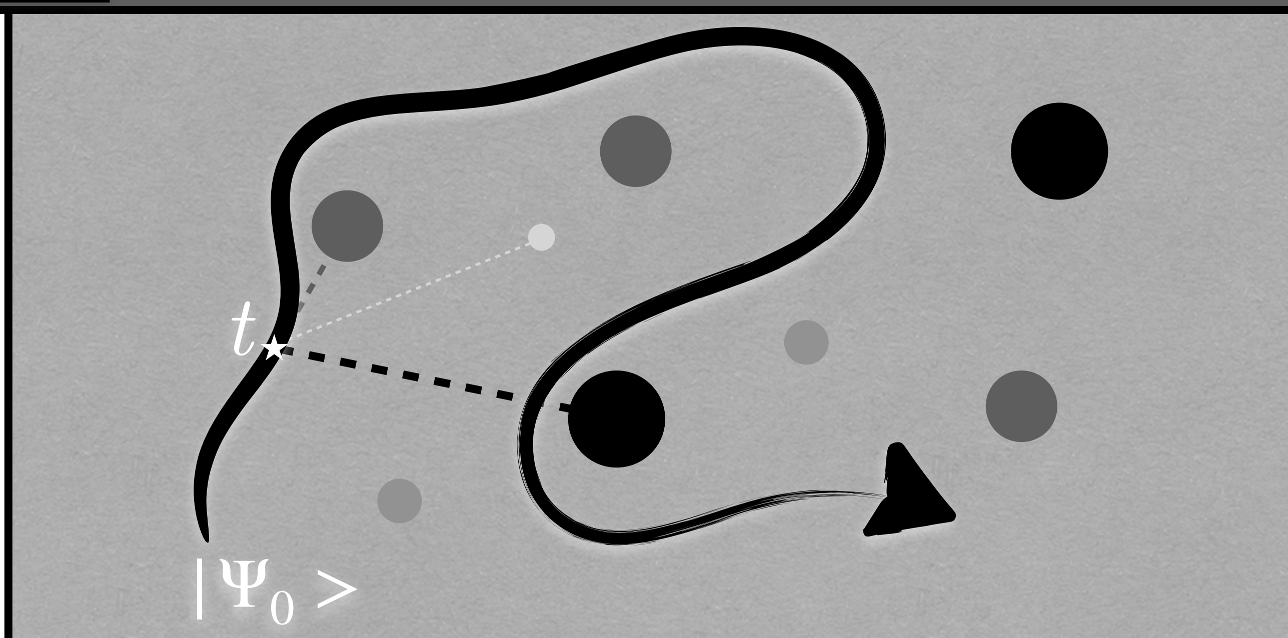
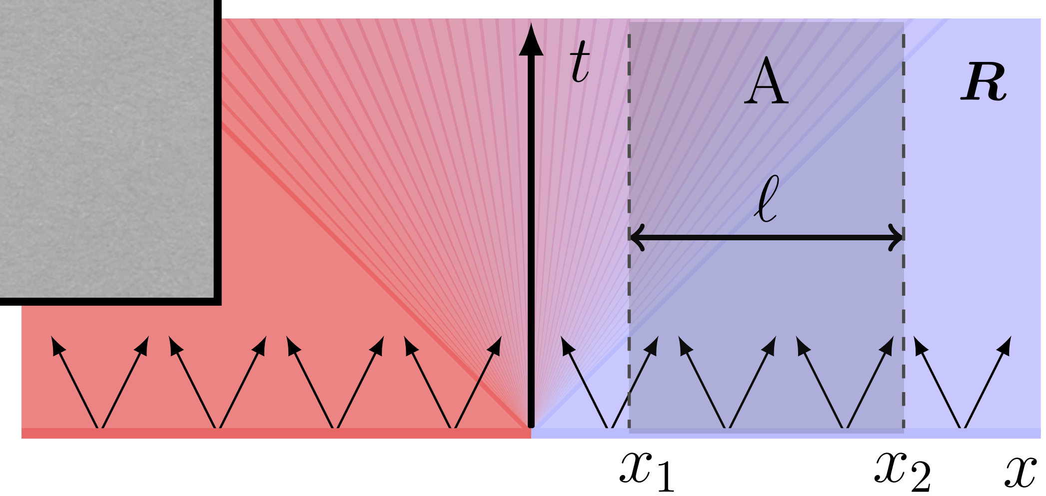
Entanglement evolution and generalised hydrodynamics

Maurizio Fagotti





September 23 2019, Quantum Transport and Universality From Topological Materials to Quantum Hydrodynamics - Roma

Summary

- ✦ Introduction
  - Infinite time in infinite systems vs infinite time average in finite systems
- ✦ Entropy
  - … of the state
  - … of the time averaged state
  - thermodynamic entropy
  - entanglement entropy
- ✦ Conclusions

Quench dynamics

a many-body system time evolves unitarily

$$|\Psi_t> \ = e^{-i\boldsymbol{H}t}|\Psi_0> \qquad (\boldsymbol{\rho} = |\Psi><\Psi|)$$

$$\boldsymbol{\rho}_t = e^{-i\boldsymbol{H}t}\boldsymbol{\rho}_0 e^{i\boldsymbol{H}t}$$

coined by J. Cardy

QUANTUM QUENCH $g_0 \to g$

$$\boldsymbol{H}(g_0)|\Psi_0> \ = E_{\mathrm{GS}}|\Psi_0>$$

$$\boldsymbol{H} \ = \boldsymbol{H}(g)$$

typical examples

- *spin lattice systems*
- *quantum field theories*

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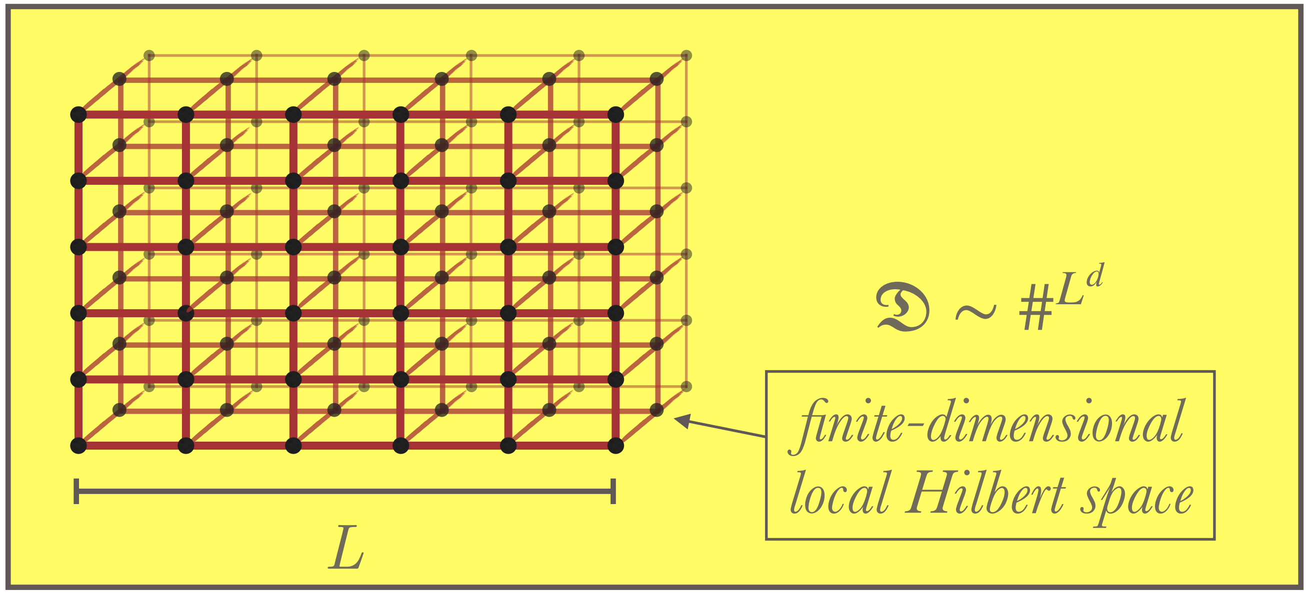

PHYSICAL REVIEW VOLUME 107, NUMBER 2 JULY 15, 1957

Quantum Recurrence Theorem

P. Bocchieri and A. Loinger
Istituto di Fisica dell'Università, Pavia, Italy, and Istituto Nazionale di Fisica Nucleare, Sez. di Milano, Italy

(Received October 9, 1956)

A recurrence theorem is proved, which is the quantum analog of the recurrence theorem of Poincaré. Some statistical consequences of the theorem are stressed.

IT is well known that in classical mechanics the following recurrence theorem holds, due to Poincaré (1890)[1]: "Any phase-space configuration (q,p) of a system *enclosed in a finite volume* will be repeated as accurately as one wishes after a finite (be it possibly very long) interval of time."

In this paper we shall show that a similar recurrence theorem holds in quantum theory; it can be formulated as follows: "Let us consider a system with *discrete* energy eigenvalues E_n; if $\Psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least one time T will exist such that the norm $\|\Psi(T)-\Psi(t_0)\|$ of the vector $\Psi(T)-\Psi(t_0)$ is smaller than ϵ."[2]

The proof of this theorem is simple and can be sketched in the following way: The equation of motion is

$$i(\partial\Psi(t)/\partial t)=H\Psi(t); \tag{1}$$

the formal solution is

$$\Psi(t)=\sum_{n=0}^{\infty} r_n \exp(i\varphi_n-iE_nt)u(E_n), \tag{2}$$

(the r_n's being real positive numbers). From (2),

$$\|\Psi(T)-\Psi(t_0)\|=2\sum_{n=0}^{\infty} r_n^2(1-\cos E_n\tau);\quad (\tau\equiv T-t_0), \tag{3}$$

Furthermore it is easy to prove that this quantum recurrence theorem does not hold in general if the system has a continuous energy spectrum. The situation here is quite similar to the classical one: the quantum systems having a continuous energy spectrum correspond to classical systems not bounded to a finite volume. The analogy with the classical case is even deeper, since it is easy to prove (see Appendix) that also for the expectation values of the q's and p's a recurrence theorem holds, which in the classical limit goes over into the theorem of Poincaré.

The quantum recurrence theorem has statistical consequences rather similar to those of the Poincaré's theorem in the classical case.

Using Poincaré's theorem, Zermelo (1896) was able to invalidate the unrestricted (nonstatistical) formulation of the Boltzmann H-theorem and to conclude that the "Stosszahlansatz" is, strictly speaking, in contradiction with the dynamical laws, the effect of the "Stosszahlansatz" being that of averaging out the fluctuations.[4]

The quantum analog to the "Stosszahlansatz" is the assumption about the number of transitions,[5] which is obtained by using the quantum-dynamical equations of motion and the conventional statistical postulate of equal *a priori* probabilities and random *a priori* phases.

Analogously to the classical case, the quantum theorem holds in quantum theory; it can be formulated as follows: "Let us consider a system with *discrete* energy eigenvalues E_n; if $\Psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least one time T will exist such that the norm $\|\Psi(T)-\Psi(t_0)\|$ of the vector $\Psi(T)-\Psi(t_0)$ is smaller than ϵ."[2]

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and, if N is suitably chosen,

$$\sum_{n=N}^{\infty} r_n^2(1-\cos E_n\tau)<\epsilon. \tag{4}$$

Consequently, it is sufficient to prove that there is a value of τ such that

$$\sum_{n=0}^{N-1}(1-\cos E_n\tau)<\epsilon. \tag{5}$$

But this is actually the case according to a standard result of the theory of the almost-periodic functions.[3]

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The quantum analog to the "Stosszahlansatz" is the assumption about the number of transitions,[5] which is obtained by using the quantum-dynamical equations of motion and the conventional statistical postulate of equal *a priori* probabilities and random *a priori* phases.

Analogously to the classical case, the quantum recurrence theorem shows that we cannot hope to obtain the assumption about the number of transitions without postulates of statistical nature.

Our theorem shows furthermore that a similar conclusion is valid also for the probability transport equation.

Finally we would like to emphasize that (contrary to a wide-spread belief) the expectation values of the macroscopic observables will *not* maintain indefinitely their equilibrium values, once they have attained them.

APPENDIX. PROOF OF THE SIMULTANEOUS RECURRENCE OF THE EXPECTATION VALUES OF THE p's AND THE q's

The state vector is

$$\Psi(t)=\sum_m r_m \exp(i\varphi_m-iE_mt)u(E_m).$$

[1] For a modern formulation of this theorem see A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, 1947), p. 90.

[2] Besides this recurrence theorem, a quasi-ergodic theorem for $\Psi(t)$ exists [J. von Neumann, Z. Physik **57**, 30 (1929), Sec. 4, p. 35]. However, it holds under very restrictive hypotheses, which most probably cannot be satisfied by any system having physical interest.

[3] See, e.g., Harald Bohr, *Fastperiodische Funktionen* (Verlag Julius Springer, Berlin, 1932), p. 31.

[4] See, e.g., W. Pauli, "Gekuerzte Vorlesung ueber statistische Mechanik," lecture notes, Zurich, 1951 (unpublished), p. 5; and also L. Rosenfeld, Acta Phys. Polonica, 14, 3 (1955); D. ter Haar, Revs. Modern Phys. 27, 289 (1955).

[5] Formula (D1.30) of the review article by ter Haar quoted in reference 3.

