \frac{1}{2}\left(\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \mathbb{1} \right)$$

boundary terms $= \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \otimes \mathbb{1}_{2^{N-1}} + \mathbb{1}_{2^{N-1}} \otimes \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$

$$\hat{\mathcal{L}} = \sum_{j=1}^{N-1} \frac{1}{2}\left(\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - \mathbb{1}\right) + b.t.$$

$\frac{1}{2}(\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - \mathbb{1}) = H_{\text{Heisenberg}}$!

b.t. ← boundary magnetic fields

<u>Steady state:</u>

$$|\rho^\infty\rangle = \lim_{t\to\infty} |\rho(t)\rangle = \lim_{t\to\infty} e^{t\hat{\mathcal{L}}} |\rho(0)\rangle$$

$e^{t\hat{\mathcal{L}}}$ ← imaginary time evolution

$\hat{\mathcal{L}} |\rho_\infty\rangle = 0$ $\quad |\rho_\infty\rangle$ ground state of $H_{\text{Hes}} + h_L + h_R$

<u>Matrix product ansatz (MPA):</u>

$$\rho^\infty_{s_1 s_2 \dots s_N} = \langle L| A_{s_1} A_{s_2} \cdots A_{s_N} |R\rangle$$

A_0, A_1 a pair of matrices for the auxiliary Hilbert space $\mathcal{H}_a$, $|L\rangle, |R\rangle \in \mathcal{H}$

$$\rho^\infty = \langle L| \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{\otimes N} |R\rangle$$

(Derrida et al. J. Phys. A 1993)

Lemma:

$$\hat{\mathcal{L}} |p^\infty\rangle = 0$$

Follows from a pair of sufficient conditions

(i) $$P_{1,2} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \otimes \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ -\mathbb{1} \end{pmatrix} \otimes \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} - \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1} \\ -\mathbb{1} \end{pmatrix}$$

(ii) $$\langle L| (\alpha A_0 - \gamma A_1) = \langle L|$$

$$(\beta A_0 - \delta A_0) |R\rangle = |R\rangle$$

Proof: Multiply (i) from the left by $\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{\otimes(j-1)} \otimes$

and from the right by $\otimes \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{\otimes(N-j-1)}$

and sum $\sum_{j=1}^{N-1}$

All terms cancel, except for boundary terms which are cancelled by (ii)

i) is equivalent to an algebra

$$[A_1, A_0] = A_0 + A_1$$ which admits an explicit inf. dim. tridiagonal (oscillator) representation

Just the algebra or concrete representation can be used to compute all physical properties of the steady state.

E.g. density profile

$$C = A_0 + A_1$$

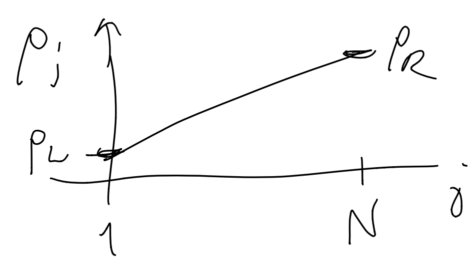
$$\langle L | C^N | R \rangle = 1$$ normalization of probability

or:

$$\rho_j = \sum_{s_1 \dots s_{j-1}, s_{j+1} \dots s_N} P_{s_1 \dots s_{j-1}, 1, s_{j+1} \dots s_N} = \frac{\langle L | C^{j-1} A_1 C^{N-j} | R \rangle}{\langle L | C^N | R \rangle}$$

Exercise: Use just the algebra to write a linear difference equation for ρ_j

$\rho_j = \rho_{j-1} + \delta$ which yields <u>Fick's law</u>



$$A_1 C = C A_1 + C$$

$$\Rightarrow \rho_j = \rho_{j+1} + \frac{\langle L | C^{N-1} | R \rangle}{\langle L | C^N | R \rangle}$$

$$= \rho_{j+1} + \delta$$

$\rho_j = \rho_1 - (j-1)\delta$ linear profile!

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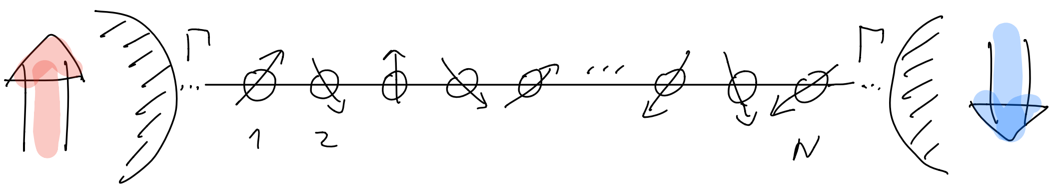
A similar problem for a quantum spin chain, but with coherent (unitary / deterministic) dynamics in the bulk:

Lindblad driven XXX spin $\frac{1}{2}$ chain



Lindblad equation with boundary incoherent processes (dissipation + driving)

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{\mu=1}^{M} \mathcal{D}_{L_\mu}(\rho)$$

Lindblad '76
Gorini-Kossakowski-Sudarshan '76

$$\mathcal{D}_L(\rho) = 2L\rho L^\dagger - \{L^\dagger L, \rho\}$$

Boundary driven XXZ:

$$H = \sum_{j=1}^{N-1} \underbrace{(2\sigma_j^+\sigma_{j+1}^- + 2\sigma_j^-\sigma_{j+1}^+ + \Delta\sigma_j^z\sigma_{j+1}^z)}_{h_{j,j+1}}$$

$L_1 = \sqrt{\Gamma}\,\sigma_1^+,\ L_2 = \sqrt{\Gamma}\,\sigma_N^-$

pure "source / sink" boundaries

Method of solution:

Cholesky - Factorization + MPA (Prosen PRL '11)

Nonequilibrium steady state (NESS)

$$i[H, \rho_\infty] = \Gamma\left(\mathcal{D}_{\sigma_1^+}(\rho_\infty) + \mathcal{D}_{\sigma_N^-}(\rho_\infty)\right) \qquad (*)$$

Cholesky-Factorization:

$$\rho_\infty = \Omega_N \Omega_N^+ \qquad \Omega_N \text{ upper triangular } 2^N \times 2^N \text{ matrix}$$

in computational basis

Lemma: $(*)$ is equivalent to

$$i[H, \Omega_N] = \Gamma\left(\sigma^z \otimes \Omega_{N-1} - \Omega_{N-1} \otimes \sigma^z\right)$$

&

$$\Omega_N = \sigma^0 \otimes \Omega_{N-1} + \sigma^+ \otimes \Omega_{N-1}^+$$

$$= \Omega_{N-1} \otimes \sigma^0 + \Omega_{N-1} \otimes \sigma^-$$

exercise: prove it!

