Prethermalization beyond high-frequency regime

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with my former master student **Victor Verreet**
====> soon (?) on arxiv

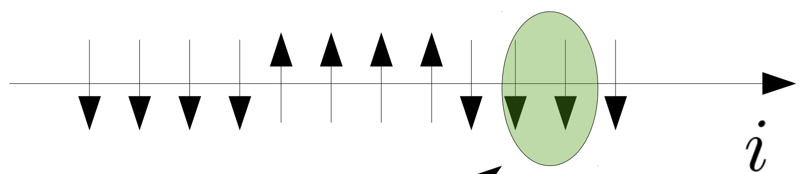
(….. waiting for numerics)

Mathematical conventions

- local (many-body) Hamiltonians H spin chain of length L

$$H = \sum_{i=1}^{L} h\sigma_i^z + J\sigma_i^x\sigma_{i+1}^x$$

$$\mathcal{H}_L = (\mathbb{C}^2)^{\otimes L} \qquad \sigma_i^z = 1 \otimes \ldots \otimes \sigma^z \otimes \ldots \otimes 1$$



site i

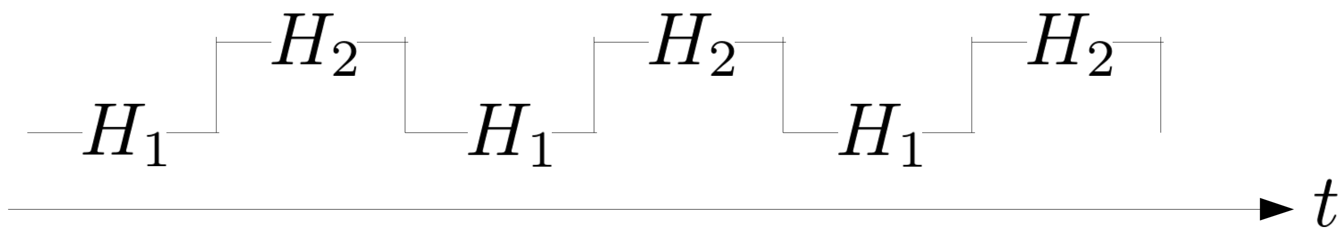
local (operator) O

- Norm of Hamiltonian $||H|| \sim L$

- More useful is “local norm” $H = \mathcal{O}_{\mathrm{loc}}(2|J| + |h|)$

Thermodynamic intuition

- local (many-body) Hamiltonians H_1, H_2 chain of length L



- Evolution after $t = n$ $\quad U(n) \equiv U^n, \qquad U = e^{-iH_2} e^{-iH_1}$

…… should heat up to infinite temp.

$$\lim_{L} \langle O(t) \rangle \underset{t \to \infty}{\to} \lim_{L} \mathrm{tr}(O) \qquad \text{for local } O$$

- Possible obstruction:

emergent local Ham H_{E}

$$U H_{\mathrm{E}} U^* = H_{\mathrm{E}} + \mathcal{O}_{\mathrm{local}}(\epsilon)$$

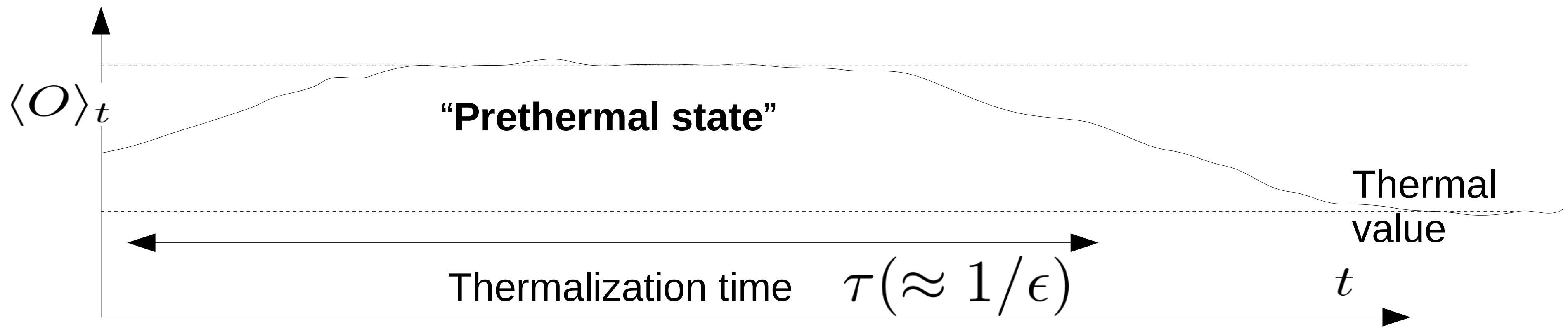
$$U^n H_{\mathrm{E}} U^{-n} = H_{\mathrm{E}} + \mathcal{O}_{\mathrm{loc}}(n\epsilon)$$