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➡ no relaxation in systems with discrete energy eigenvalues

ways to circumvent it in spin lattice systems

relaxation "on average"

thermodynamic limit

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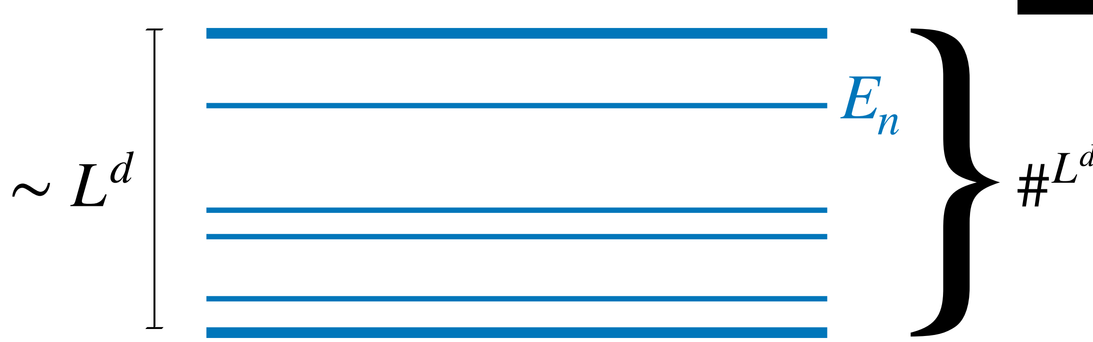
→ **relaxation "on average"**

at a random time t, expectation values almost always close to a particular value

→ **thermodynamic limit** ➡

- *thermalization in generic systems*
- *relaxation to GGEs in integrable systems*



Entropy

1. … of the state

2. … of the time averaged state

3. thermodynamic entropy

4. entanglement entropy (half chain/subsystem)

5. …

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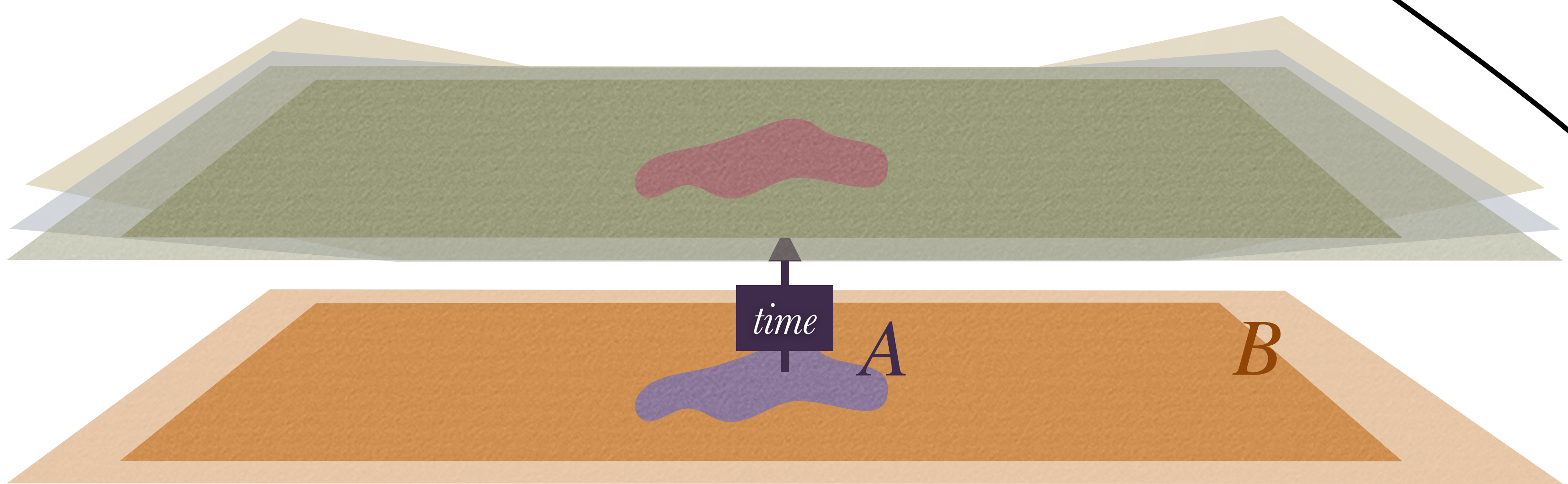
$$(\exists)\lim_{t\to\infty}\lim_{|B|\to\infty}\rho_A(t) = \operatorname{Tr}_B[\rho^{MS}]$$

macro-state

underdetermined state (only the local properties are fixed)



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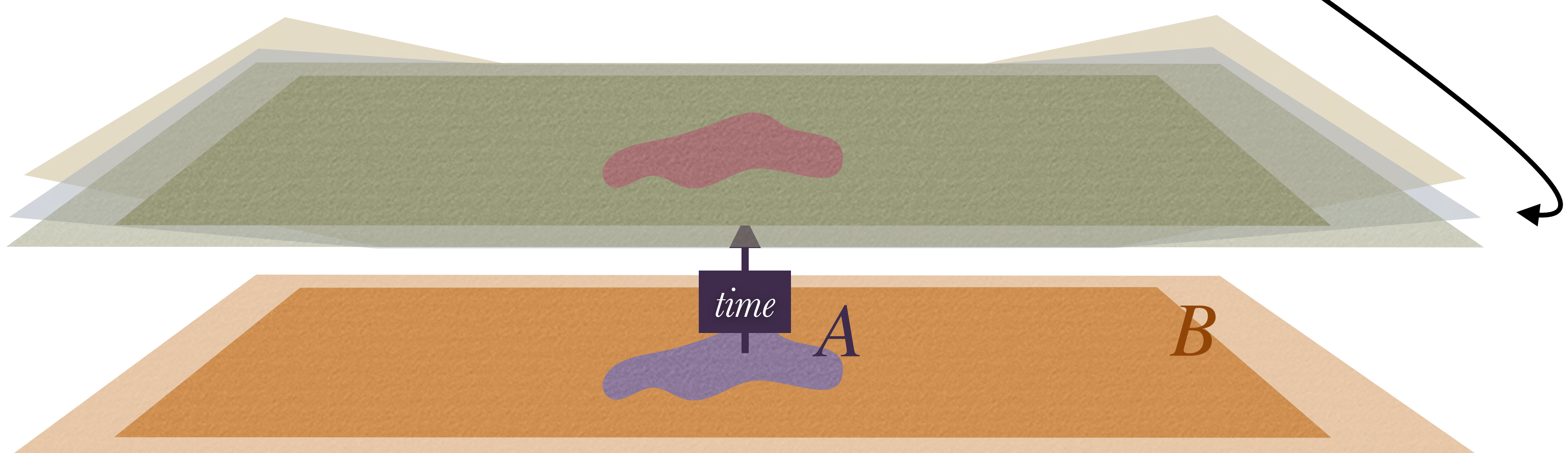

4. **entanglement entropy (half chain/subsystem)** $\boxed{S_{vN}[A] = -\operatorname{tr}[\rho_A(t)\log\rho_A(t)]}$

5. **…**

Entropy

1. **... of the state** $\boxed{S_{vN} = -\operatorname{tr}[\rho(t)\log\rho(t)] = 0}$

   pure state

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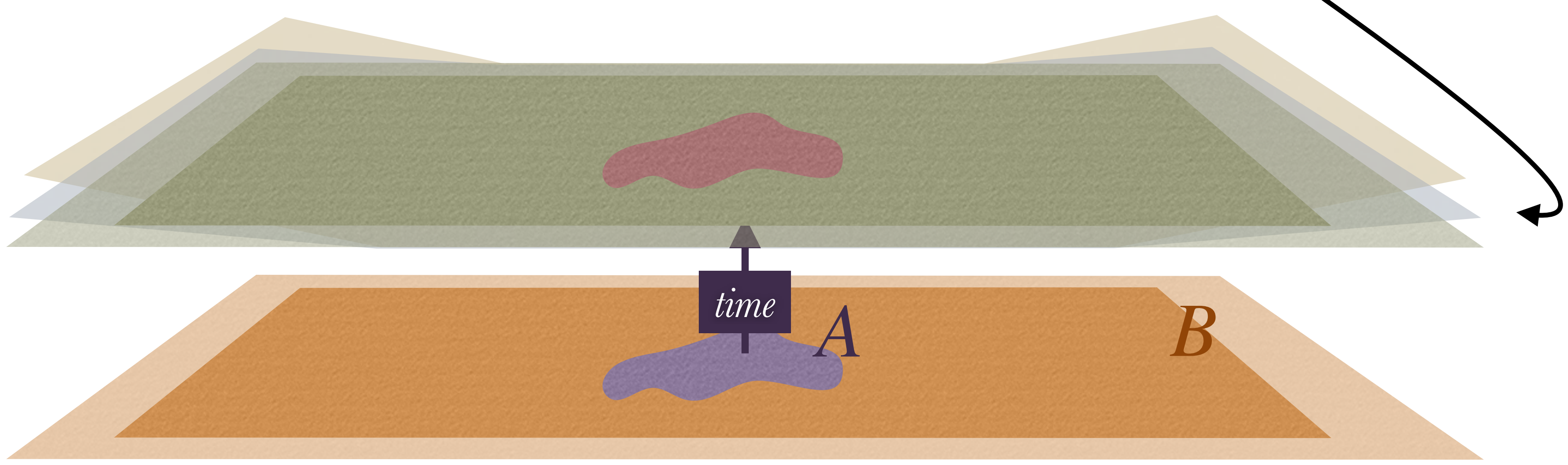

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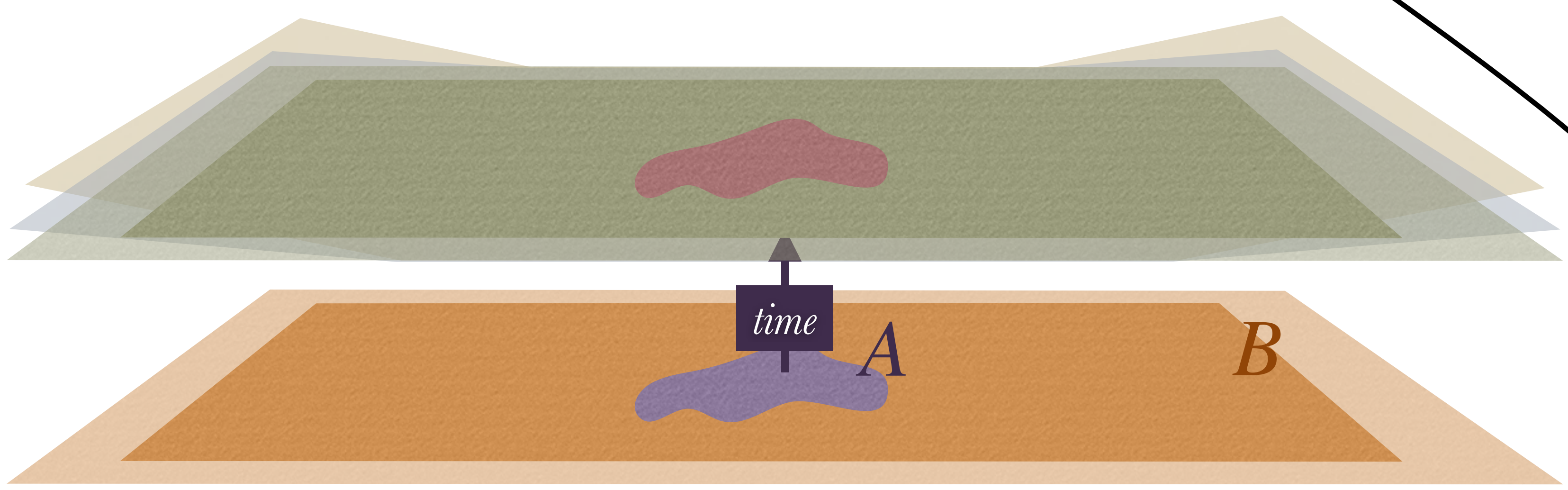

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$$\kappa_n = \partial_t^n \log\Big|_{t=0} < \Psi_0 | e^{t\boldsymbol{H}} | \Psi_0 > \qquad\qquad < \Psi_0 | \Psi_t > \overset{t\approx 0}{=} \exp\Big(\sum_{n=1}^{\infty} (-i)^n \frac{\kappa_n t^n}{n!} \Big)$$

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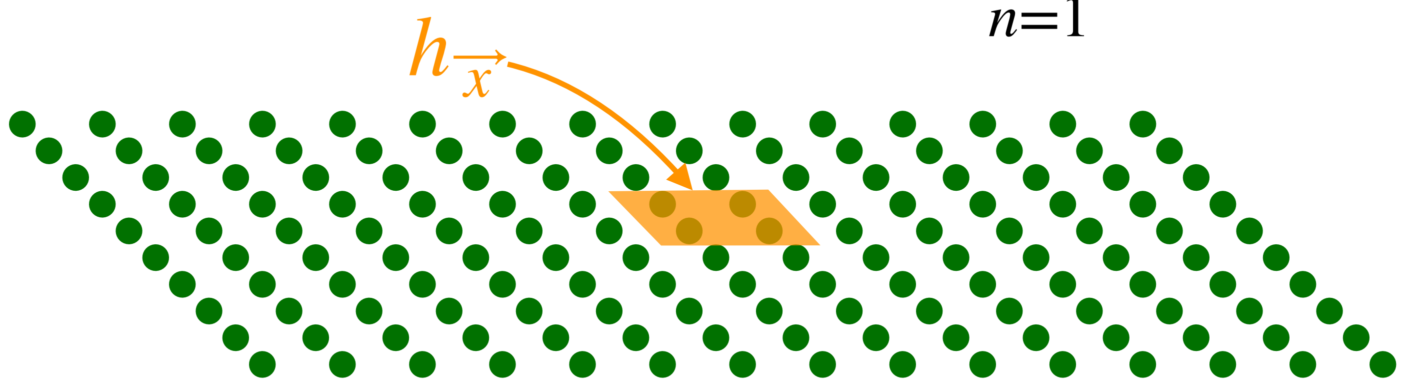
(Quasi)local Hamiltonian

$$\boldsymbol{H} = \sum_{\vec{x}} \boldsymbol{h}_{\vec{x}}$$



Ground state in the generic case *(finite correlation lengths)*

$$< \Psi_0 | \boldsymbol{O}_{\vec{x}} \boldsymbol{O}_{\vec{x}+\vec{r}} | \Psi_0 > - < \Psi_0 | \boldsymbol{O}_{\vec{x}} | \Psi_0 > < \Psi_0 | \boldsymbol{O}_{\vec{x}+\vec{r}} | \Psi_0 > \sim e^{-\frac{r}{\xi}}$$