The ansatz

$$\Omega_N = \langle 0| \begin{pmatrix} A_0 & A_+ \\ A_- & A_0 \end{pmatrix}^{\otimes N} |0\rangle$$

Sufficient condition is to find $A_0, A_\pm$ (8)

which satisfy

$$[h_{12}, L \otimes L] = \sigma^z \otimes L - L \otimes \sigma^z \quad (**)$$

$$L = \begin{pmatrix} A_0 & A_+ \\ A_+ & A_0 \end{pmatrix}$$

Written in components $A_0, A_\pm$ should be generators of $SU(2)$ algebra ($\Delta = 1$)

or $U_q(SU(2))$ ($\Delta \neq 1$)

Matching the boundaries:

$A_+ |0\rangle = 0$ $\quad |0\rangle$ highest weight state

$A_0 |0\rangle = s|0\rangle$

$$\boxed{s = \frac{2i}{\Gamma}}$$ Complex spin

$\Rightarrow$ infinitely-dimensional non-unitary irrep of angular momentum algebra!

These NESS solutions gave birth to Quasi-local conservation laws which settled some fundamental questions about transport and relaxation in integrable quantum systems.

(See eg. reviews Ilievski et al JSTAT 2016, Prosen J. Phys. A, Topical Review 2015)

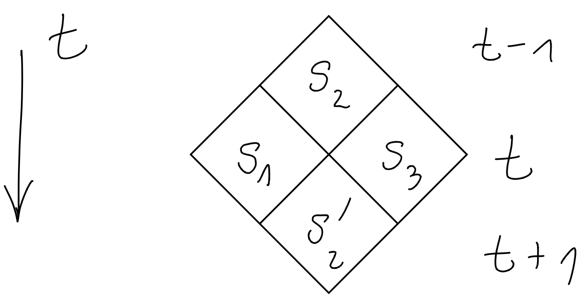
But now, let's discuss something much simpler about which we can say even much more:

Deterministic - reversible classical interacting systems with stochastic boundary driving

- The Rule 54 -

Simplest examples: binary reversible (2nd order in time) cellular automata

[Classification in Bobenko et al. 1993]



$$s_2' = \chi(s_1, s_2, s_3)$$

binary function of 3 arguments

s_1	s_2	s_3	χ	
0	0	0	0	1
0	0	1	1	2
0	1	0	1	4
0	1	1	0	8
1	0	0	1	16
1	0	1	1	32
1	1	0	0	64
1	1	1	0	128
				54

$$s_2' = s_1 + s_2 + s_3 + s_1 s_3 \pmod 2$$

(Show some pictures)

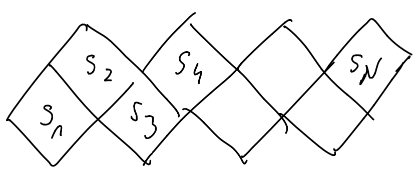
Boundary-driven discrete-time Markov process

$p_{s_1 s_2 \dots s_N}$ probabilities of 2^N configurations

N even

$$P_{s_1's_2's_3', s_1s_2s_3} = \delta_{s_1 s_1'} \delta_{\chi(s_1s_2s_3), s_2'} \delta_{s_3 s_3'}$$

(control bits)

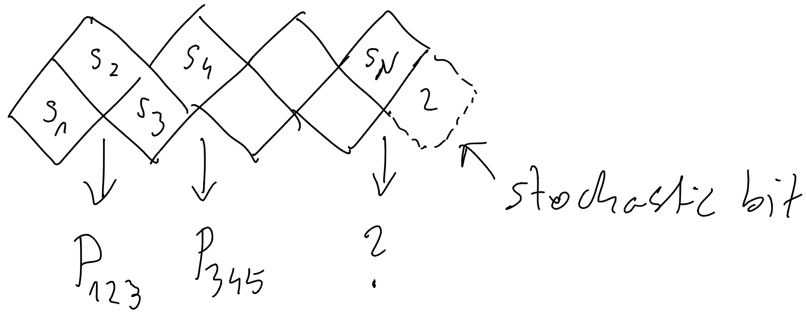


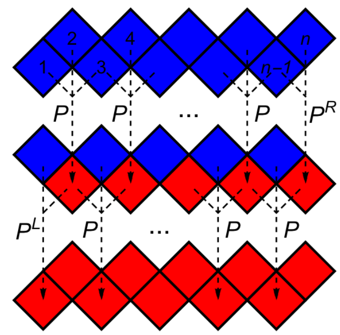
$$[P_{i-1,i,i+1}, P_{j-1,j,j+1}] = 0$$

$$|i-j| \geq 2$$

e.g. $[P_{123}, P_{345}] = 0$

$[P_{123}, P_{234}] \neq 0$





$$U_e = P_{123} P_{345} \cdots P_{N-3\,N-2\,N-1} P^R_{N-1\,N}$$

$$U_o = P^L_{12} P_{234} P_{456} \cdots P_{N-2\,N-1\,N}$$

$$P^L = \begin{pmatrix} \alpha & 0 & \alpha & 0 \\ 0 & \beta & 0 & \beta \\ 1-\alpha & 0 & 1-\alpha & 0 \\ 0 & 1-\beta & 0 & 1-\beta \end{pmatrix}$$

probabilities $\alpha, \beta, \gamma, \delta \in [0,1]$

$$P^R = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ 1-\gamma & 1-\gamma & 0 & 0 \\ 0 & 0 & \delta & \delta \\ 0 & 0 & 1-\delta & 1-\delta \end{pmatrix}$$

Propagator $\quad \underline{p}(t+1) = U\,\underline{p}(t)$

$U = U_o U_e \quad$ Markov matrix

(Prosen & Mejía-Monasterio, J. Phys. A 2016)

Ergodicity and mixing of BD-Rule 54 (13)

"Holographic ergodicity"

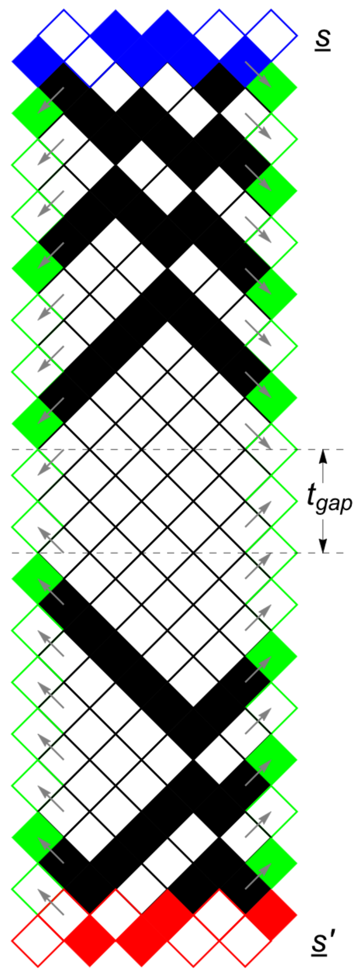
According to Perron-Frobenius theory:

U is ergodic & mixing, i.e. $\underline{p}_\infty$, $U\underline{p}_\infty = \underline{p}_\infty$, is unique, and any $\underline{p}(0)$ relaxes exponentially fast to $\underline{p}_\infty$, if $\exists\, t > 0$, $\forall\ \underline{s}, \underline{s}' \in \{0,1\}^N$, $(U^t)_{\underline{s},\underline{s}'} > 0$.

geometric proof:

$\underline{s} \xrightarrow[\text{boundaries}]{\text{absorbing}} \underline{0} \xleftarrow[\text{boundaries}]{\text{absorbing}} \underline{s}'$

$\xrightarrow[\text{reversal}]{}$



Finitely correlated time-translation invariant (equilibrium) states (bulk, $N \to \infty$)

- Patch-state ansatz (PSA)

$$p_{\ldots s_{-1} s_0 s_1 s_2 s_3 s_4 \ldots} = \ldots X_{s_{-1} s_0, s_1 s_2} X_{s_1 s_2, s_3 s_4} X_{s_3 s_4, s_5 s_6} \ldots$$

$$U_e \, p = U_o \, p$$

$$\ldots X_{s_{-1}\chi(s_{-1}s_0 s_1), s_1 \chi(s_1 s_2 s_3)} X_{s_1 \chi(s_1 s_2 s_3), s_3 \chi(s_3 s_4 s_5)} \ldots$$

$$= X_{\chi(s_{-2} s_{-1} s_0) s_0, \chi(s_0 s_1 s_2) s_2} X_{\chi(s_0 s_1 s_2) s_2, \chi(s_2 s_3 s_4) s_4} \ldots$$

$\Rightarrow$ unique solution, up to gauge

$$X_{\underline{s}, \underline{s}'} \to g_{\underline{s}} X_{\underline{s}, \underline{s}'} g^{-1}_{\underline{s}'}$$

with 2 free parameters

$$X(\xi, \omega) = \begin{pmatrix} 1 & 1 & \omega & 1 \\ \xi\omega & \xi\omega & 1 & \xi \\ \xi & \xi & \xi\omega & \xi \\ \omega & \omega & \omega & \xi\omega \end{pmatrix}$$

$\mathcal{X}_X(\xi, w)$ $\quad \xi, w > 0$ is an analogue to Gibbs ensemble

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Two local conserved quantities

$$Q_\xi = \frac{\partial}{\partial \xi} \log \mathcal{X}_X(\xi, w)\Big|_{\xi = w = 1}$$

$$Q_w = \frac{\partial}{\partial w} \log \mathcal{X}_X(\xi, w)\Big|_{\xi = w = 1}$$

$$\frac{\partial}{\partial \xi} \log X_{s_1 s_2, s_3 s_4}\Big|_{\xi = w = 1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{s_1 s_2, s_3 s_4}$$

$$=: q^{\xi}_{s_1 s_2 s_3 s_4}$$

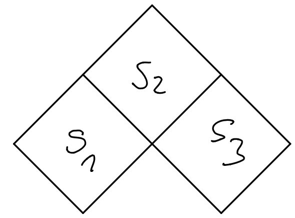
$$\frac{\partial}{\partial w} \log X_{s_1 s_2, s_3 s_4}\Big|_{\xi = w = 1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}_{s_1 s_2, s_3 s_4}$$

$$=: q^{w}_{s_1 s_2 s_3 s_4}$$

$$Q_{\xi/w}(\underline{s}) = \sum_j q^{\xi/w}_{s_{2j-1} s_{2j} s_{2j+1} s_{2j+2}}, \quad Q_\pm = Q_w \pm Q_\xi$$

Upto gradient terms in densities:

$$Q_- = \sum_j (-1)^j s_j s_{j+1}$$



$$Q_+ = \sum_j \left(2 s_j - 3 s_j s_{j+1} + 2 s_j s_{j+1} s_{j+2}\right)$$

NESS AND DIAGONALIZATION OF BD RULE 54

(Prosen & Buča, J. Phys. A 2017)

Lemma: Suppose there exist two pairs of operators over auxiliary space $\mathcal{H}_a$

$$W_0, W_1 \qquad \underline{W} = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}$$

$$W_0', W_1' \qquad \underline{W}' = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}$$

and delimiter operator S

which satisfy a cubic algebra

$$P_{123}\, \underline{W}_1 S \underline{W}_2 \underline{W}_3' = \underline{W}_1 \underline{W}_2' \underline{W}_3 S$$

$$P_{123}\, \underline{W}_1 \underline{W}_2' \underline{W}_3 S = \underline{W}_1' S \underline{W}_2 \underline{W}_3'$$

(or in components

$$W_s S W_{x(ss's'')} W'_{s''} = W_s W'_{s'} W_{s''} S$$

$$W_s W'_{x(ss's'')} W_{s''} S = W'_s S W_{s'} W'_{s''}$$)

Suppose there exist additionally sets of vectors $|\underline{r}_{12}\rangle$, $|\underline{r}_1'\rangle$, $\langle \underline{l}_{12}'|$, $\langle \underline{l}_1|$ from $\mathcal{H}_n$ which satisfy boundary relations:

$$P_{123} \langle \underline{l}_1 | \underline{W}_2 \underline{W}_3' = \langle \underline{l}_{12}' | \underline{W}_3 S$$

$$P_{12}^2 \langle \underline{l}_{12}' | = \lambda_2 \langle \underline{l}_1 | \underline{W}_2 S$$

and

$$P^R_{12} |\underline{r}_{12}\rangle = \underline{W}_1' S |\underline{r}_2'\rangle$$

$$P_{123} \underline{W}_1' \underline{W}_2 |\underline{r}_3'\rangle = \lambda_R \underline{W}_1' S |\underline{r}_{23}\rangle$$

then

$$p = \langle \underline{l}_1 | \underline{W}_2 \underline{W}_3' \underline{W}_4 \cdots \underline{W}_{N-3}' \underline{W}_{N-2} | \underline{r}_{N-1,N} \rangle$$

$$p' = \langle \underline{l}_{12}' | \underline{W}_3 \underline{W}_4' \underline{W}_5 \cdots W_{N-2}' W_{N-1} | r_N' \rangle$$

generate eigenvectors of $U = U_o U_e$

$$U p = \lambda p$$

of eigenvalue

$$\lambda = \lambda_L \lambda_R .$$

In particular:

$$(*) \quad U_e p = \lambda_L p' \qquad U_o p' = \lambda_R p$$

<u>Proof</u>: Apply:

$$V_e = P_{123} P_{345} \cdots P_{N-4\,N-3\,N-2} P^R_{N-1\,N}$$

$$V_o = P^L_{12} P_{234} \cdots P_{N-2\,N-1\,N}$$

on (*) with bulk + boundary algebra

A simple representation of the algebra is:

$$W_0(\xi, \omega) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad W_1(\xi, \omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega \end{pmatrix}$$

$$W_s'(\xi, \omega) := W_s(\omega, \xi)$$

Left eqs express ξ, ω in terms of λ_L

$$\xi = \frac{(\alpha+\beta-1) - \beta\lambda_L}{(\beta-1)\lambda_L^2}, \quad \omega = \frac{\lambda_L(\alpha - \lambda_L)}{(\beta-1)}$$

and yield $\langle \underline{\ell}_1 |, \langle \underline{\ell}_{23} |$

Similarly, right eqs. yield

$$\xi = \frac{\lambda_R(\gamma-\lambda_R)}{(\delta-1)}, \quad \omega = \frac{\gamma+\delta-\nu-\delta\lambda_R}{(\delta-1)\lambda_R^2}$$

equating $\xi_L = \xi_R$ & $\omega_L = \omega_R$

$\Rightarrow$ pair of eqs for λ_L and λ_R

Exercise: Show that $\lambda = \lambda_L\lambda_R = 1$ is always an eigenvalue!