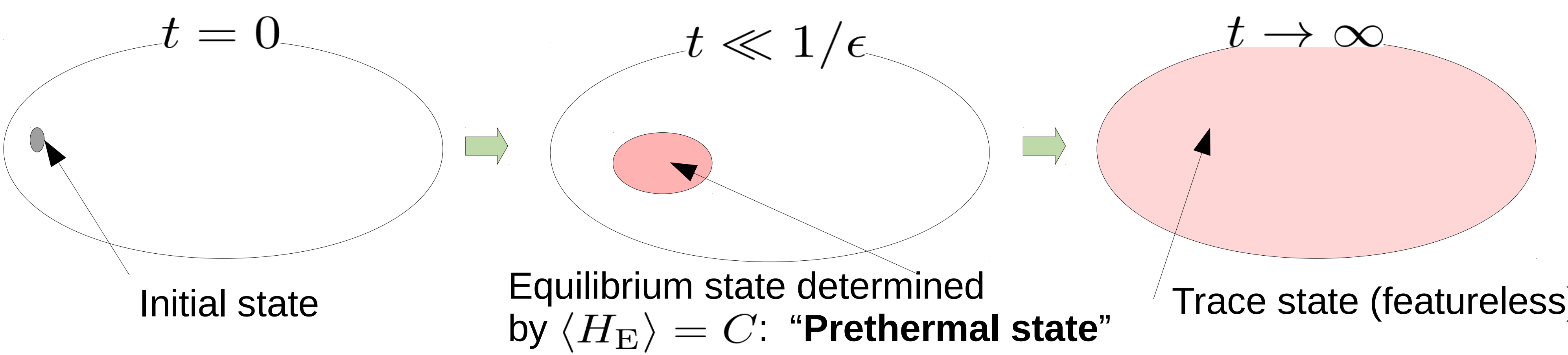
Obstruction…but usually also prethermalization

Possible obstruction:

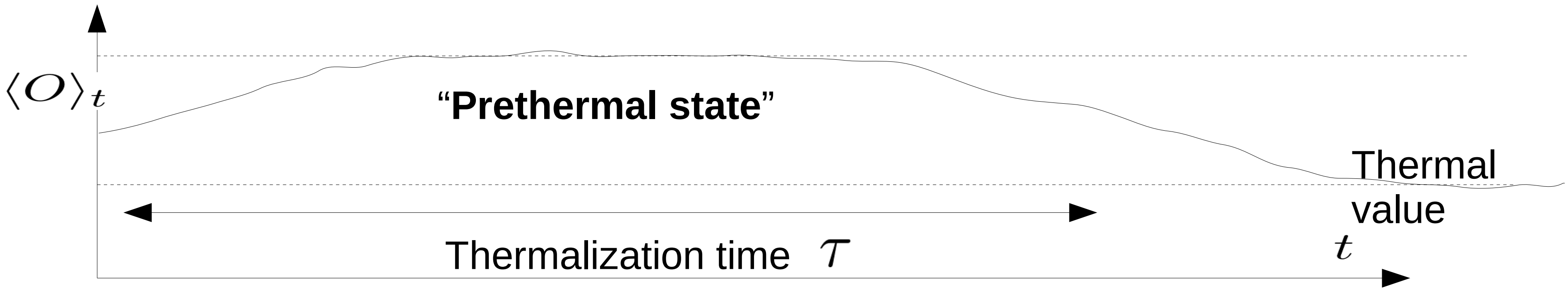
emergent local Ham H_E

$$\boxed{\begin{aligned} U H_\mathrm{E} U^* &= H_\mathrm{E} + \mathcal{O}_\mathrm{local}(\epsilon) \\ U^n H_\mathrm{E} U^{-n} &= H_\mathrm{E} + \mathcal{O}_\mathrm{loc}(n\epsilon) \end{aligned}}$$