Ground state in critical systems *(e.g. infinite correlation length in 1d)*

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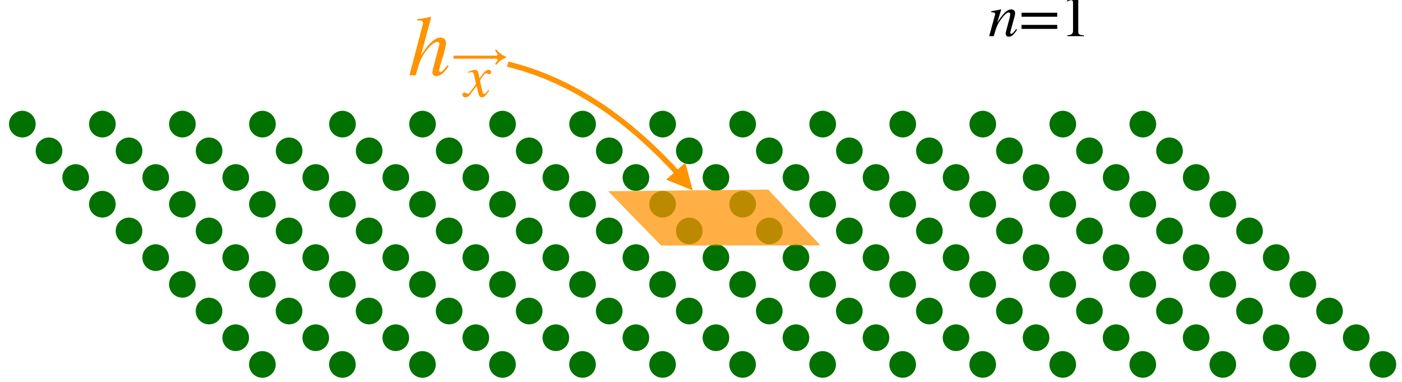

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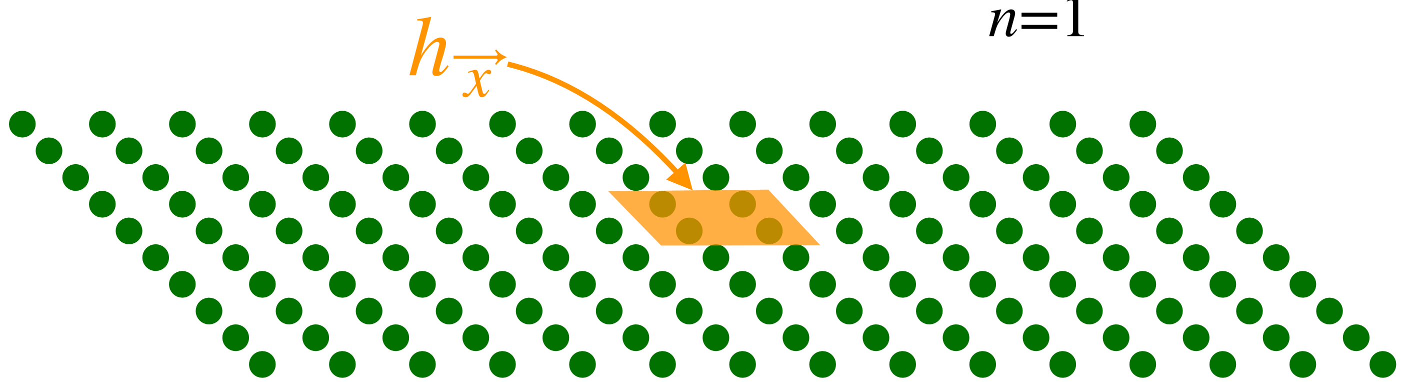

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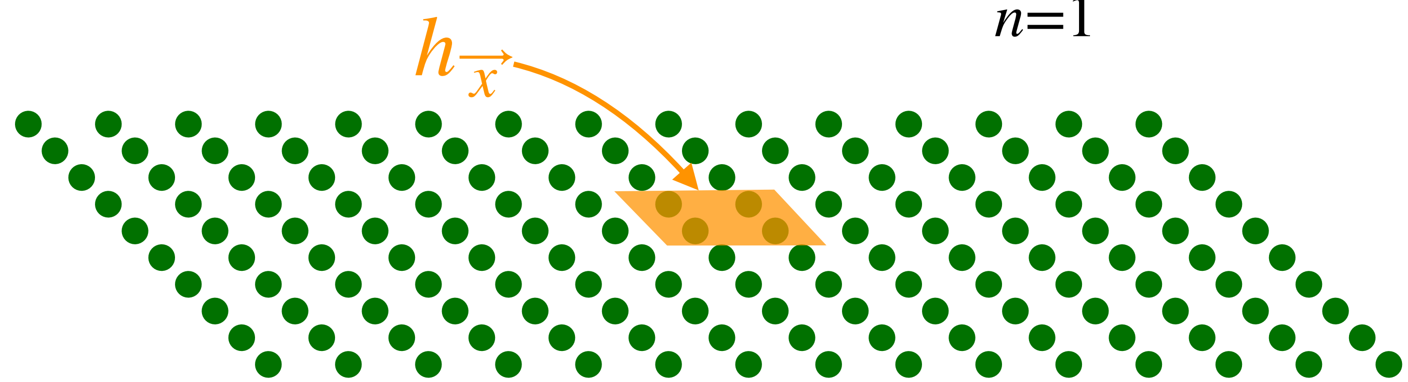

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(von Neumann entropy: $S_{vN} = -\,\mathrm{tr}[\bar{\rho}_{0,t} \log \bar{\rho}_{0,t}]$**)**

complete characterisation of the eigenvalue distribution *(Hausdorff moment problem)*

$$e^{-L^d f(t_1-t_2)} = < \Psi_{t_1} | \Psi_{t_2} > = < \Psi_0 | e^{iH(t_1-t_2)} | \Psi_0 > \sim \exp\left(L^d \sum_{n=1}^{\infty} (-i)^n \frac{\mathbf{e}_n (t_2 - t_1)^n}{n!} \right)$$

Loschmidt echo

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Loschmidt echo

Useful properties of $f(t)$ in the thermodynamic limit:

- existence of the limit of infinite time
- $f(t)=0$ has only the solution $t=0$

see ,e.g, Karrasch and Schuricht, Phys.Rev. B **87**, 195104 (2013)

MF, SciPost Phys. **6**, 059 (2019)

$$\mathrm{tr}[\bar{\boldsymbol{\rho}}_{0,t}^{\alpha}] = \mathrm{tr}[\cdots\bar{\boldsymbol{\rho}}_{0,t}^{2}\cdots] = \mathrm{tr}[\cdots\iint_0^t \frac{\mathrm{d}\tau}{t^2}\,|\,\Psi_{\tau_1} > < \Psi_{\tau_1}\,|\,\Psi_{\tau_2} > < \Psi_{\tau_2}\,|\cdots]$$

exponentially small in the system's size for any nonzero $\tau_1 - \tau_2$

the integration domain can be reduced into a region where $\log < \Psi_{\tau_1}\,|\,\Psi_{\tau_2} >$ **can be series expanded about** $\tau_1 \approx \tau_2$

MF, SciPost Phys. **6**, 059 (2019)

$$\mathrm{tr}[\bar{\rho}^{\alpha}_{0,t}] = \mathrm{tr}[\cdots\bar{\rho}^{2}_{0,t}\cdots] = \mathrm{tr}[\cdots\iint_0^t \frac{\mathrm{d}\tau}{t^2}|\Psi_{\tau_1}><\Psi_{\tau_1}|\Psi_{\tau_2}><\Psi_{\tau_2}|\cdots]$$

exponentially small in the system's size for any nonzero $\tau_1-\tau_2$

the integration domain can be reduced into a region where $\log<\Psi_{\tau_1}|\Psi_{\tau_2}>$ can be series expanded about $\tau_1\approx\tau_2$

asymptotic expansion in the limit of a large number of sites

$$\mathrm{tr}[\bar{\rho}^{\alpha}_{t}] \sim \iiint_{[0,t\sqrt{L}]^\alpha} \frac{\mathrm{d}^\alpha\tau}{t^\alpha L^{d\frac{\alpha}{2}}} e^{-\mathrm{e}_2\frac{(\tau_\alpha-\tau_1)^2+\sum_{j=1}^{\alpha-1}(\tau_j-\tau_{j+1})^2}{2}} \sim \alpha^{-\frac{1}{2}}(\frac{\mathrm{e}_2}{2\pi})^{\frac{1-\alpha}{2}}t^{1-\alpha}L^{d\frac{1-\alpha}{2}}$$

$$S_\alpha[\bar{\rho}_t] = \frac{d}{2}\log L + \frac{1}{2}\log\frac{\mathrm{e}_2 t^2}{2\pi} + \frac{\log\alpha}{2(\alpha-1)} + O(L^{-\frac{d}{2}})$$

$$S_{vN}[\bar{\rho}_t] \sim \frac{d}{2}\log L + \frac{1}{2}\log\frac{\mathrm{e}_2 t^2}{2\pi} + \frac{1}{2}$$

An interesting consequence

MF, SciPost Phys. **6**, 059 (2019)

approximate support of $e^{iHt}\boldsymbol{O}e^{-iHt}$

supp($\boldsymbol{O}$)

$\ell = |\mathrm{supp}(O)|$

Lieb and Robinson, Comm. Math. Phys. **28**, 251 (1972)

Lieb-Robinson bound

L **can be replaced by** $\tilde{L} \leq \ell + 2v_{\mathrm{LR}}t + 2\xi$

correlation length in the initial state

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correlation length in the initial state

typical time step $\delta t = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}_t^{(\epsilon)}\right)^{-1} \sim const \frac{t^{-\frac{d}{2}}}{\sqrt{\mathfrak{e}_2 v_{\mathrm{LR}}^{\frac{d}{2}}}}$

(at fixed accuracy)

in integrable systems, v_{LR} can be replaced by the maximal velocity of the excitations in that particular state

Entropy

1. **… of the state** $S_{vN} = -\operatorname{tr}[\rho(t)\log\rho(t)] = 0$

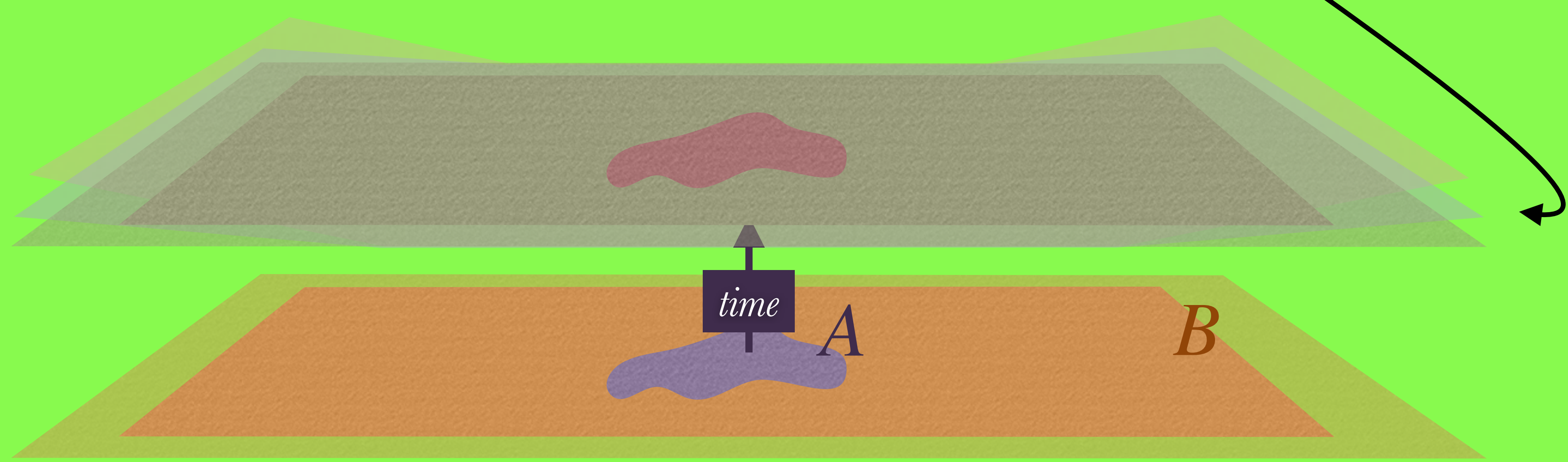
2. **… of the time averaged state** $S_{vN} = -\operatorname{tr}[\bar\rho_{0,t}\log\bar\rho_{0,t}] \sim \frac{1}{2}\left(\log L^d + \log\frac{ee_2t^2}{2\pi}\right)$

3. **thermodynamic entropy** $S_{vN} = \sup_{\rho^{MS}}(-\operatorname{tr}[\rho^{MS}\log\rho^{MS}])$

$$(\exists)\lim_{t\to\infty}\lim_{|B|\to\infty}\rho_A(t) = \mathrm{Tr}_B[\rho^{MS}]$$

macro-state

underdetermined state (only the local properties are fixed)



4. **entanglement entropy (half chain/subsystem)** $S_{vN}[A] = -\operatorname{tr}[\rho_A(t)\log\rho_A(t)]$

