Spectral statistics and many-body localization

Jan Šuntajs, Lev Vidmar, Janez Bonča

September 16, 2019

Jožef Stefan Institute, Ljubljana, Slovenia
Many-body localization (MBL) - what is it about?

1. Occurring in **INTERACTING** quantum systems with **DISORDER**

2. An **IDEAL INSULATOR** → at **ANY** temperature

3. Explains the **FAILURE** of some systems to **THERMALIZE**
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The properties of MBL systems

Introduction of the physical model

Numerical analysis of the spectral statistics
  - A brief introduction to spectral statistics
  - Spectral form factor (SFF)

Our recent results and conclusion
The outline of the presentation

1. The properties of MBL systems
2. Introduction of the physical model
3. Numerical analysis of the spectral statistics
   - A brief introduction to spectral statistics
   - Spectral form factor (SFF)
4. Quantum chaos challenges many-body localization

Jan Šuntajs,\textsuperscript{1} Janez Bonča,\textsuperscript{2,1} Tomaž Prosen,\textsuperscript{2} and Lev Vidmar\textsuperscript{1}

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Closed quantum systems

Nandkishore, Huse, 2015
- **Closed** quantum systems

Presence of interactions

Nandkishore, Huse, 2015
MBL - a quick recap

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- Presence of **interactions**
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The properties of MBL systems

- The absence of ergodicity

Abanin, Altman, Bloch, Serbyn, 2018
The properties of MBL systems

- The **absence of ergodicity**

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- **THE ENTANGLEMENT ENTROPY:**
  - Area law scaling for all eigenstates
  - Logarithmic growth in time
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THE ENTANGLEMENT ENTROPY:
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Many-body localization in the Heisenberg XXZ magnet in a random field

Marko Žnidarič,1 Tomaž Prosen,1 and Peter Prelovšek1,2
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(Received 31 August 2007; revised manuscript received 8 November 2007; published 25 February 2008)

Special properties of the energy spectra
- The subject of our numerical analysis
The properties of MBL systems

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The model

The hamiltonian - paradigmatic ‘quantum chaotic’/ergodic:

Heisenberg-like 1D

\[ \hat{H} = \sum_{j \in \{1,2\}} J_j \sum_{\ell} \left( \hat{s}_\ell^x \hat{s}_{\ell+j}^x + \hat{s}_\ell^y \hat{s}_{\ell+j}^y + \Delta_j \hat{s}_\ell^z \hat{s}_{\ell+j}^z \right) + \sum_{\ell=1}^{L} w_\ell \hat{s}_\ell^z \]

- \( l \) - site, \( L \) - chain length

- \( w_\ell \): randomly disordered potential
The model

\[ J_1 = J_2 = 1 \]

\[ \Delta_1 = \Delta_2 = 0.55 \]

Disorder probability distribution:

\[ p(w) \]

\[ W \] - the disorder strength parameter
Our (numerical) analysis of the MBL systems

- We perform **full** or **partial** diagonalization of the Hamiltonians
  
  **Spectrum:** \( \{ E_1 \leq E_2 \leq \cdots \leq E_D \} \)

- **Partial diagonalization:** \( \approx 500 \) eigenstates from the middle of the spectra

- Maximum Hilbert space dimensions:
  
  \( D = 48620 \) (full)
  
  \( D = 184756 \) (partial)

- Between \( 10^2 - 10^3 \) different disorder realizations for each model parameter
Why do we study energy spectra?

- Quantum chaos conjecture (Bohigas, Giannoni, Schmidt, 1984):
  - Quantum systems
  - Spectral properties match the predictions of the random matrix theory (RMT).
  - Corresponding classical systems
  - The dynamics are completely chaotic.
What about systems without a classical analogue?
Quantum chaos and energy spectra

- Many-body quantum chaos
  - GENERIC systems $\rightarrow$ RMT-like spectral statistics
    - Montambaux et. al. (1993), Prosen (1999), Santos and Rigol (2008)

- RMT statistics $\rightarrow$ hallmarks of **ergodicity** and **thermalization** in an isolated quantum system.
  - D’Alessio, Kafri, Polkovnikov, Rigol (2016)
We analyse the **statistical properties** of the energy spectra.

We rely on the findings of the **RMT**: 

- **Ergodic** systems: spectral statistic match the **Gaussian orthogonal ensemble (GOE)**
- **MBL** systems: nearest levels distributed in accordance with the **Poisson** distribution

We compare our **RESULTS** with the above cases.
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We compare our **RESULTS** with the **above cases**.
The mean ratio of the level spacings

- The spacings between the **nearest** energy levels:
  \[
  \delta_n = E_{n+1} - E_n \geq 0
  \]

- We define the **level spacing ratio**:
  \[
  0 \leq \tilde{r}_n = \min\{\delta_n, \delta_{n-1}\} / \max\{\delta_n, \delta_{n-1}\} \leq 1
  \]

**KEYNOTE:** the limiting values of \( \langle \tilde{r} \rangle \) are well known:

- **Ergodic:** \( \langle \tilde{r} \rangle_{\text{GOE}} = 0.5307 \)
- **MBL:** \( \langle \tilde{r} \rangle_{\text{P}} = 2 \ln 2 - 1 \approx 0.3863 \)
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The mean ratio of the level spacings

\[\langle r \rangle \]

\(W\)

\(L = 12\) \(L = 14\) \(L = 16\) \(L = 18\) \(L = 20\)

GOE

Poisson
Pros and cons of $\langle \tilde{r} \rangle$

Mean level spacings ratio:

- a commonly used indicator of a given system’s ergodicity
  + straightforward implementation
  - only considers correlation between the nearest energy levels

We would like to consider correlations between all the levels

- This is why we implement the spectral form factor (SFF)
  - the implementation is more demanding
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The spectral form factor (SFF)

Definition:

\[ K(\tau) := \left\langle \frac{1}{D} \sum_{i,j} e^{-i(\varepsilon_i - \varepsilon_j)\tau} \right\rangle; \quad K(0) = D, \quad K(\tau \to \tau_H) = 1 \]

**D** - Hilbert space dimension \( \tau \to \) an external parameter

\( \langle \ldots \rangle \) over disorder realizations

Heisenberg time \( \tau_H \propto \) inverse mean level spacing
(largest sensible timescale of a system)

\( \{ \varepsilon_i \} \to \) energy levels after spectral unfolding
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\( \{\varepsilon_i\} \to \) energy levels after *spectral unfolding*
The spectral form factor (SFF)

- A quick introduction to unfolding

![Spectral plots](image-url)
The spectral form factor (SFF)

- **Unfolding** → mean level spacing $= 1$

![Spectral plots](image)
SFF - KEYNOTES:

- we again expect different behaviour for ergodic and uncorrelated spectra

- we investigated the behaviour of the Thouless time $\tau_{Th}$

- Thouless time $\tau_{Th} \rightarrow$ the onset of UNIVERSAL DYNAMICS
The spectral form factor (SFF)

\[ K(\tau) \]

- \( W = 0.5 \)
- \( W = 8.0 \)
- \( K_{\text{GOE}}(\tau) \)

MBL

ERG

\[ \tau_{\text{Th}} \]

\[ \tau_{H} \]
The spectral form factor (SFF)
The spectral form factor (SFF)

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- \( W = 8.0 \)

\[ K_{\text{GOE}}(\tau) \]

- MBL
- ERG

DECAY

RAMP

\[ \tau_{\text{Th}} \]

\[ \tau_{\text{H}} \]
The spectral form factor (SFF)

We see universal behaviour after some transient time $\tau_{Th}$

$K(\tau)$ for uncorrelated spectra

$$K_P(\tau) = 1$$

$K(\tau)$ in ergodic systems

$$K_{\text{GOE}}(\tau) = 2 - \tau \log \left( \frac{2\tau + 1}{2\tau - 1} \right)$$
**Thouless time** $\tau_{\text{Th}}$: determines the energy scale at which the spectral correlations are universally determined by the GOE predictions (e.g., when the RAMP appears)

- large($r$) $\tau_{\text{Th}} \rightarrow$ small(er) spectral correlation length $E_{\text{Th}}$
Thouless time $\tau_{\text{Th}}$: determines the energy scale at which the spectral correlations are universally determined by the GOE predictions (e.g., when the RAMP appears)

Large(r) $\tau_{\text{Th}} \rightarrow$ small(er) spectral correlation length $E_{\text{Th}}$
We set out to find out different scalings

- How does $\tau_{Th}$ scale with the system size $L$?

- How does $\tau_{Th}$ scale with disorder strength parameter $W$?
We noticed some surprising results along the way

Quantum chaos challenges many-body localization

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arXiv:1905.06345 [cond-mat.str-el], 15 May 2019
We first checked for consistency of the $K(\tau)$ and $\langle \tilde{r} \rangle$ results.
We then numerically extracted $\tau_{Th}$ values and performed a scaling analysis w.r.t. both $L$ and $W$.

In the subsequent scaling analysis, we introduce the **physical Thouless time** $t_{Th}$, rescaling $\tau_{Th}$ by the mean level spacing $\delta E$ of the raw spectra:

$$t_{Th} = \tau_{Th}/\delta E$$
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$$t_{Th} = \tau_{Th}/\delta E$$
The results of the scaling analysis

We obtain the following double scaling:

\[ t_{Th} = t_0 e^{W/\Omega} L^2 \]
Verification of the scaling: data collapse

\( W = 2.0 \)

\( L = 12 \)

\( t_{\text{HH}}/L^2 \)

\( K(\tau) \tau H e^{-W/\Omega} \)

\( W = 3.5 \)

\( L = 18 \)

\( \tau t_{\text{HH}} e^{-W/\Omega} \)

Nonuniversal

Universal
... Is there a MBL transition at all?

- a **SUPPOSED** MBL transition occurs for some **CRITICAL** disorder $W^*$ when the energy spectrum becomes uncorrelated:

  $$ t_{\text{Th}}(W^*) = t_H $$

- $t_H$ scaling is given by:

  $$ t_H \propto \exp(L \ln 2) $$

- combining these results gives us

  $$ W^* \approx \Omega \ln(2) L \propto L $$

- **IMPLIES ABSENCE OF MBL IN THE THERMODYNAMIC LIMIT!**
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SFF - results

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- We tested our conclusions against more commonly used statistics.
- However, we interpreted our results DIFFERENTLY.
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![Graph showing spectral statistics and MBL](image-url)
Verification of our results

- **BOTTOM LINE:** results for different spectral statistics seem consistent
Conclusions and further work

- Our results for the **PARADIGMATIC** class of models expected to give **MBL** show **NO INDICATIONS** of the MBL transition.

- The emergence of **QUANTUM CHAOS** for **ANY** disorder strength in the TD limit.
Conclusions and further work

- We need to test our assumptions on other models in which MBL is predicted.

- We need to examine and better understand the relationship between our results and the transport properties.
Thank you for your attention!