There is much more:

- Whole spectrum of U [TP, Buča JPA17]
- Exact large deviations for arbitrary local-sum observables [Buča, Garrahan, TP, Vanicat, PRE19]
- Exact time dependent MPA of local observables $\Rightarrow$ Exact dynamical structure factor [Klobas, Medenjak, TP, Vanicat, CMP19]

II Many-Body Quantum Chaos

Question: Why does Random Matrix Theory work so well in describing spectra of simple many body systems, such as spin $\frac{1}{2}$ chains, even with local interactions?

- No small parameter ($\hbar_{eff} \sim 1$)
- No disorder / randomness needed!?

Best simple candidates for Q Chaos! Systems with no conserved quantities, even H;

Periodically driven - Floquet systems:

$$H \longrightarrow \quad U = \hat{T} \exp\left(-i \int_0^T dt\, H(t)\right), \quad H(t+T) = H(t)$$

What is RMT and what are the best signatures of spectral fluctuations?

Spectral Form Factor (SFF)

$$U|n\rangle = e^{-i\varphi_n}|n\rangle \qquad N = 2^L$$

$$\rho(\varphi) = \frac{2\pi}{N}\sum_n \delta(\varphi - \varphi_n)$$

$$\langle \rho(\varphi)\rangle = 1$$

$$\varphi_+ = \varphi + \frac{\vartheta}{2}$$
$$\varphi_- = \varphi - \frac{\vartheta}{2}$$
$$\vartheta = \varphi_+ - \varphi_-$$

$$R(\vartheta) = \langle \rho(\varphi + \tfrac{\vartheta}{2})\rho(\varphi - \tfrac{\vartheta}{2})\rangle - \langle\rho\rangle^2$$

$$K(t) = \frac{N^2}{2\pi}\int_0^{2\pi} d\vartheta\, R(\vartheta)\, e^{i\vartheta t}$$

$$= \frac{N^2}{2\pi}\int_0^{2\pi} d\vartheta \left[\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\frac{(2\pi)^2}{N^2}\sum_{n,n'}\delta(\varphi + \tfrac{\vartheta}{2} - \varphi_n)\,\delta(\varphi - \tfrac{\vartheta}{2} - \varphi_{n'}) - 1\right] e^{i\vartheta t}$$

$$= -\delta_{t,0}N^2 + \int_0^{2\pi} d\varphi_+ \int_0^{2\pi} d\varphi_- \sum_n \delta(\varphi_+ - \varphi_n)\, e^{i\varphi_+ t}\sum_{n'}\delta(\varphi_- - \varphi_{n'})\, e^{-i\varphi_- t}$$

$$= \sum_n e^{i\varphi_n t}\sum_{n'} e^{-i\varphi_{n'} t} - \delta_{t,0}N^2$$

$$= (\mathrm{tr}\, U^t)(\mathrm{tr}\, U^{-t}) - \delta_{t,0}N^2$$

$$\boxed{K(t) = |\mathrm{tr}\, U^t|^2 - \delta_{t,0}N^2, \quad K(0) = 0}$$

SFF is not self-averaging!
– ensemble average or moving time average needed! $\overline{K}(t) = \mathbb{E}(K(t))$

SFF in Random Matrix Theory

Dyson's Three fold way

- $CUE(N) = U(N)$ with Haar measure; for: systems with no anti-unitary symmetry

$$K(t) = \int_{U(N)} dU \, |\mathrm{tr}\, U|^2 = \begin{cases} t & ; t < N \\ N & ; t \geq N \end{cases}$$

- $COE(N) \ni V$, $V = UU^T$ $U \in U(N)$ Haar distributed

for systems with anti-unitary symmetry T

$$\exists T, \quad TV = V^{-1}T, \quad T^2 = +\mathbb{1}$$

$$K(t) = \int_{U(N)} dU \, |\mathrm{tr}(UU^T)^t| = \begin{cases} 2t - t\log\left(1 + \frac{2t}{N}\right) & ; t < N \\ 2N - t \log \frac{2t+N}{2t-N} & ; t \geq N \end{cases}$$

- $CSE(N) \ni V$, $V = UU^R$ $U \in U(2N)$ Haar distributed

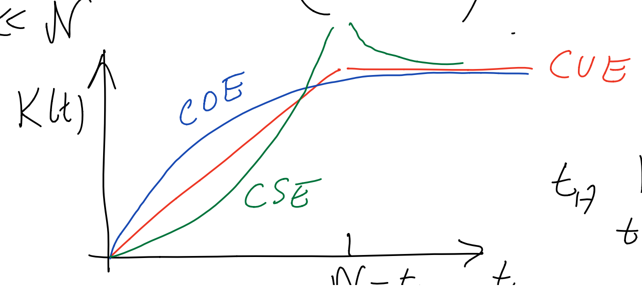
$$U^R = J U^T J$$

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$, \exists T, \; TV = V^{-1}T$

$T^2 = -\mathbb{1}$

$$K(t) \simeq \frac{t}{2} \quad t \ll N$$



t_H Heisenberg time

Semiclassical chaos or periodic orbit theory

$K(t)$ can be heuristically evaluated for classically chaotic and hyperbolic systems (with all orbits exp. unstable)

(Berry 1985 / Sieber-Richter 2001 / Müller et al 2004)

Floquet systems (finite but large $N \simeq (2\pi\hbar)^{-d}$

$$\operatorname{tr} U^t = \int_{q(0)=q(t)} \mathcal{D}q(t)\, e^{-iS[q]/\hbar}$$

$$= \sum_{p,\ \text{periodic orbits of length } t} A_p\, e^{-iS_p/\hbar}$$

$t > 0$

$$2\pi K(t) = \langle |\operatorname{tr} U^t|^2 \rangle$$

$$\sum_{p,p'} \langle A_p A_{p'}^* \, e^{-i(S_p - S_{p'})/\hbar} \rangle$$

Random phase argument (RPA):

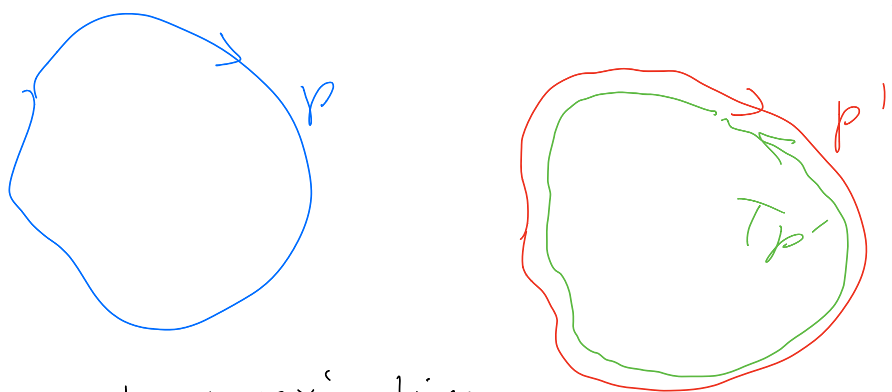
$$\langle e^{-i(S_p - S_{p'})/\hbar} \rangle = 0 \quad \text{unless} \quad S_p = S_{p'}$$