5. **…**

Integrable systems with a TBA description

$$H = \sum_{\ell} s^x_\ell s^x_{\ell+1} + s^y_\ell s^y_{\ell+1} + \Delta s^z_\ell s^z_{\ell+1}$$

	free-fermion system *(e.g., transverse-field Ising chain)*	**interacting integrable system** *(e.g., XXZ spin-1/2 chain)*
excited state	$\lvert \lambda_1, \lambda_2, \ldots \rangle = [b^\dagger_{\lambda_1} b^\dagger_{\lambda_2} \cdots] \lvert \varnothing \rangle$ $\quad \{b^\dagger_\lambda, b_\mu\} = \delta_{\lambda\mu}$, $\{b^\dagger_\lambda, b^\dagger_\mu\} = 0$	$\lvert \lambda_1, \lambda_2, \ldots \rangle = [B(\lambda_1) B(\lambda_2) \cdots] \lvert \varnothing \rangle$
energy	$E = \sum_{\lambda' \in \{\lambda\}} e(\lambda')$	
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excitations		

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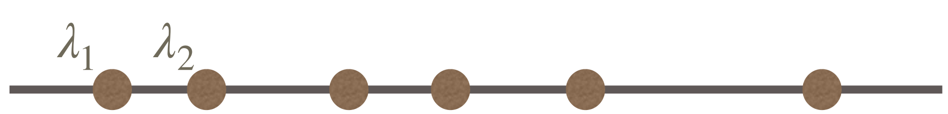
excitations		

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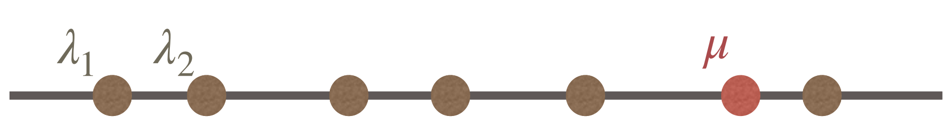
excitations	 *particle excitation*	

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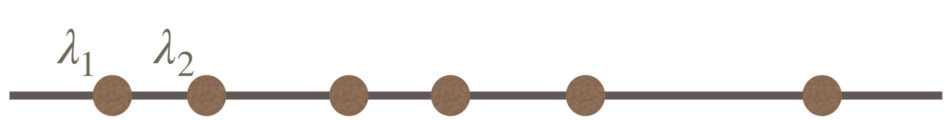
excitations		

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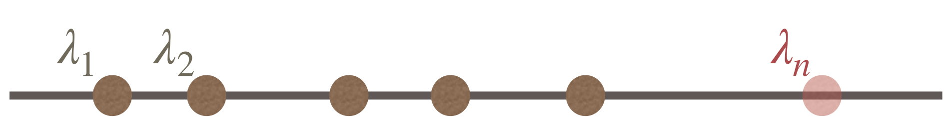
excitations	 *hole excitation*	

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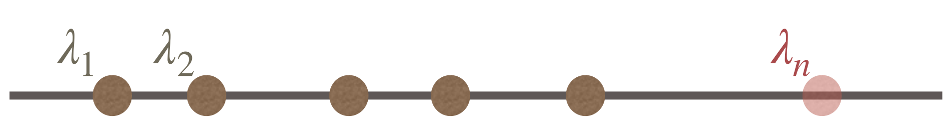
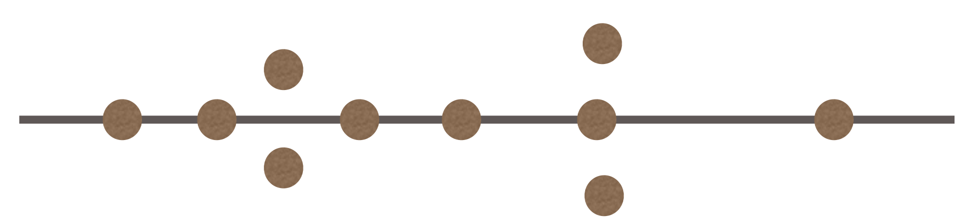
excitations	 *hole excitation*	

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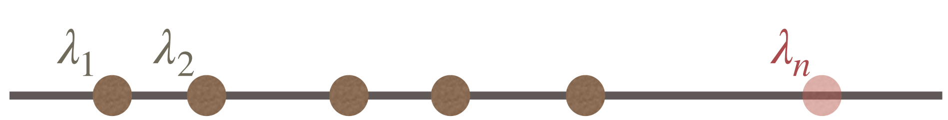
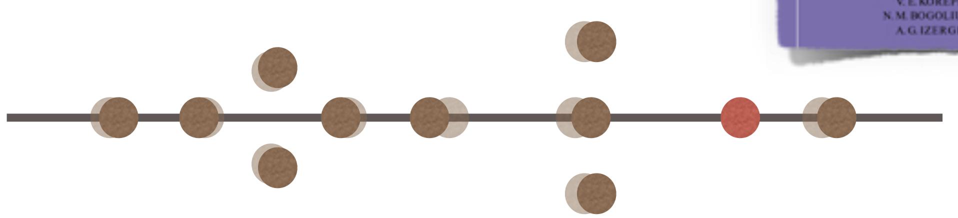
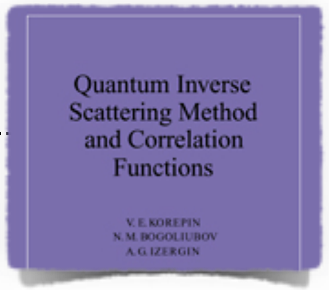
excitations	 *hole excitation*	 *particle excitation*



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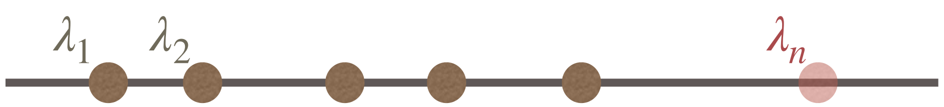
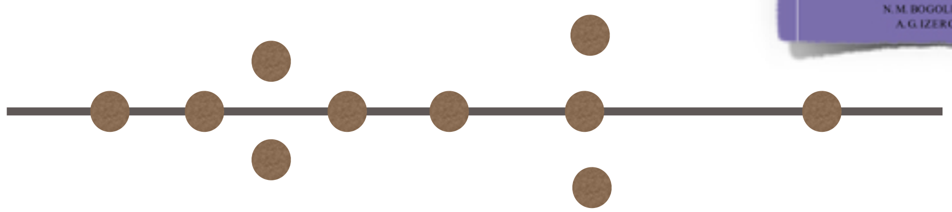
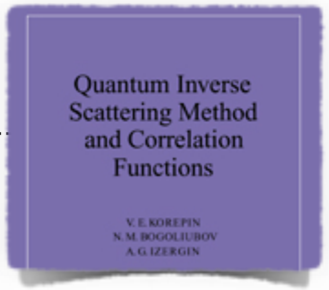
excitations	 *hole excitation*	



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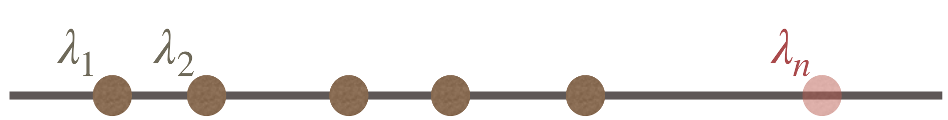
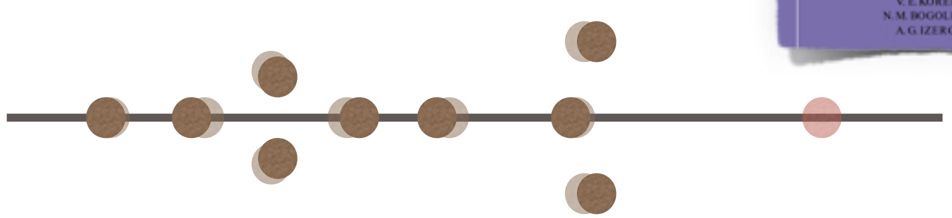
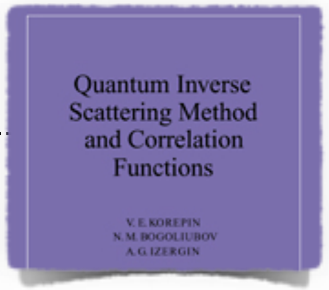
excitations	 *hole excitation*	 *hole excitation*



Integrable systems with a TBA description

Takahashi, "Thermodynamics of one-dimensional solvable models"

Thermodynamic limit

density of rapidities/particles excitations

$$Q\,\big|\lambda_1,\lambda_2,\ldots\big\rangle = \sum_{\lambda'\in\{\lambda\}} q(\lambda')\big|\lambda_1,\lambda_2,\ldots\big\rangle \longrightarrow \sum_n \int \mathrm{d}\lambda\, q_n(\lambda)\rho_n(\lambda) \equiv \vec{q}\cdot\vec{\rho}$$

the rapidities are organised in strings

(groups of equidistant rapidities with the same real part)

bare charge

Integrable systems with a TBA description

Takahashi, "Thermodynamics of one-dimensional solvable models"

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bare charge

TBA equations for an excited state represented by $\dfrac{e^{-Q}}{\mathrm{tr}[e^{-Q}]}$

XXZ spin-1/2 chain

$$u_n(\lambda) = 1$$

bare charge associated with the state

$$\log(\hat{\vartheta}^{-1}-\hat{1})\vec{u} = -\vec{q} - \hat{T}\hat{\sigma}\log(\hat{1}-\hat{\vartheta})\vec{u}$$

$$\boxed{\vec{a}} = (\boxed{\hat{\sigma}}\,\hat{\vartheta}^{-1} + \boxed{\hat{T}})\vec{\rho}$$

fixed by the Hamiltonian

$$\vartheta_\ell(\lambda) = \frac{\rho_\ell(\lambda)}{\rho_\ell(\lambda)+\rho^h_\ell(\lambda)}$$

filling function

(interacting generalisation of the occupation number)

density of hole excitations

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 7 JULY 1969

Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction

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Ohio State University, Columbus, Ohio

(Received 10 October 1968)

The equilibrium thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction is shown to be derivable from the solution of a simple integral equation. The excitation spectrum at any temperature T is also found.

I. INTRODUCTION

The ground-state energy of a system of N bosons with repulsive delta-function interaction in one dimension with periodic boundary condition was calculated by Lieb and Liniger.[1] The Hamiltonian for the system is

$$H = -\sum_1^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i>j} \delta(x_i - x_j), \quad c > 0, \qquad (1)$$

and the periodic box has length L. Using Bethe's hypothesis[2] they showed that the k's in the hypothesis satisfy

$$(-1)^{N-1} \exp(-ikL) = \exp\left[i \sum_{k'} \theta(k' - k)\right], \qquad (2)$$

where

$$\theta(k) = -2\tan^{-1}(k/c), \quad -\pi < \theta < \pi. \qquad (3)$$

Now, for any set of real I's, $I_1, I_2, \cdots, I_N$, Eq. (4) has a unique real solution for the k's, $k_1, k_2, \cdots, k_N$. The proof of this statement (similar to but simpler than the proof of a corresponding statement[3] for the Heisenberg–Ising problem) follows. Let

$$\theta_1(k) = \int_0^k \theta(k)\, dk.$$

Define

$$B(k_1, \cdots, k_N) = \tfrac{1}{2} L \sum_1^N k_j^2 - 2\pi \sum_1^N I_j k_j - \tfrac{1}{2} \sum_{j,S} \theta_1(k_j - k_S). \qquad (6)$$

Equation (4) is the condition for the extrema of B. Now the second-derivative matrix B_2 of B is positive-definite. [The first sum in (6) contributes a positive-

By a continuity argument with respect to c^{-1} we obtain the following:

Theorem: For any set of I's satisfying (5), no two of which are identical, there is a unique set of real k's satisfying (4), with no two k's being identical. With this set of k's, one eigenfunction of H, of Bethe's form, can be constructed. The totality of such eigenfunctions form a complete set for the boson system.

The numbers I are quantum numbers for the problem.

III. ENERGY AND ENTROPY FOR A SYSTEM WITH $N = \infty$

We now consider the problem for $N = \infty$ and $L = \infty$ at a fixed density $D = N/L$. For the ground state, the quantum numbers I/L form[1] a uniform lattice between $-D/2$ and $D/2$. The k's then form[1] a nonuniform distribution between a maximum k and a minimum k. For an excited state, (5) shows that the quantum numbers I/L are still on the same lattice, but not all lattice sites are taken, and the limits $-D/2$ and $D/2$ are no longer respected. We shall call the omitted lattice sites J_j/L. We would want to define corresponding "omitted k values" to be called holes. This can be easily done: Given the I's, Eq. (4) defines the set of k's as proved in the last section. Now,

$$Lh(p) \equiv pL - \sum_{k'} \theta(p - k') \tag{8}$$

is a continuous monotonic function of p. At $p = \pm\infty$, it is equal to $\pm\infty$. Those values of p where $Lh(p) = 2\pi I$ are k's. Those values of p where $Lh(p) = 2\pi J$ will be defined as holes.