Obstruction…but usually also prethermalization



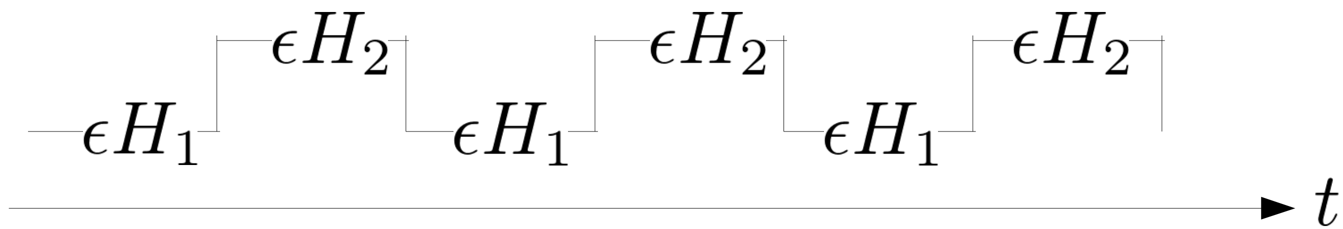
- **Prethermal state**: "Quasi-stationary Noneq state" (Berges, Gasenzer, 2008-...)

- Phenomenon in near-integrable systems $H = H_{\text{integrable}} + gV$
  - $\tau \sim 1/g^2$
  - Mechanism not H_{E} but simply Fermi Golden Rule
  - Not to be discussed here

- This talk: situations where $\exists\, H_{\mathrm{E}}$ and $\tau = \mathcal{O}(g^{-\infty})$

- Only the obstruction is sometimes rigorous, **not** the thermalization and prethermalization (but Kos, Bertini, Prosen 2018)

Simplest example of obstruction: high frequency



$$U = e^{-i\epsilon H_2} e^{-i\epsilon H_1} \stackrel{?}{=} e^{-i\epsilon(H_2+H_1)+i\mathcal{O}_{\mathrm{loc}}(\epsilon^2)}$$

Baker-Campbell-Hausdorf?
No, converges only for $||H_i|| \sim 1$

Still, can construct $\quad H_{\mathrm{E}} = H_{\mathrm{eff}} = \epsilon(H_1 + H_2) + \mathcal{O}_{\mathrm{loc}}(\epsilon^2)$

$$\boxed{U H_{\mathrm{eff}} U^* = H_{\mathrm{eff}} + \mathcal{O}_{\mathrm{loc}}(e^{-1/\epsilon})}$$

Exponentially slow heating! $\boxed{\tau \sim e^{1/\epsilon}}$

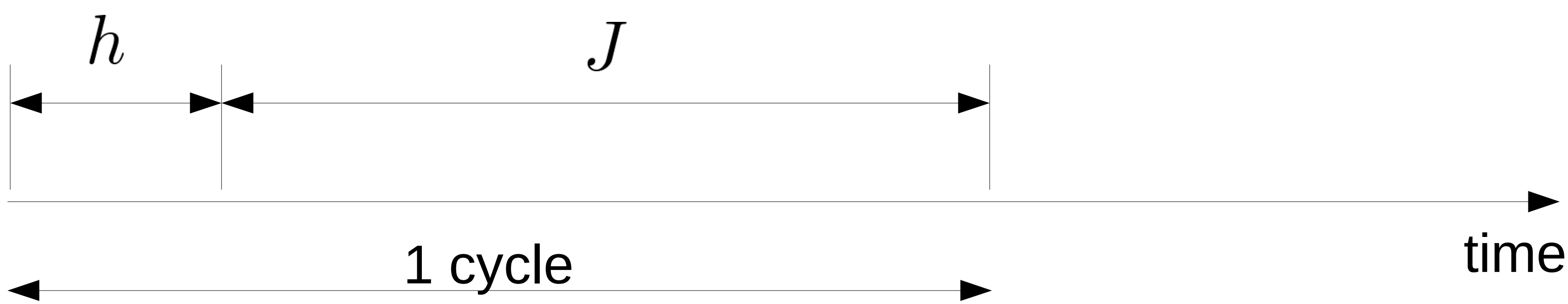
(Magnus, …..D'Alesio et al,….. Rigourous 2017: Kuwahara et al, Abanin et al)