For TRI systems $p = p'$ or $p = Tp'$

For non-TRI systems $p = p'$

$S_p = S_{Tp}$

$A_p = A_{Tp}$



Diagonal approximation

$$2\pi K(t) = (2) \sum_{p(t)} |A_p|^2 = (2)\, t$$

Hannay-Ozorio de Almeida sum rule

<=> classical ergodicity

Next order: Sieber-Richter pairing

Some p.orbits come close to themselves before they actually close in phase space

SFF in spin chains

Example of non-interacting random spins (l-bits)

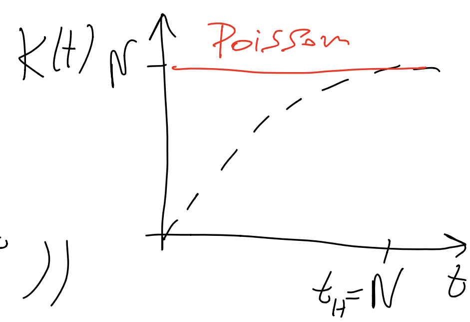
$\Rightarrow$ Poisson $K(t) = 2^L$

SFF for independent disordered $U(1)$ spins

$$U_h = e^{-ih\sigma^z} \qquad \underline{h} = (h_1, h_2, \dots h_L)$$

$$U = U_{h_1} \otimes U_{h_2} \otimes \dots \otimes U_{h_L} = \exp(-iH) \qquad H = \sum_{i=1}^{L} h_i \sigma_i^z$$

$$U^t = U_{h_1}^t \otimes U_{h_2}^t \otimes \dots \otimes U_{h_L}^t$$



$$\bar{K}(t) = \mathbb{E}_{\underline{h}}\left(\operatorname{tr} U^t \operatorname{tr} U^{-t}\right)$$

$$= \mathbb{E}_{\underline{h}}\left(\operatorname{tr}\left(U^t \otimes U^{-t}\right)\right)$$

$$= \left(\mathbb{E}_h\left(\operatorname{tr} e^{-ith\sigma^z} \otimes e^{ith\sigma^z}\right)\right)^L$$

$$= \mathbb{E}_h\left(e^{-2ith} + e^{2ith} + 1 + 1\right)^L$$

$$= 2^L \text{ if } t \neq 0, \qquad \mathbb{E}_h\left(e^{\pm 2ith}\right) \underset{t \neq 0}{=} 0$$

Kicked Ising models (long-ranged)

$$H(t) = H_I + \sum_{m\in\mathbb{Z}} \delta(t-m)\, H_K$$

$$H_I = \sum_{j=1}^{N} h_j \sigma_j^z + \sum_{j,j'=1}^{N} J_{j,j'} \sigma_j^z \sigma_{j'}^z + \sum_{j,j',j''=0}^{N} J'_{jj'j''} \sigma_j^z \sigma_{j'}^z \sigma_{j''}^z + \dots$$

$$H_K = b \sum_{j=1}^{N} \sigma_j^x$$

$$W = \exp(-i H_I)$$

$$U = W V \qquad V = \exp(-i H_K)$$

Nice structure in computational basis, σ_j^z eigenbasis

$$|\underline{s}\rangle = |s_1, \dots, s_N\rangle \qquad s_j \in \{0,1\}$$

$$\sigma_j^z |\underline{s}\rangle = (-1)^{s_j} |\underline{s}\rangle$$

$$W|\underline{s}\rangle = e^{-i\theta_{\underline{s}}} |\underline{s}\rangle \qquad \theta_{\underline{s}} = \sum_j h_j (-1)^{s_j} + \sum_{j,j'} J_{jj'} (-1)^{s_j + s_{j'}} + \dots$$

$$V = \begin{pmatrix} \cos b & i \sin b \\ i \sin b & \cos b \end{pmatrix}^{\otimes N}$$

Let us assume that phases $\theta_{\underline{s}}$ can be treated as pseudo-random (explained later!)

"Clean" (non-disordered) example which works well: (27)

$$H_I = h N_1 \sum_{j=1}^{N} \frac{\sigma_j^z}{j^\alpha} + J N_2 \sum_{j<j'} \frac{\sigma_j^z \sigma_{j'}^z}{(j'-j)^\alpha}$$

$$N_1 = \left(\sum_j \frac{1}{j^\alpha} \right)^{-1}$$

$$N_2 = N \left(\sum_{j<j'} \frac{1}{(j'-j)^\alpha} \right)^{-1}$$

} Kac normalization

$\alpha < 1$ effectively mean field

$\alpha > 2$ effectively short ranged

$\alpha \in [1, 2]$ interesting region

$\langle \mathrm{tr} U^t \, \mathrm{tr} U^{-t} \rangle$

↖ "slight" averaging over parameters $J, h, \ldots$

or, computing integrated SFF over time

$$K_{int}(t) = \sum_{t'=1}^{t} |\mathrm{tr} U^{t'}|^2$$

which is self-averaging for large t

(Ljubotina, Kos, Prosen, PRX 2018)

Now, let us write:

$$\mathrm{tr}\, U^t = \mathrm{tr}\, W V\, W V \cdots W V$$

$$= \sum_{\underline{s}_1, \ldots, \underline{s}_t} e^{i\theta_{\underline{s}_1}} V_{\underline{s}_1, \underline{s}_2}\, e^{i\theta_{\underline{s}_2}} V_{\underline{s}_2 \underline{s}_3} \cdots e^{i\theta_{\underline{s}_t}} V_{\underline{s}_t \underline{s}_1}$$

$$V_{\underline{s}, \underline{s}'} = \prod_{j=1}^{N} v_{s_j, s_j'} \qquad \begin{array}{l} v_{00} = v_{11} = \cos b \\ v_{01} = v_{10} = -i \sin b \end{array}$$

$$K(t) = \sum_{\underline{s}_1 \underline{s}_2 \cdots \underline{s}_t} \sum_{\underline{s}_1' \underline{s}_2' \cdots \underline{s}_t'} \left\langle e^{i(\theta_{\underline{s}_1} + \cdots + \theta_{\underline{s}_t} - \theta_{\underline{s}_1'} - \cdots - \theta_{\underline{s}_t'})} \right\rangle \times$$

$$\times \prod_{j=1}^{N} \prod_{\tau=1}^{t} v_{s_{j,\tau}, s_{j,\tau+1}}\, \bar{v}_{s'_{j,\tau}, s'_{j,\tau+1}}$$