For a large system, there is thus a density distribution of holes as well as one of k's:

The energy per particle for the state is

$$E/N = D^{-1}\int_{-\infty}^{\infty} \rho(k)k^2\, dk, \tag{12}$$

where

$$D = N/L = \int_{-\infty}^{\infty} \rho(k)\, dk. \tag{13}$$

The entropy of the "state" is not zero since the existence of the omitted quantum numbers J_j allows many wavefunctions of approximately the same energy to be described by the same ρ and ρ_h. In fact, for given ρ and ρ_h, the total number of k's and holes in dk is $L(\rho + \rho_h)\, dk$, of which $L\rho\, dk$ are k's and $L\rho_h\, dk$ are holes. Thus the number of possible choices of states in dk consistent with given ρ and ρ_h is

$$\frac{[L(\rho + \rho_h)\, dk]!}{[L\rho\, dk]!\,[L\rho_h\, dk]!}.$$

The logarithm of this gives the contribution to the entropy from dk. Thus, the total entropy is, putting the Boltzman constant equal to 1,

$$S = \sum \{(L\rho\, dk + L\rho_h\, dk) \ln(\rho + \rho_h) - L\rho\, dk \ln \rho - L\rho_h\, dk \ln \rho_h\}$$

or

$$S/N = D^{-1}\int_{-\infty}^{\infty} [(\rho + \rho_h)\ln(\rho + \rho_h) - \rho \ln \rho - \rho_h \ln \rho_h]\, dk. \tag{14}$$

IV. THERMAL EQUILIBRIUM

At temperature T, we should maximize the contribution to the partition function from the states described by ρ and ρ_h. In other words, given ρ, ρ_h is defined by (11). One then computes the contribution

Yang-Yang entropy

$$\lim_{L\to\infty} \frac{S^{YY}}{L} = -\sum_{\alpha} \int \mathrm{d}\lambda [\rho_\alpha(\lambda) + \rho_\alpha^h(\lambda)] \left[\rho_\alpha(\lambda) \log \rho(\lambda) + \rho_\alpha^h(\lambda) \log \rho^h(\lambda)\right]$$

entropy of the state $\dfrac{e^{-Q}}{\mathrm{tr}[e^{-Q}]}$

maximises the entropy at fixed expectation value of Q

Yang-Yang entropy

$$\lim_{L\to\infty}\frac{S^{YY}}{L} = -\sum_{\alpha}\int d\lambda[\rho_\alpha(\lambda)+\rho^h_\alpha(\lambda)]\Big[\rho_\alpha(\lambda)\log\rho(\lambda)+\rho^h_\alpha(\lambda)\log\rho^h(\lambda)\Big]$$

entropy of the state $\dfrac{e^{-Q}}{\mathrm{tr}[e^{-Q}]}$ *maximises the entropy at fixed expectation value of* Q

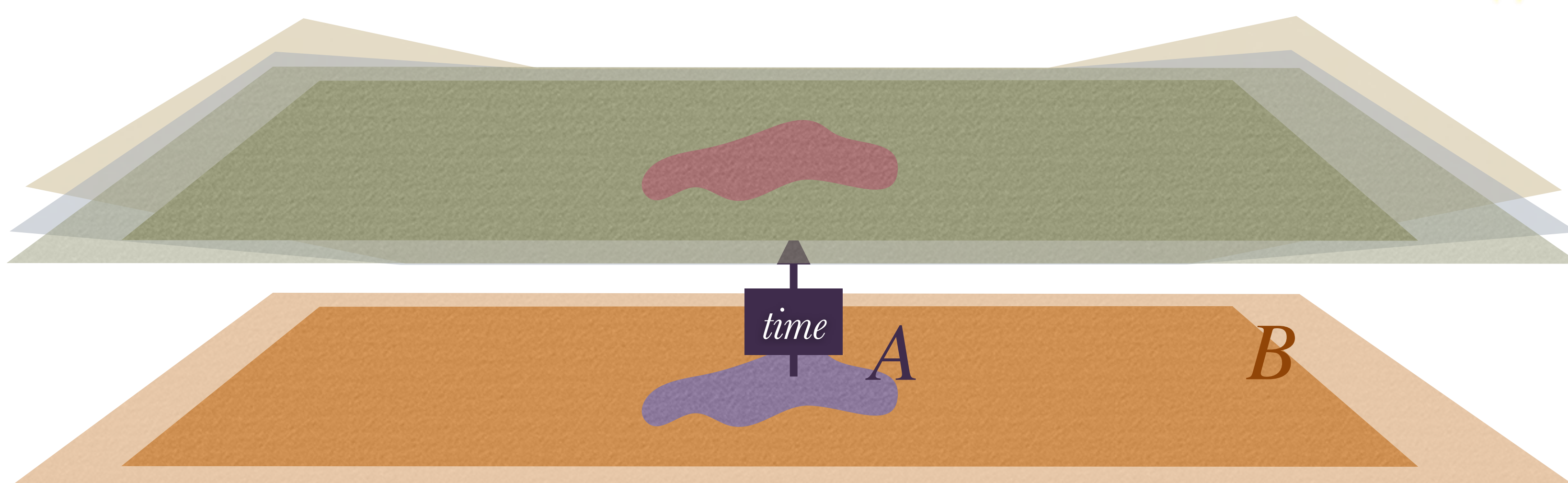
this comes to hand for determining the **"thermodynamic entropy"**
after a quantum quench from a homogeneous state

$$\boxed{S_{vN} = \sup_{\rho^{MS}}(-\mathrm{tr}[\rho^{MS}\log\rho^{MS}])}$$

$$(\exists)\lim_{t\to\infty}\lim_{|B|\to\infty}\rho_A(t) = \mathrm{Tr}_B[\rho^{MS}]$$

$$\rho^{MS}\to\begin{cases}\rho_{Gibbs}\propto e^{-\beta H} & \textbf{generic } H\\ \rho_{GGE}\propto e^{-\sum_n \lambda_n Q_n} & \textbf{integrable } H\end{cases}$$

conserved (quasi)local charges



Integrable systems with a TBA description

TBA equations for an excited state represented by $\dfrac{e^{-Q}}{\mathrm{tr}[e^{-Q}]}$

bare charge associated with the state

$$u_n(\lambda) = 1$$

$$\log(\hat{\vartheta}^{-1} - \hat{1})\vec{u} = -\vec{q} - \hat{T}\hat{\sigma}\log(\hat{1} - \hat{\vartheta})\vec{u}$$

$$\boxed{\vec{a}} = (\boxed{\hat{\sigma}}\,\hat{\vartheta}^{-1} + \boxed{\hat{T}})\vec{\rho}$$

fixed by the Hamiltonian

$$\vartheta_\ell(\lambda) = \frac{\rho_\ell(\lambda)}{\rho_\ell(\lambda) + \rho_\ell^h(\lambda)}$$

density of hole excitations

filling function

(interacting generalisation of the occupation number)

Integrable systems with a TBA description

Castro-Alvaredo, Doyon, Yoshimura, Phys. Rev. X **6**, 041065 (2016)

Bertini, Collura, De Nardis, MF, Phys. Rev. Lett. **117**, 207201 (2016)

TBA equations for an excited state represented by $Z^{-1}\exp\Big(-\sum_n q_n^{(n/\xi)}\Big) \to \Big|\rho_{\alpha;\frac{x}{\xi}}(\lambda)\Big\rangle$

in the limit $\xi \to \infty$**, when** $[H, \sum_n q_n^x] = 0$

$$\partial_t \rho_{\alpha;x,t}(\lambda) + \partial_x v_{\alpha;x,t}(\lambda)\rho_{\alpha;x,t}(\lambda) = o(\xi^0)$$

$$e'_\alpha(\lambda) = 2\pi\sigma_\alpha[\rho_{\alpha;x,t}(\lambda) + \rho^h_{\alpha;x,t}(\lambda)] + 2\pi\sum_\beta \int \mathrm{d}\mu T_{\alpha,\beta}(\lambda-\mu)v_{\alpha;x,t}(\mu)\rho_{\alpha;x,t}(\mu)$$

dressed velocity

(defined as $\dfrac{\Delta E}{\Delta P}$ *after the addition of a rapidity)*

Bonnes, Essler, Läuchli, Phys. Rev. Lett. **113**, 187203 (2014)

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Bonnes, Essler, Läuchli, Phys. Rev. Lett. **113**, 187203 (2014)

this comes to hand for determining the **"thermodynamic entropy"** after a quantum quench from an inhomogeneous state

$$\Big|\Psi_0\Big\rangle=\Big|\Psi_L\Big\rangle\otimes\Big|\Psi_R\Big\rangle\xrightarrow[x=\zeta\xi \atop Jt=\xi]{\xi\to\infty}\Big|\rho_{\alpha;\zeta}(\lambda)\Big\rangle$$

locally quasistationary state (LQSS)

Bertini, MF, Phys. Rev. Lett. **117**, 130402 (2016)

Entropy

1. … of the state $S_{vN} = -\,\mathrm{tr}[\rho(t)\log\rho(t)] = 0$

2. … of the time averaged state $S_{vN} = -\,\mathrm{tr}[\bar{\rho}_{0,t}\log\bar{\rho}_{0,t}] \sim \frac{1}{2}\left(\log L^d + \log\frac{e e_2 t^2}{2\pi}\right)$

3. thermodynamic entropy

$$S_{vN} = \sup_{\rho^{MS}}(-\mathrm{tr}[\rho^{MS}\log\rho^{MS}])$$

$$\sim \begin{cases} S_{Gibbs} & \text{generic } H, \text{ homogeneous } |\Psi_0> \\ S_{GGE} & \text{integrable } H, \text{ homogeneous } |\Psi_0> \\ S_{LQSS} & \text{integrable } H, \text{ inhomogeneous } |\Psi_0> \end{cases}$$

entropy "*per unit rapidity*" ➔ ***Yang-Yang entropy***

$$\partial_t S^{YY}_{x,t}(\lambda) + \partial_x[v_{x,t}(\lambda) S^{YY}_{x,t}(\lambda)] \sim 0$$

4. entanglement entropy (half chain/subsystem) $S_{vN}[A] = -\,\mathrm{tr}[\rho_A(t)\log\rho_A(t)]$

5. …

Bipartite entanglement

complete orthogonal bases of the corresponding spaces

Schmidt decomposition

$$\left|\Psi\right\rangle = \sum_{n=1}^{M} \sqrt{p_n} \left|\Psi_n^A\right\rangle \otimes \left|\Psi_n^B\right\rangle$$

pure state

no classical (thermal) correlations

quantum correlations *between A and B*

$$\boxed{\textbf{entanglement entropy} \quad S_{vN} = -\sum_n p_n \log p_n}$$

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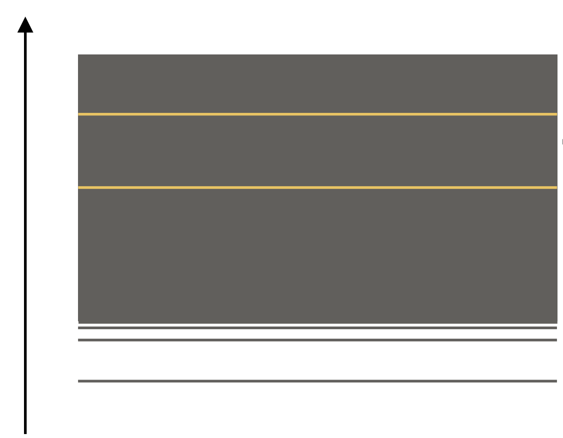
critical

E

non-critical



volume law $S_{vN}(A) \sim |A|$

area law $S_{vN}(A) \sim |\partial A|$

Eisert, Cramer, Plenio, Rev. Mod. Phys. **82**, 277 (2010)

Bipartite entanglement

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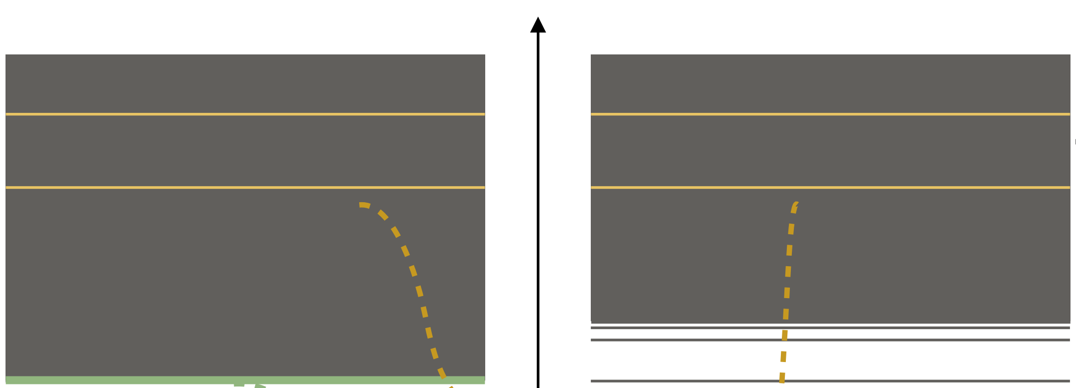

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area law $S_{vN}(A) \sim |\partial A|$

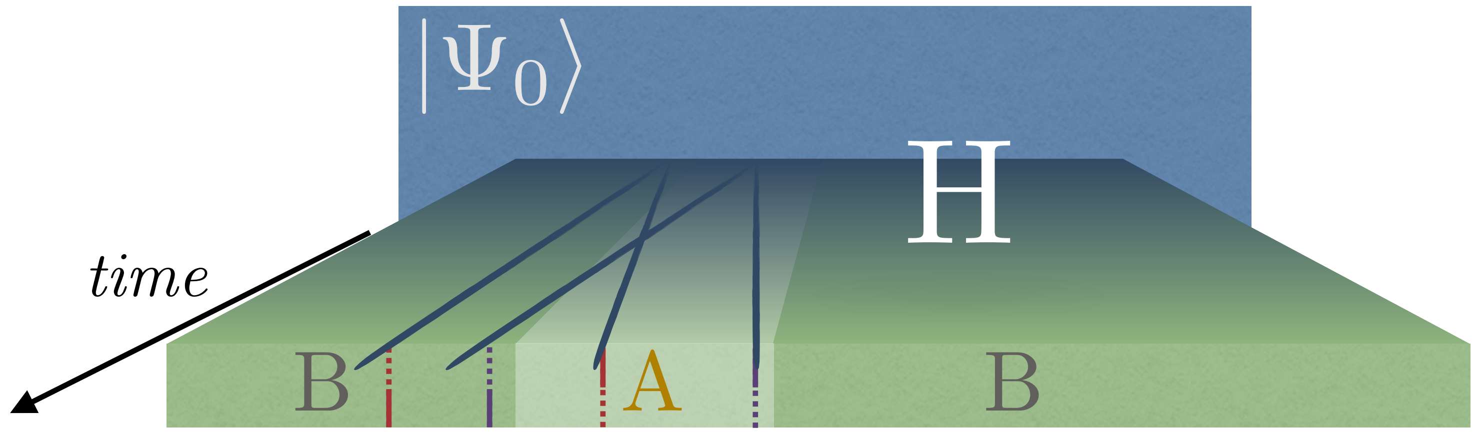
(integrable systems)