Motivation for this work

Replica resummation of the Baker-Campbell-Hausdorff series
(Vajna, Klobas, Prosen, Polkovnikov, PRL 2018)

Kicked many-body model: One-cycle unitary is

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i(\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$



- $|h| + |J| \ll 1$ =====> High-frequency regime

  Exponentially Slow heating

- $|h| \ll |J| \sim 1$ =====> Moderate frequency but weak driving

  ?????????

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

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Exponentially Slow heating !

- $|J| \ll |h| \sim 1$ =====> Moderate frequency but weak driving

Exponentially Slow heating !

Numerics and Replica Resummation suggest

$$U^* H_{\mathrm{eff}} U = H_{\mathrm{eff}} + \mathcal{O}_{\mathrm{loc}}(e^{-1/\sqrt{\epsilon}}), \quad \epsilon = h \text{ or } J$$

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \, e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

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  Exponentially Slow heating !

Weak driving always gives exponentially slow heating? **No, in general**

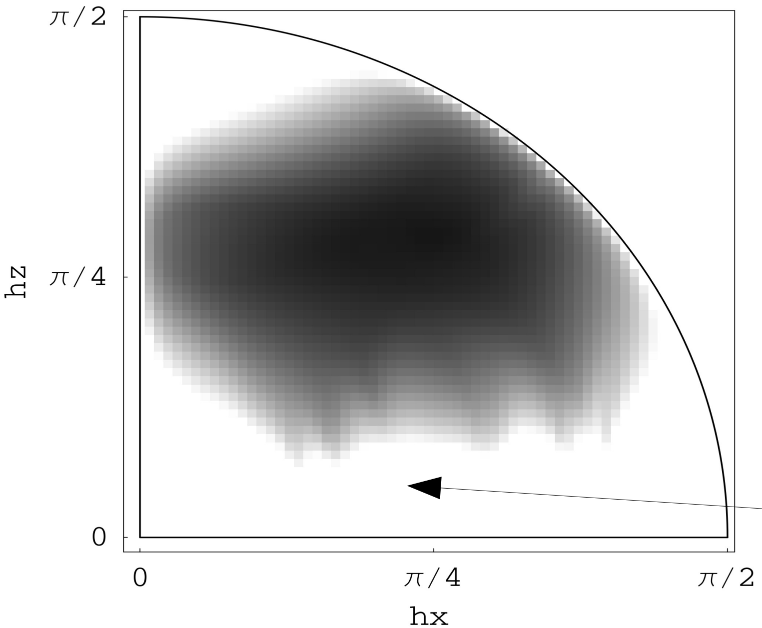
$\mathrm{rate} \sim (\mathrm{driving})^2$

weakly interacting phonons or fermions $\Rightarrow$ kinetic equation

Is there some simple special structure to these models? **Yes: this talk**

A-posteriori motivation

$$U = e^{\mathrm{i} J \sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i} \sum_i (h_x \sigma_i^x + h_z \sigma_i^z)}$$



Numerics by Prosen 2007:

'Minimal decay rate' Δ of local Ham

$$U H_{\mathrm{E}} U^* = H_{\mathrm{E}} + \mathcal{O}_{\mathrm{loc}}(\Delta)$$

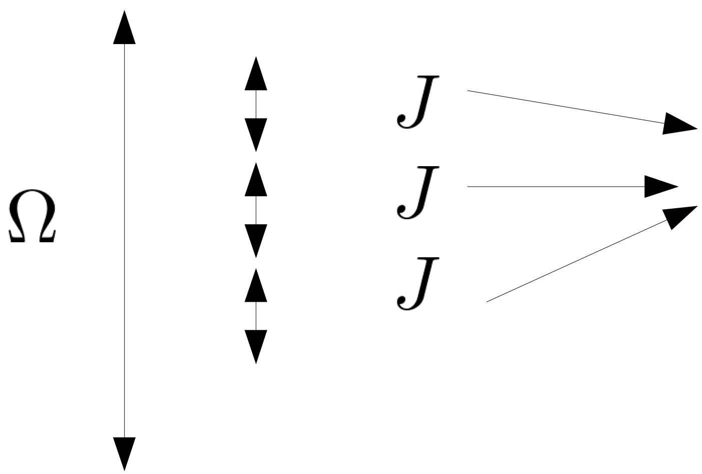
White: $\Delta \leq 10^{-6}$

So it really matters whether h_x or h_z is small

$\rightarrow$ special structure ?

Recall high-frequency regime

$$H = JH_0 + h\cos(\Omega t)H_1 \qquad\qquad J, h \ll \Omega$$



Many local events needed to absorb one photon of frequency Ω

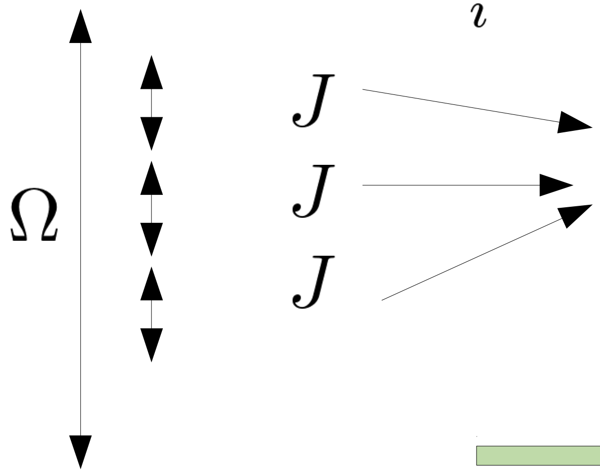
Dissipation only visible in order of PT $\sim \Omega/J$

$$\text{heating rate} \sim e^{-\Omega/J}$$

(Magnus, …..D'Alesio et al,….. Rigourous 2017: Kuwahara et al, Abanin et al)

Same logic: stability of doublons D $(n_i = 2)$

$$H = \Omega \sum_i n_i(n_i - 1) + J H_1 \qquad J \ll \Omega$$



Many local events needed to provide D-energy Ω

$$\mathrm{D - annihilation\ rate} \sim e^{-\Omega/J}$$

$$(\tau \sim \mathcal{O}(e^{\Omega/J}))$$

$$H_{\mathrm{E}} = H_{\mathrm{eff}} = \sum_i n_i(n_1 - 1) + \mathcal{O}(J/\Omega)$$ conserves dressed doublons

(Sensarma et al, ….. Rigorous: Abanin et al, Else et al 2017)

Same logic: stability of doublons D$(n_i = 2)$

$$H = \Omega \sum_i n_i(n_i - 1) + J H_1 \qquad J \ll \Omega$$

Ω

J

J

J

Many local events needed to provide D-energy Ω

$$\mathrm{D-annihilation\ rate} \sim e^{-\Omega/J}$$

Wait... enough to have two distinct energy scales?

No, crucial propery is: $\Omega \sum_i n_i(n_i - 1)$ can absorb only a discrete small set of energies locally.

Simplest examples:

- Sum of commuting local terms with integer gaps (as here)
- MBL systems (stability of MBL)

So: do we have “sums of commuting local terms with integer gaps” ?

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

Yes, both terms have this property =====>

$$\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)$$

$$\sum_i \sigma_i^x \sigma_{i+1}^x \quad \approx \text{doublons } D$$

Choose this one to continue

J

$2\pi \; (\text{frequency})$

To absorb doublon D, need to match frequency up to error of $\mathcal{O}(h)$

n’th order PT: $\min_{m\in\mathbb{Z}} \left| n - m\frac{J}{2\pi} \right| \geq Ch^n$

Mechanism of exp. slow dissipation is there !