Key assumption: RPA

$$\left\langle e^{i(\theta_{\underline{s}_1} + \cdots + \theta_{\underline{s}_t} - \theta_{\underline{s}_1'} - \cdots - \theta_{\underline{s}_t'})} \right\rangle = \delta_{\langle \underline{s}_1 \underline{s}_2 \cdots \underline{s}_t \rangle, \langle \underline{s}_1' \underline{s}_2' \cdots \underline{s}_t' \rangle}$$

$$+ \text{ fluctuations}$$

iff all 2^L phases $\theta_{\underline{s}}$ were i.i.d. uniform on $[0, 2\pi)$ then the above is exact (fluctuations $= 0$) and we refer to this as <u>random phase model</u> (RPA)

For $t \ll 2^L = t_H$ probability of repetitions in the sequence

$\underline{s}_1 \, \underline{s}_2 \, \cdots \, \underline{s}_t$ can be neglected

and

$$K(t) = \sum_{\pi \in S_t} \left(Z_\pi \right)^L$$

1D twisted Ising model

$$Z_\pi = \sum_{\underline{s} \in \{0,1\}^t} v_{s_1 s_2} v_{s_2 s_3} \cdots v_{s_t s_1} \bar{v}_{s_{\pi_1} s_{\pi_2}} \bar{v}_{s_{\pi_2} s_{\pi_3}} \cdots \bar{v}_{s_{\pi_t} s_{\pi_1}}$$

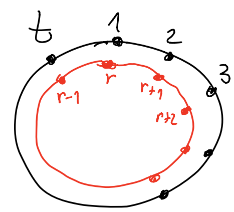
$$= \sum_{\underline{s}} (-1)^{w_{\underline{s}} - w_{\pi(\underline{s})}} (\sin b)^{w_{\underline{s}} + w_{\pi(\underline{s})}} (\cos b)^{t - w_{\underline{s}} - w_{\pi(\underline{s})}}$$

$w_{\underline{s}} = \frac{1}{2} \sum_{\tau=1}^{t} (1 - \delta_{s_\tau s_{\tau+1}})$ half number of domain walls on a periodic string

Dominant contributions:

$$\pi_{cyc} = \begin{pmatrix} 1 & 2 & \cdots & t \\ r & r+1 & \cdots & r-1 \end{pmatrix}$$

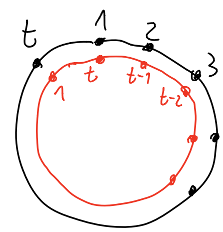
t cyclic permutations



t cyclic

or inversion

$$\pi_{inv} = \begin{pmatrix} 1 & 2 & \cdots & t \\ t & t-1 & \cdots & 1 \end{pmatrix}$$



t anti-cyclic

or $\pi_{cyc} \circ \pi_{inv}$

2t such permutations map neighbours to neighbours and hence

$$Z_\pi = Z_{id} \quad (\text{Ising p.f.})$$

From these we get the leading contribution

$$K(t) \simeq 2t \left(Z_{id}\right)^L$$

$$Z_{id} = \sum_{\underline{s}} |v_{s_1 s_2}|^2 |v_{s_2 s_3}|^2 \cdots |v_{s_t s_1}|^2$$

$$= \operatorname{tr} \begin{pmatrix} \cos^2 b & \sin^2 b \\ \sin^2 b & \cos^2 b \end{pmatrix}^t$$

$$= \operatorname{tr} T^t \qquad T^t = \begin{pmatrix} \frac{1}{2}(1+x^t) & \frac{1}{2}(1-x^t) \\ \frac{1}{2}(1-x^t) & \frac{1}{2}(1+x^t) \end{pmatrix}$$

$$= 1 + x^t$$

$$x = \cos 2b$$

$$K(t) \simeq 2t \left(1 + x^t\right)^L \simeq 2t\, e^{-L x^t} \quad ; \; x^t \ll 1$$

$$K(t) \longrightarrow 2t$$

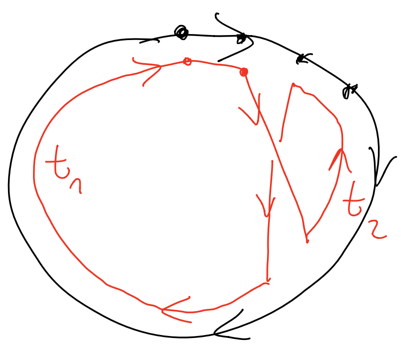
for $t > t^*$

$$L x^{t^*} \simeq 1 \qquad t^* = \frac{\log N}{\log(1/x)} = -\frac{\log N}{\log \cos 2b}$$

Ehrenfest / Thouless time / Ramp starts here

Systematic corrections:

It can be shown that the leading corrections come from permutations with minimal number (2) of deffects (neighbourship breaking points), X-diagrams

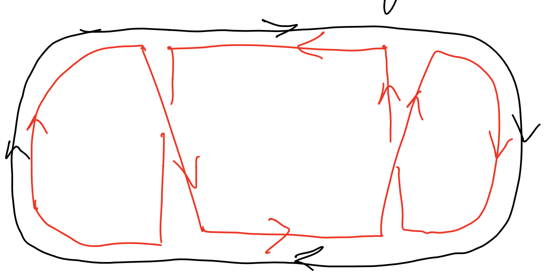


$t_1 + t_2 + 2 = t$

$$Z_X = \frac{1}{2}\left(1 + x^{t_1} + x^{t_2} - x^{t-2} + x^{t}\right)$$

↑ gives exponentially subdominant contributions to Z_X^L w.r.t. dominant terms Z_{id}^L, for most t_1, t_2, as t grows

Next come 2X diagrams:



etc ...

Locally Interacting Kicked Ising Model

$$H_{KI}[\underline{h};t] = H_I[\underline{h}] + \delta_p(t)\, H_K$$

$$H_I[\underline{h}] = \sum_{j=1}^{L} \{ J\, \sigma_j^z \sigma_{j+1}^z + h_j \sigma_j^z \} \qquad \vec{\sigma}_{L+1} \equiv \vec{\sigma}_1$$

$$H_K = b \sum_{j=1}^{L} \sigma_j^x \qquad \underline{h} = (h_1, h_2, \ldots, h_N)$$

$$\delta_p(t) = \sum_{m \in \mathbb{Z}} \delta(t-m)$$

$$U_{KI}[\underline{h}] = e^{-iH_K}\, e^{-iH_I[\underline{h}]}$$

$$K(t) = |\operatorname{tr} U_{KI}^t[\underline{h}]|^2 \qquad t > 0,$$ averaging shall be specified later

$$\operatorname{tr} U_{KI}^t[\underline{h}] = \sum_{\{\underline{s}_\tau\}} \prod_{\tau=1}^{t} \langle \underline{s}_{\tau+1} | e^{-iH_K} e^{-iH_I[\underline{h}]} | \underline{s}_\tau \rangle$$

$$\underline{s}_{t+1} \equiv \underline{s}_1$$

$$= \sum_{\{\underline{s}_\tau\}} \prod_{\tau=1}^{t} \left(\prod_{j=1}^{L} v_{s_{\tau,j},\, s_{\tau+1,j}}\; e^{-i \sum_{j=1}^{L} (J s_{\tau,j} s_{\tau,j+1} + h_j s_{\tau,j})} \right)$$