Eisert, Cramer, Plenio, Rev. Mod. Phys. **82**, 277 (2010)

log-breaking of area law $S_{vN}(A) \sim \log |A|$

Semi-classical theory

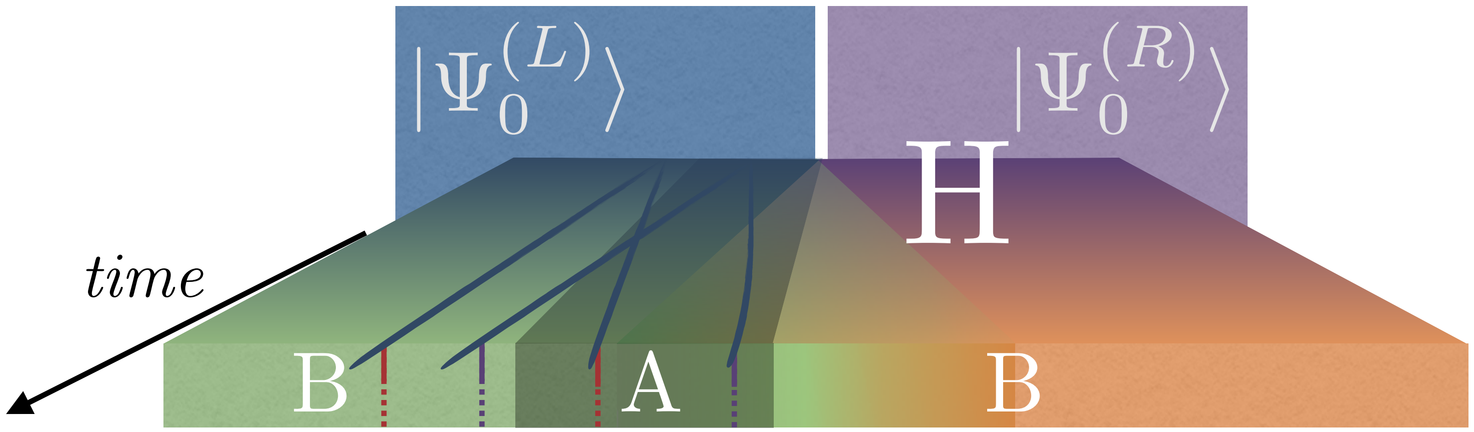
Calabrese, Cardy, J.Stat. Mech. P04010 (2005)



A. Low-entangled initial state
B. Strong correlations only between quasiparticle excitations with opposite velocities
C. Low entanglement *(entangled semiclassical excitations are close to one another)*
D. Entanglement not transferred when excitations scatter
E. Translational invariance *(semiclassical excitations originated everywhere)*

Semi-classical theory

Bertini, MF, Piroli, Calabrese, J. Phys. A **51**, 39LT01 (2018)



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E. Translational invariance *(semiclassical excitations originated everywhere)*

F. Bipartite entanglement equivalent to "particle entanglement" of the set of semiclassical excitations in A

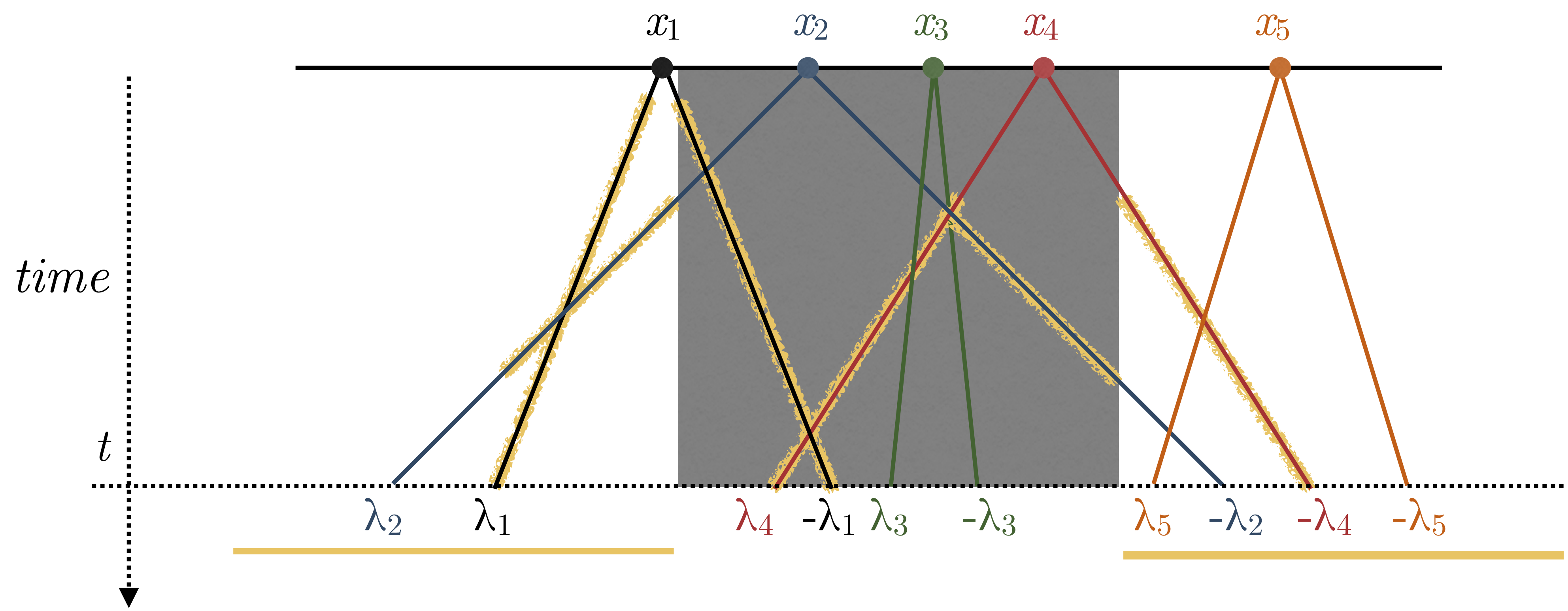
very effective in noninteracting spin chains

still not exploited in the presence of interactions

(several issues to be understood and solved)

Semi-classical theory: no interaction

initial state → tensor product of pairs (or groups) of semiclassical quasiparticles



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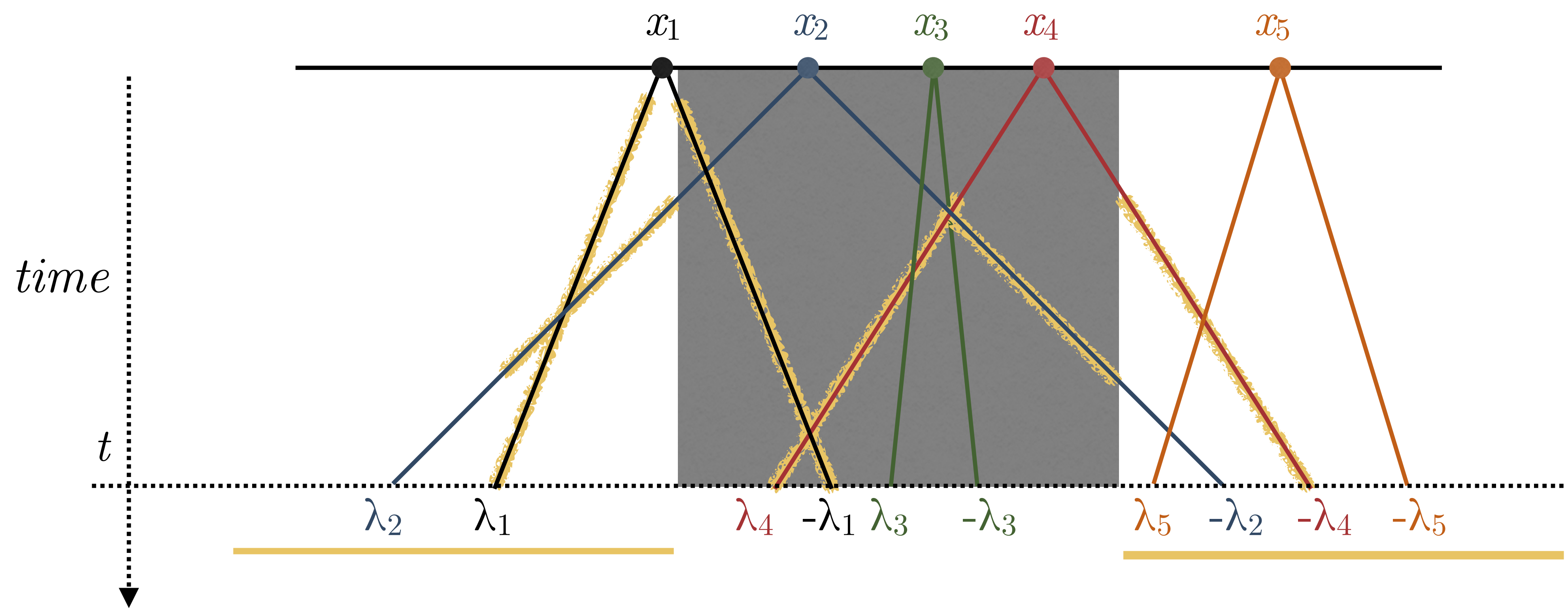

the spatial bipartite entanglement is equivalent to the particle bipartite entanglement

$$\rho_A(t) \sim \mathrm{tr}_{\{x_2,\lambda_2\},\{x_1,\lambda_1\},\{x_5,\lambda_5\},\{x_2.-\lambda_2\},\{x_5,\lambda_5\},\{x_5,-\lambda_5\},\dots}[|\Psi_0\rangle\langle\Psi_0|]$$

$$\downarrow$$

$$\cdots \otimes |\Psi_0^{(x_1)}\rangle\langle\Psi_0^{(x_1)}| \otimes |\Psi_0^{(x_2)}\rangle\langle\Psi_0^{(x_2)}| \otimes \cdots$$

Bertini, MF, Piroli, Calabrese, J. Phys. A **51**, 39LT01 (2018)

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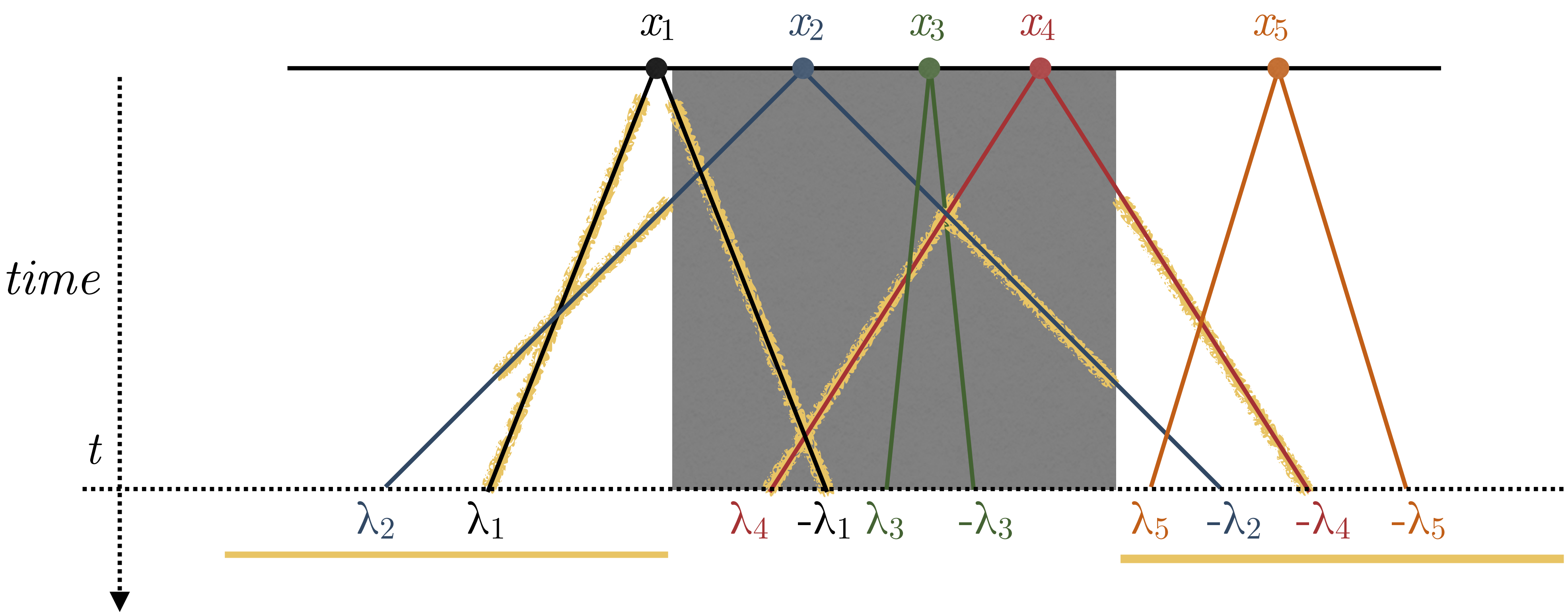