Our Theorem

$$H(t) = J(t)D + hW(t), \quad t \in [0,1]$$

Assumptions $x := \frac{1}{2\pi}\int_0^1 dt J(t)$ is “sufficiently Diophantine”

D is sum of commuting local terms with integer gaps

periodicity $H(t) = H(t+1)$

Result Take h small, $\epsilon := e^{-C/h^{1/10}}$ and go to $\widetilde{\text{rotated frame}}$

$$\boxed{\begin{array}{ll} i\partial_t \widetilde{U}(t) = \widetilde{H}(t)\widetilde{U}(t) & [\widetilde{H}(t), D] = \mathcal{O}_{\text{loc}}(\epsilon) \\ \Rightarrow D = H_{\text{E}} \text{ quasi-conserved} & \Rightarrow \tau \sim e^{C/h^{1/10}} \end{array}}$$

Alternative formulation $\boxed{U\widetilde{D}U^* = \widetilde{D} + \mathcal{O}_{\text{loc}}(\epsilon)}$

Can expect $\Rightarrow$ **Prethermalization at** $\widetilde{D} = \text{constant}$

What means $x := \frac{1}{2\pi}\int_0^1 dt J(t)$ is "sufficiently Diophantine"

Recall: we need n'th order PT: $\min_{m\in\mathbb{Z}} |n - mx| \geq h^n$

$$\boxed{\textbf{Def: } x \textbf{ is Diophantine: } \min_{m\in\mathbb{Z}} |n - mx| \geq \frac{a(x)}{n^{-p}}} \qquad p > 1$$

Most numbers are Diophantine:

$$\text{size } \{x \in [0,1] : a(x) > \epsilon\} \geq 1 - C\epsilon$$

Our case: $\frac{h}{a(x)}$ **is the real small parameter**

Proof idea: Schrieffer-Wolf to exhibit conserved quantity

$$H(t) = JD + hW(t), \quad t \in [0,1]$$

- Goal (first order) for some W_d s.t. $[W_d, D] = 0$

$$e^{hA}(-i\partial_t + H)e^{-hA} = -i\partial_t + JD + hW_d + \mathcal{O}_{\mathrm{loc}}(h^2)$$

Brings us to (in rotated frame)

$$i\partial_t \widetilde{U}(t) = \widetilde{H}(t)\widetilde{U}(t) \qquad [\widetilde{H}(t), D] = \mathcal{O}_{\mathrm{loc}}(h^2)$$

===> D conserved up to time h^{-2}

Proof idea: Schrieffer-Wolf to exhibit conserved quantity

$$H(t) = JD + hW(t), \quad t \in [0,1]$$

- Goal (first order) for some W_d s.t. $[W_d, D] = 0$

$$e^{hA}(-i\partial_t + H)e^{-hA} = -i\partial_t + JD + hW_d + \mathcal{O}_{\text{loc}}(h^2)$$

- Suffices to solve linear ODE with periodic A

$$-i\partial_t A + (W - W_d) + [A, D] = 0$$

- Solution: (write $\mathcal{N} \equiv iJ[D, \cdot]$)

$$A(t) = e^{t\mathcal{N}} A(0) + \int_0^t ds e^{(t-s)\mathcal{N}} (W - W_d)(s)$$

- Imposing periodicity at time t=1 : $A(0) = \boxed{\frac{1}{e^{\mathcal{N}} - 1}} (\dots)$

Resonance Denominator

Crux: locality preservation of map

$$A(0) = \boxed{\frac{1}{e^{\mathcal{N}} - 1}}(\ldots)$$

Resonance denominator

$$\mathcal{N} \equiv iJ[D, \cdot]$$

Use crucially that $D = \sum \text{commuting terms}$

$$\frac{1}{e^{\mathcal{N}} - 1}(O) = \frac{1}{e^{\mathcal{N}_{\overline{X}}} - 1}(O)$$

$$\overline{X} = \text{fattening of } X$$

$$X = \text{supp}(O)$$

Smallest denominator-value on such O is $\min_{n \leq |\overline{X}|} (nJ) \,\text{MOD}\, (2\pi)$

Diophantine condition ===> smallest denominator $|\overline{X}|^{-\tau}$ not $\exp(-|\overline{X}|)$