Now we write $s_{\tau j} \in \{+1, -1\}$ $\quad \sigma_j^z |\underline{s}\rangle = s_j |\underline{s}\rangle$

$$v_{+,+} = v_{-,-} = \cos b$$
$$v_{+,-} = v_{-,+} = -i \sin b$$

$$\operatorname{tr} U_{KI}^{t}[\underline{h}] = \left(\frac{\sin 2b}{2i}\right)^{\frac{Lt}{2}} \sum_{\{s_{\tau,j}\}} e^{-i\mathcal{E}[\{s_{\tau,j}\},\underline{h}]}$$

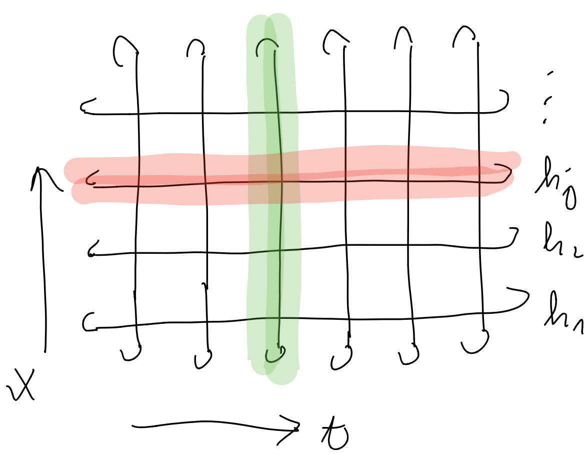
$$\mathcal{E}[\{s_{\tau,j}\},\underline{h}] = \sum_{\tau=1}^{t}\sum_{j=1}^{L}\left(J s_{\tau,j} s_{\tau,j+1} + \tilde{J} s_{\tau,j} s_{\tau+1,j} + h_j s_{\tau,j}\right)$$

$$\tilde{J} = -\frac{\pi}{4} - \frac{i}{2}\log\tan b$$

Partition function of a classical 2D Ising model on a rectangular $t \times L$ lattice and with stripe modulated field (and complex weights)

$$U_{KI}[\underline{h}; J, b, L]$$

$$J = -\frac{\pi}{4} - \frac{i}{2}\log\tan\tilde{b}$$



$$\tilde{U}_{KI}[h_j;\underline{\epsilon}] \equiv U_{KI}[h_j\underline{\epsilon}; \tilde{J}, \tilde{b}, t]$$

$$\underline{\epsilon} = (\overbrace{1, 1, \ldots, 1}^{t})$$

"dual" transfer matrix $\tilde{U}_{KI}$ has exactly the same algebraic form but for periodic chain of t sites, and $J, \tilde{J}$ exchanged

Duality formula

$$\mathrm{tr}\, U_{KI}^{t}[\underline{h}] = \mathrm{tr} \prod_{j=1}^{L} \tilde{U}_{KI}[h_j, \underline{t}]$$

$\tilde{J} = \tilde{J}(J, b)$

$\tilde{b} = \tilde{b}(J, b)$

at points $J = \pm \frac{\pi}{4}$, $b = \mp \frac{\pi}{4}$ both

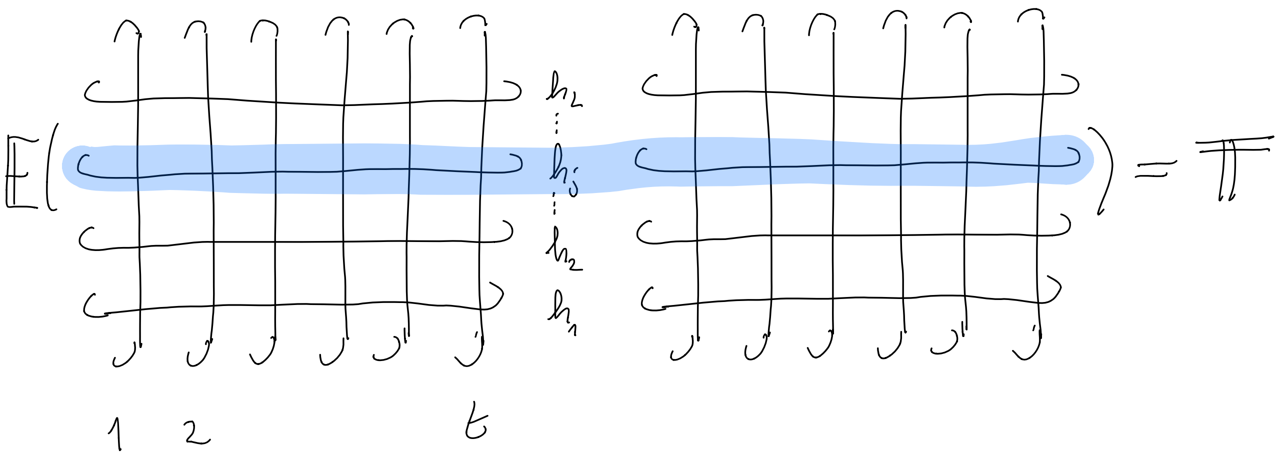
U_{KI} & $\tilde{U}_{KI}$ are unitary

We call these regime self-dual

Ensemble (disordered) averaged SFF

Assume that h_j are i.i.d random variables with as yet unspecified distribution

$\mathrm{tr}\, U_{KI}^{t}$ $\qquad$ $\left(\mathrm{tr}\, U_{KI}^{t}\right)^{*}$



$$\bar{K}(t) = \mathbb{E}_{\underline{h}}\left(\mathrm{tr}\, U_{KI}^{t}[\underline{h}]\ \mathrm{tr}\, U_{KI}^{*t}[\underline{h}]\right)$$

$$= \mathbb{E}_{\underline{h}}\left(\mathrm{tr}\prod_{j=1}^{L}\tilde{U}_{KI}[h_j\,\underline{\epsilon}]\ \mathrm{tr}\prod_{j=1}^{L}\tilde{U}^{*}_{KI}[h_j\,\underline{\epsilon}]\right)$$

$$= \mathbb{E}_{\underline{h}}\left[\mathrm{Tr}\prod_{j=1}^{L}\left(\tilde{U}_{KI}[h_j\,\underline{\epsilon}]\otimes\tilde{U}^{*}_{KI}[h_j\,\underline{\epsilon}]\right)\right]$$

$$= \mathrm{Tr}\prod_{j=1}^{L}\mathbb{E}_{h_j}\left(\tilde{U}_{KI}[h_j\,\underline{\epsilon}]\otimes\tilde{U}^{*}_{KI}[h_j\,\underline{\epsilon}]\right)$$

$$= \mathrm{Tr}\ \mathbb{T}^{L}$$

where

$$\mathbb{T} = \mathbb{E}_{h}\left(\tilde{U}_{KI}[h\,\underline{\epsilon}]\otimes\tilde{U}^{*}_{KI}[h\,\underline{\epsilon}]\right)$$