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$$\downarrow$$

$$\cdots \otimes |\Psi_0^{(x_1)}\rangle\langle\Psi_0^{(x_1)}| \otimes |\Psi_0^{(x_2)}\rangle\langle\Psi_0^{(x_2)}| \otimes \cdots$$

Example *(quantum Ising model, ...)*:

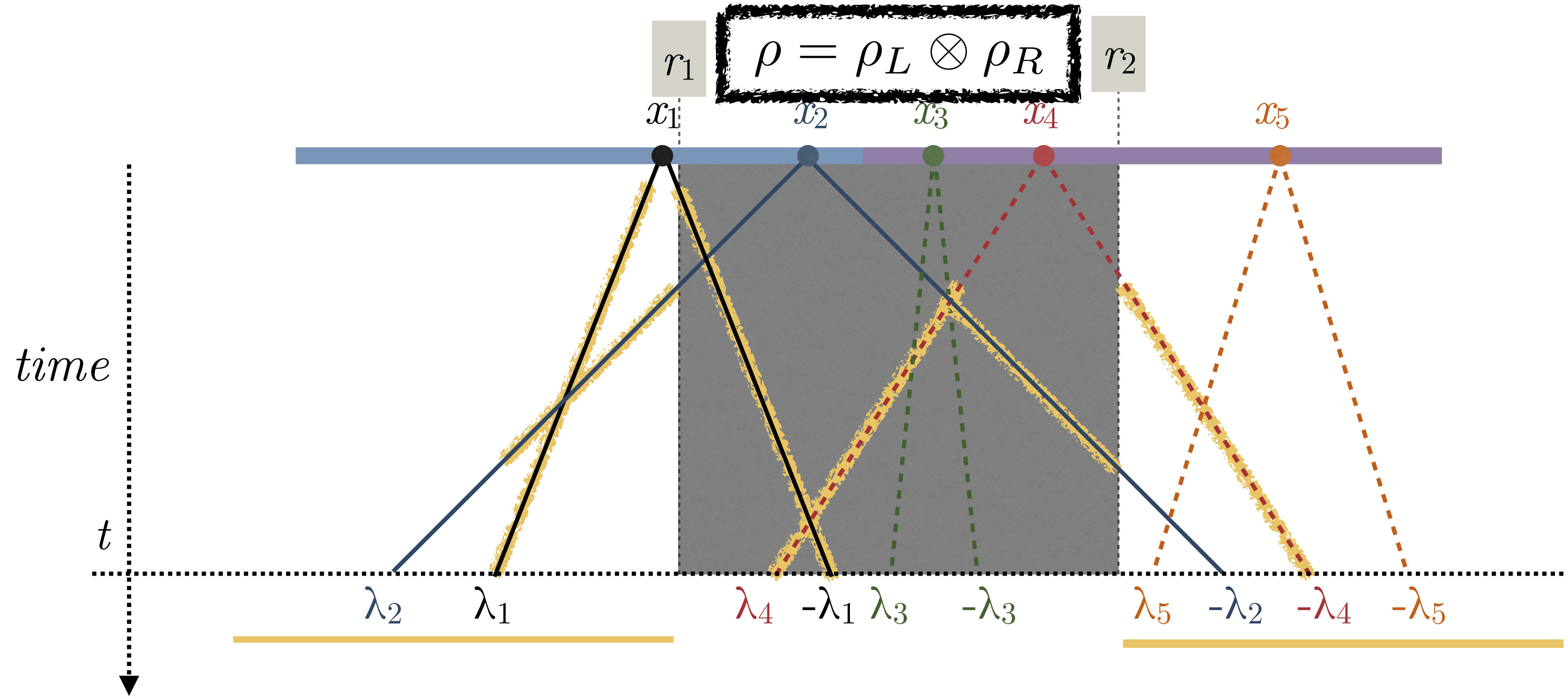
pair structure

$$\rho_\lambda = \mathrm{tr}_{-\lambda}[\rho_{\lambda,-\lambda}] = (1-\vartheta_\lambda)b_\lambda b_\lambda^\dagger + \vartheta_\lambda b_\lambda b_\lambda^\dagger$$

$$\frac{\delta S_{vN}}{\delta x \delta \lambda} = s(\vartheta_\lambda) \equiv \frac{-\vartheta_\lambda \log(\vartheta_\lambda) - (1-\vartheta_\lambda)\log(1-\vartheta_\lambda)}{2\pi}$$

Bertini, MF, Piroli, Calabrese, J. Phys. A **51**, 39LT01 (2018)

Semi-classical theory: no interaction



1. Time evolution encoded in the population of particles in the subsystem
2. The entanglement is computed at the initial time

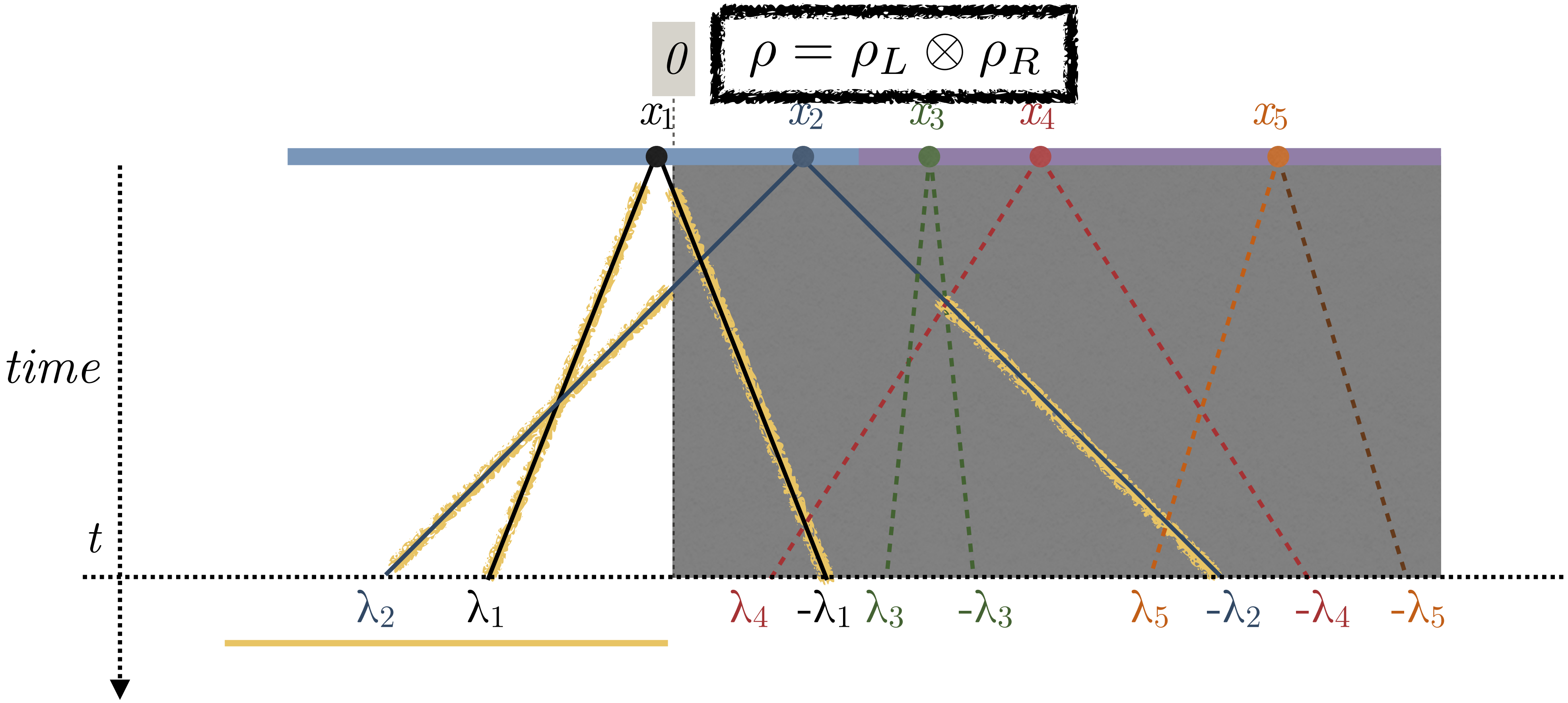
$$S_{[r_1,r_2]}(t) = \int d\lambda \theta_H(-v_\lambda) \int_{\max(r_2+2v_\lambda)t, r_1}^{r_2} dx f_{x-v_\lambda t}(\lambda) + \int d\lambda \theta_H(v_\lambda) \int_{r_1}^{\min(r_1+2v_\lambda)t, r_2} dx f_{x-v_\lambda t}(\lambda)$$

$$f_x(\lambda) = \theta_h(x) s[\vartheta_\lambda^R] + \theta_H(-x) s[\vartheta_\lambda^L]$$

Bertini, MF, Piroli, Calabrese, J. Phys. A **51**, 39LT01 (2018)

Half-chain entropy

no interaction



$$\frac{\mathrm{d}}{\mathrm{d}t}S_{[0,\infty]}(t) = \int \mathrm{d}\lambda \, | v_\lambda(0) | \, S_\lambda^{YY}(0) = \int \mathrm{d}\lambda \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \mathrm{d}x S_\lambda^{YY}(x) \right|$$

it was interpreted as the rate at which the two parts exchange thermodynamic entropy

(through attributing a thermodynamic entropy to each particle)

Alba, Phys. Rev. B **97**, 245135 (2018)

Semi-classical theory: *interaction*

nontrivial trajectories:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{x(t)}{t} = \frac{v(x(t)/t)}{t} - \frac{x(t)}{t^2}$$

$$\int_{\frac{x(t_0)}{t_0}}^{\frac{x(t)}{t}} \frac{\mathrm{d}\zeta}{v(\zeta)-\zeta} = \log\frac{t}{t_0}$$

Doyon, Spohn, Yoshimura, Nucl. Phys. B **926**, 570 (2018)

Alba, Bertini, MF, Scipost **7**, 005 (2019)

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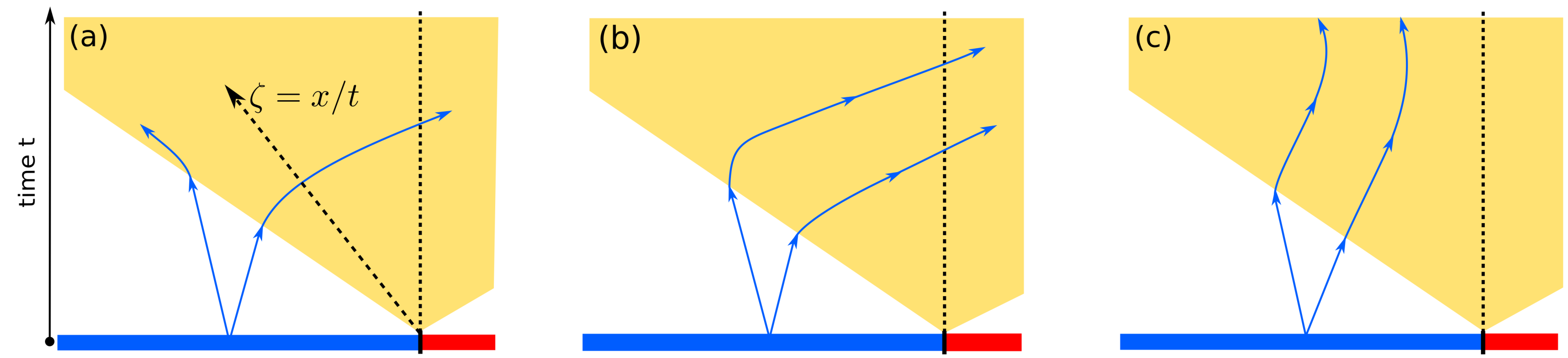
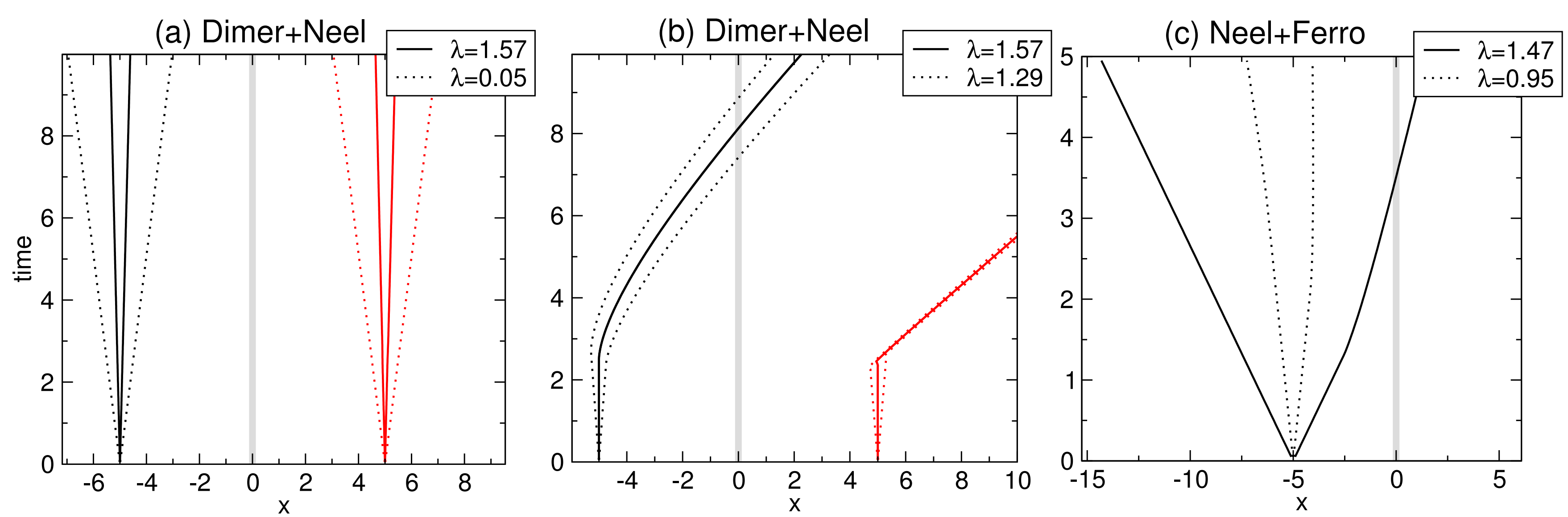
Alba, Bertini, MF, Scipost **7**, 005 (2019)

initial states with a pair structure $(\lambda, -\lambda)$

Piroli, Pozsgay, Vernier, Nucl. Phys. B **925**, 362 (2017)