===> smallness of perturbation wins! $h^{|X|}$

Example and Extension

Recall $$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}\sum_i (h_z \sigma_i^z + h_x \sigma_i^x)}$$

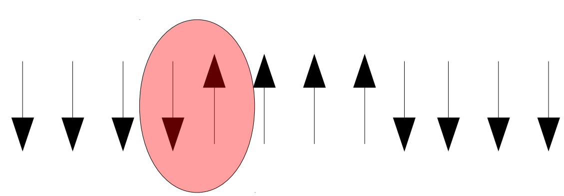
(but same applies for static $H = \sum_i J\sigma_i^x \sigma_{i+1}^x + h_z \sigma_i^z + h_x \sigma_i^x$)

Assume now: $\boxed{h_z \ll h_x \sim J \sim 1}$ instead of $\boxed{h_x^2 + h_z^2 \ll J^2 \sim 1}$

New Diophantine condition: $$\min_{|n_1|+|n_2|+|n_3| \leq n} |n_1 J + n_2 h_x + 2\pi n_3| \geq \frac{a}{n^{-\tau}}$$

Then: Both $D = \sum_i \sigma_i^x \sigma_{i+1}^x$ $M = \sum_i \sigma_i^x$ quasi-conserved

→ **kinetically constrained model**

Example  No local move possible

Both $D = \sum_i \sigma_i^x \sigma_{i+1}^x$ $M = \sum_i \sigma_i^x$ quasi-conserved

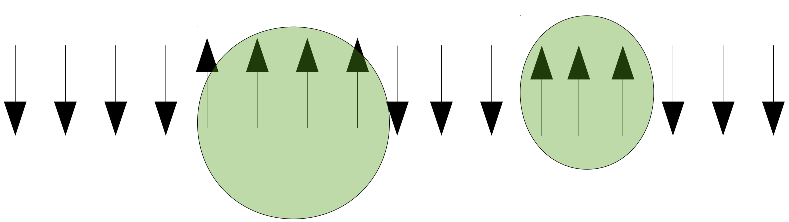
General phenomenology:

- first order in h_z : no spin flips at all
- First dissipation (spin flips) at time h_z^{-4}
- Prethermalization $h_z^{-4} \longleftrightarrow e^{1/h_z^{0.1}}$
- Actually, even at order 4: dynamics is highly constrained

further slowness depending on state

droplet mass $\sim \exp\left(\text{droplet length}\right)$

(magnetization, density of doublons)

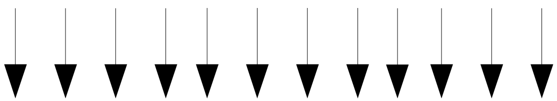


- So even prethermalization might be very slow here ------ ‘translation invariant (asymptotic) MBL’

Mathematical excursion to spectral edge

$$H = \sum_i J\sigma_i^x \sigma_{i+1}^x + h_z \sigma_i^z + h_x \sigma_i^x$$

$$\boxed{h_z \ll h_x \sim J \sim 1}$$



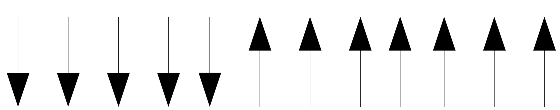
Our result: stable up to exp long time

Reality: stable for ever (groundstate close to $h_z = 0$ GS)

Proofs: Yarotsky ‘04, Michalakis-Zwollak ‘14, Del Vechhio-Frohlich-Pizzo ‘18

Perhaps also infinite-time stability in bulk? ….. Let’s have a symmetry

$$H = \sum_i J\sigma_i^x \sigma_{i+1}^x + h_z \sigma_i^z + h_x \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x$$



(in progress)

I think: gives rise to eigenstate (breather) as well, but not at spectral edge

Result is known for some integrable models

Conclusion

- Perturbative, rigorous view on slow heating in kicked Ising models
- We identified conditions for slow heating: small perturbations of Hamiltonians with commuting terms + Diophantine
- Not clear whether this indeed explains all the observed absence of heating in this model: numerics needed.
- Also mechanism for creating kinetically constrained models