Consider for simplicity a Gaussian average with mean $\bar{h}$ and standard deviation σ

$$\mathbb{E}_h(f(h)) = \int_{-\infty}^{\infty} f(h)\, e^{-\frac{(h-\bar{h})^2}{2\sigma^2}} \frac{dh}{\sqrt{2\pi}\,\sigma}$$

$$\mathbb{T} = \mathbb{E}_h\left(\tilde{U}_{KI}[\theta]\, e^{-ihM_z} \otimes \tilde{U}^*_{KI}[\theta]\, e^{ihM_z} \right)$$

$$= \left(\tilde{U}_{KI}[\theta] \otimes \tilde{U}^*_{KI}[\theta]\right) \mathbb{E}_h\left(e^{-ihM_z} \otimes e^{ihM_z} \right)$$

$$M_z = \sum_{\tau=1}^{t} \sigma^z_\tau$$

$$\mathbb{E}_h\left(e^{-ihM_z} \otimes e^{ihM_z} \right) = \mathbb{E}_h\left(e^{-ih(M_z \otimes \mathbb{1} - \mathbb{1} \otimes M_z)} \right)$$

$$= e^{-\frac{1}{2}\sigma^2 (M_z \otimes \mathbb{1} - \mathbb{1} \otimes M_z)^2 - i\bar{h}(M_z \otimes \mathbb{1} - \mathbb{1} \otimes M_z)}$$

$$\Rightarrow \boxed{\begin{aligned} \mathbb{T} &= \left(\tilde{U}_{KI}[\bar{h}\underline{\theta}] \otimes \tilde{U}^*_{KI}[\bar{h}\underline{\theta}]\right) \mathbb{D}_\sigma \\ \mathbb{D}_\sigma &= e^{-\frac{1}{2}\sigma^2 (M_z \otimes \mathbb{1} - \mathbb{1} \otimes M_z)^2} \end{aligned}}$$

$\mathbb{D}_\sigma$ is a contraction on $\mathcal{H}_t \otimes \mathcal{H}_t$ where $\mathcal{H}_t = (\mathbb{C}^2)^{\otimes t}$

SFF in TDL

$$\lim_{L\to\infty} \bar{K}(t) = \lim_{L\to\infty} \operatorname{tr} \mathbb{T}^L$$

amounts to determining multiplicity of eigenvalue 1 proving that the rest of the spectrum is gapped inside unit disk.

We achieve that by proving a "number" of nice properties of $\mathbb{T}$:

Property 1:

i) Eigenvalues of $\mathbb{T}$ have at most unit magnitude, and eigenvectors corresponding to uni-modular eigenvalues are simultaneous eigenvectors of $\mathbb{D}_6$ and $\tilde{U}_{KI} \otimes \tilde{U}_{KI}^*$.

ii) Geometric and algebraic multiplicity of any eigenvalue of magnitude 1 coincide.

Proof: i) $\mathbb{T}$ is a product of a unitary and a contraction, hence it is a contraction

For any normalized eigenvector $|A\rangle$ of eigenvalue $e^{i\phi}$, we have

$$1 = \langle A|\mathbb{T}^\dagger \mathbb{T}|A\rangle = \langle A|\mathbb{D}_6^2|A\rangle$$

$$= \sum_n |\langle A|n\rangle|^2 \sigma_{6,n}$$

$\mathbb{D}_6 |n\rangle = \sigma_{6,n}|n\rangle$

$0 < \sigma_{6,n} \leq 1$, $\sigma_{6,n} \in \mathbb{R}$

and $\sum_n |\langle A|n\rangle|^2 = 1$

$$\Rightarrow |A\rangle = \sum_{\sigma_{6,n}=1} c_n |n\rangle$$

$$\Rightarrow \mathbb{D}_6 |A\rangle = |A\rangle$$

ii) We prove by showing contradiction with assuming that $\exists$ nontrivial Jordan block of eigenvalue $e^{i\phi}$

$$\Rightarrow \begin{cases} (\tilde{U}_{KI} \otimes \tilde{U}^*_{KI}) |A\rangle = e^{i\phi} |A\rangle \\ (M_z \otimes \mathbb{1} - \mathbb{1} \otimes M_z) |A\rangle = 0 \end{cases}$$

unvectorization:

$\{|n\rangle\}$ basis of $\mathcal{H}_t$

$$|A\rangle = \sum_{n,m} A_{n,m} |n\rangle \otimes |m\rangle^* \in \mathcal{H}_t \otimes \mathcal{H}_t$$

$$\Updownarrow$$

$$A := \sum_{n,m} A_{n,m} |n\rangle\langle m| \in \mathrm{End}(\mathcal{H}_t)$$

$$\boxed{\begin{array}{c} \tilde{U}_{KI} A \tilde{U}^\dagger_{KI} = e^{i\phi} A \\ [M_z, A] = 0 \end{array}}$$

$$M_\alpha = \sum_{\tau=1}^{t} \sigma^\alpha_\tau$$

<u>Property 2:</u>

The boxed relations are equivalent to:

$$U A U^\dagger = e^{i\phi} A$$

$$[M_\alpha, A] = 0 \qquad \alpha \in \{x, y, z\}$$

$$U = \exp\left(i \frac{\pi}{4} \sum_{\tau=1}^{t} (\sigma^z_\tau \sigma^z_{\tau+1} - \mathbb{1})\right)$$ parity of half-numbers of domain walls

Since: $U^2 = \mathbb{1} \Rightarrow \phi \in \{0, \pi\}$

<u>Property 3:</u>

For odd t: $\phi = 0$ (only eigenvalue 1 possible)
hence all other eigenvalues
strictly inside unit disk!

$\Rightarrow$ <u>For odd t:</u>

$$\lim_{L\to\infty} \overline{K}(t) = \dim\{U, M_x, M_y, M_z\}'$$

<u>Theorem:</u> Any element of $\{U, M_x, M_y, M_z\}'$
(commutant algebra) is of the form

$$A = \sum_{\tau=0}^{t-1} \sum_{n=0}^{1} a_{\tau,n} \Pi^{\tau} R^{n} \qquad (*)$$

where $\Pi = \prod_{\tau=1}^{t-1} P_{\tau,\tau+1}$ $\quad R = \prod_{\tau=1}^{[t/2]} P_{\tau, t+1-\tau}$

generate Dihedral group D_{2t}

where $P_{j,k} = \frac{1}{2}(\mathbb{1} + \vec{\sigma}_j \cdot \vec{\sigma}_k)$ is
a transposition in $\mathcal{H}_t$

$$\Rightarrow \boxed{\lim_{L\to\infty} \overline{K}(t) = \begin{cases} 2t-1 \;;\; t \leq 5 \\ 2t \;;\; t \geq 7 \end{cases} \quad t \text{ odd}}$$

The idea of the proof is to show that there is no other operator in $\{U, M_\alpha\}'$ which is not of <u>the form (*)</u>.

For <u>even</u> t we generally find <u>one</u> extra operator in $\{U, M_\alpha\}'$ which is <u>not</u> in the form yielding a conjecture

$$\boxed{\lim_{L\to\infty} \overline{K}(t) = 2t+1 \;;\; t \text{ even } \& \; t \geq 12}$$

(Bertini, Kos, Prosen, PRL 2018)