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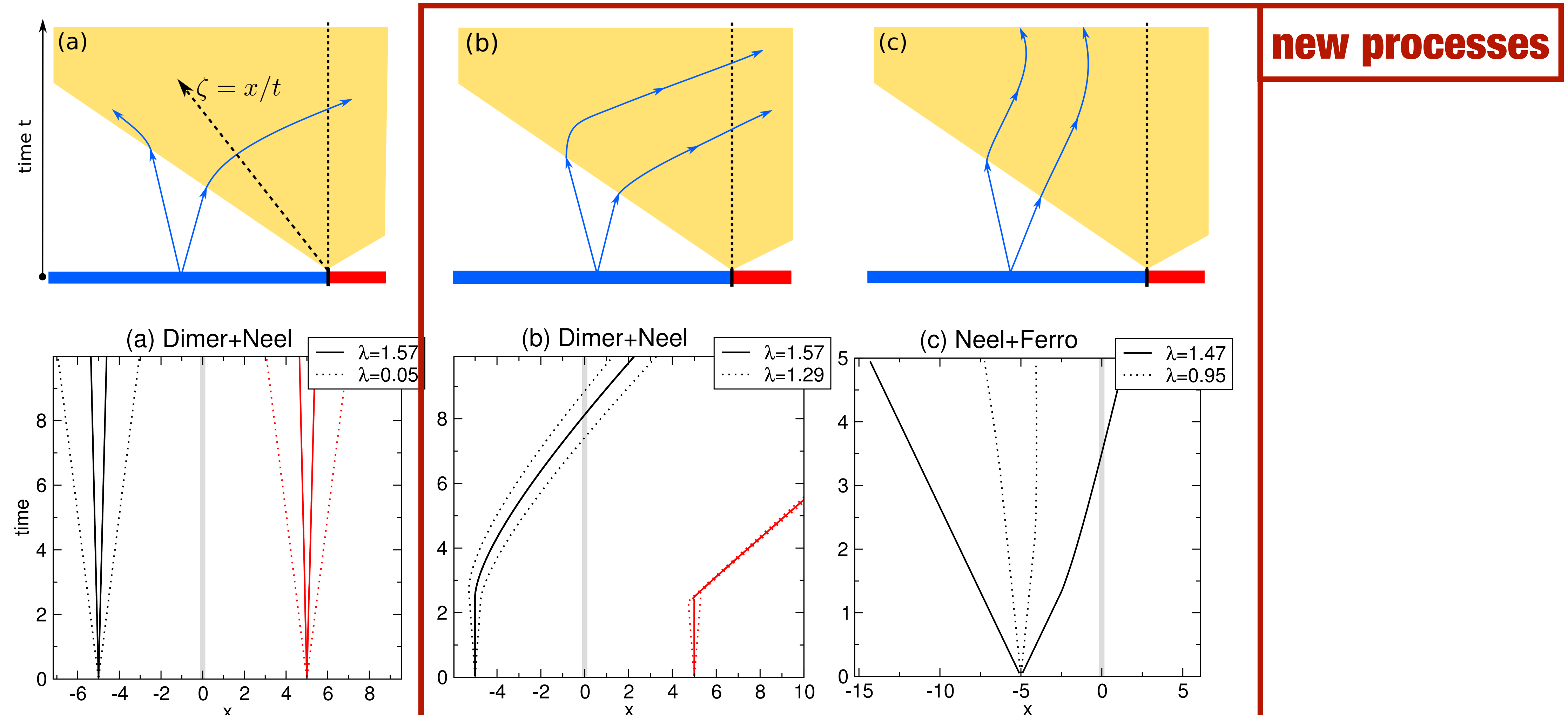


Semi-classical theory: interaction

Alba, Bertini, MF, Scipost **7**, 005 (2019)

what's the entanglement between the semiclassical particles of a pair?

Semi-classical theory: interaction

Alba, Bertini, MF, Scipost **7**, 005 (2019)

<u>what's the entanglement between the semiclassical particles of a pair?</u>

Train of thought

1. entanglement is computed at the initial time
2. being quasi-localised, a semiclassical pair can not be affected by the inhomogeneity

⬇

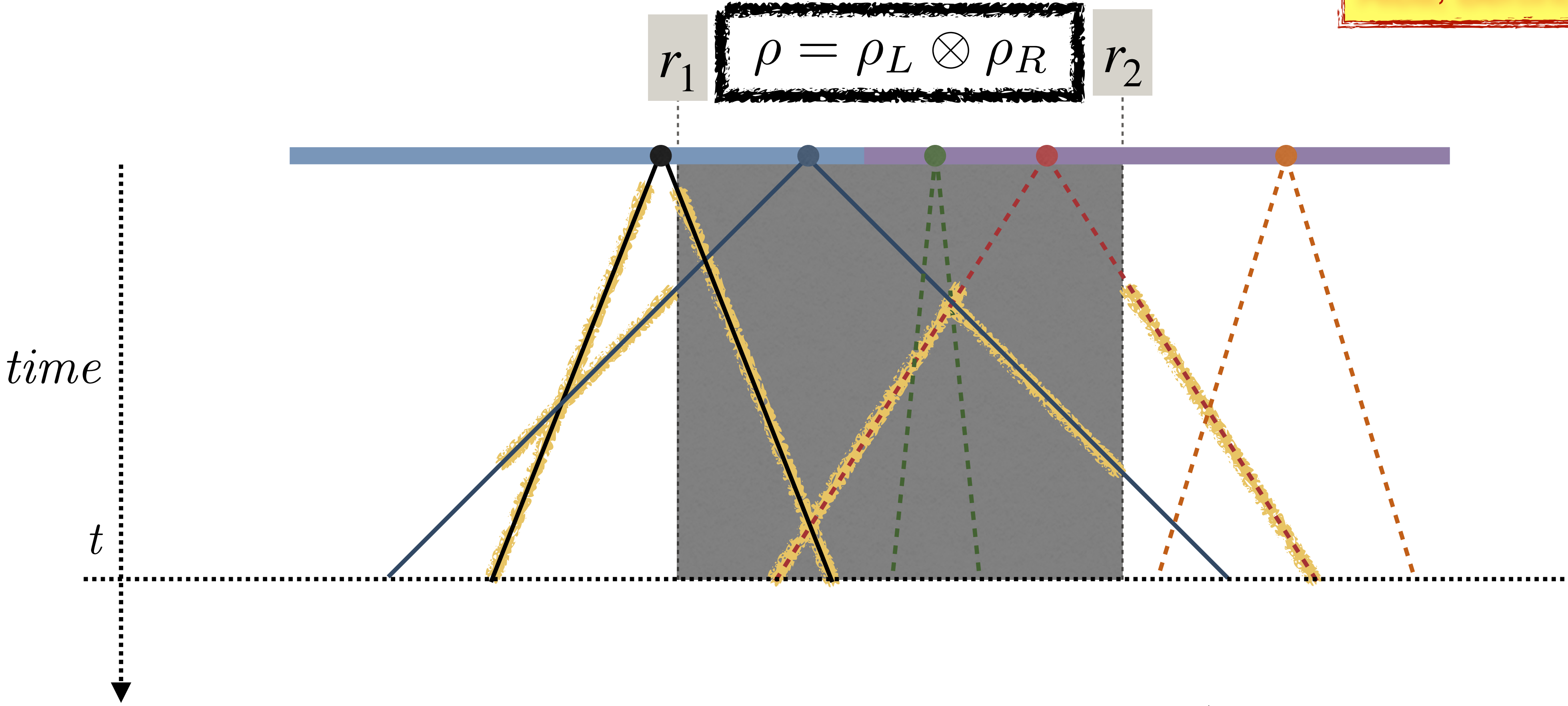
the entanglement entropy of one particle should be the same as in the corresponding homogeneous quench

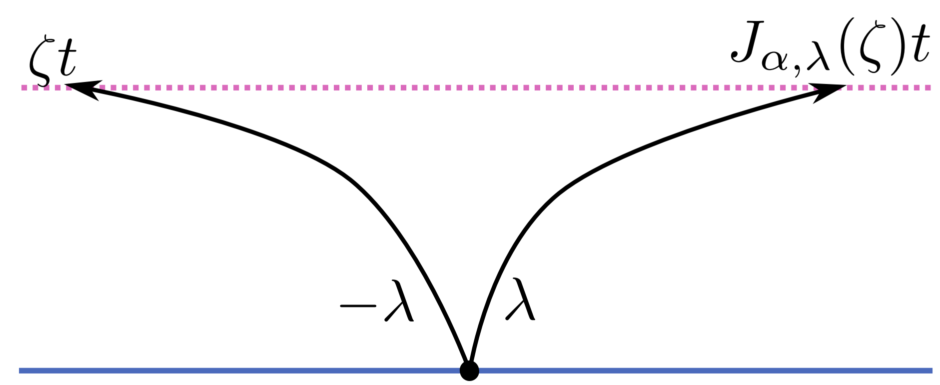
↓

thermodynamic entropy per given rapidity

Alba, Calabrese, PNAS **114**, 7947 (2017)

Alba, Bertini, MF, Scipost **7**, 005 (2019)

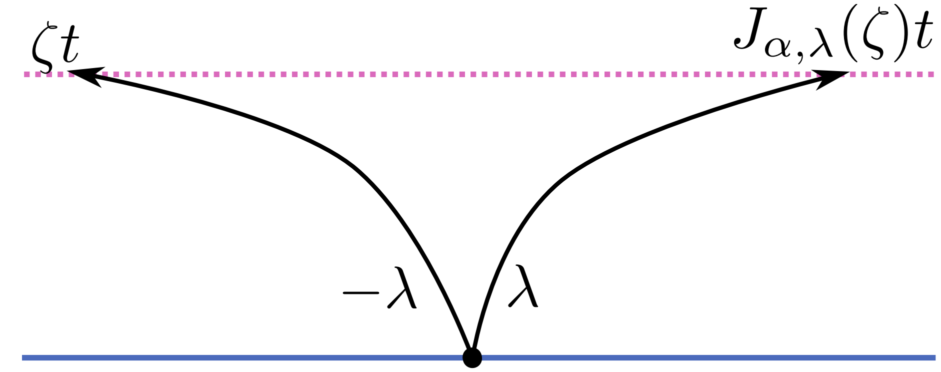




$$
\begin{aligned}
S_{[r_1,r_2]}(t) = t\sum_{\alpha}\int \mathrm{d}\lambda \,\Bigg\{ &\mathrm{sgn}(J_{\alpha,-\lambda}(\frac{r_1}{t}) - \frac{r_1}{t})\mathrm{sgn}(\frac{r_2}{t} - J_{\alpha,-\lambda}(\frac{r_1}{t}))(\frac{r_1}{t} - v_{\alpha,\lambda}(\frac{r_1}{t}))S^{YY}_{\alpha,\lambda}(\zeta_1) \\
&-\mathrm{sgn}(J_{\alpha,-\lambda}(\frac{r_2}{t}) - \zeta_1)\mathrm{sgn}(\frac{r_2}{t} - J_{\alpha,-\lambda}(\frac{r_2}{t}))(\frac{r_2}{t} - v_{\alpha,\lambda}(\frac{r_2}{t}))S^{YY}_{\alpha,\lambda}(\frac{r_2}{t})\Bigg\},
\end{aligned}
$$

Half-chain entropy
interaction

Alba, Bertini, MF, Scipost **7**, 005 (2019)



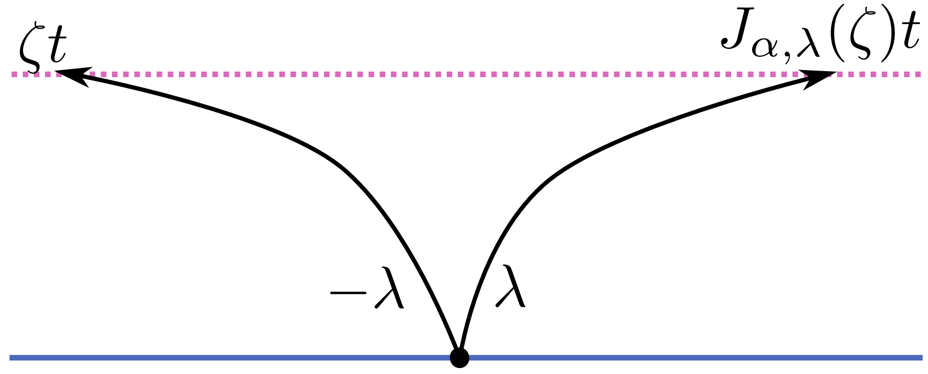
$$S_{[r,\infty]}(t) = t\sum_{\alpha}\int \mathrm{d}\lambda\ \mathrm{sgn}(J_{\alpha,-\lambda}(\frac{r}{t}) - \frac{r}{t})(\frac{r}{t} - v_{\alpha,\lambda}(\frac{r}{t}))S^{YY}_{\alpha,\lambda}(\frac{r}{t})$$

1. $v_{\alpha,\lambda}(\pm\infty)$ are differentiable, periodic functions of λ
2. $v_{\alpha,\lambda}(\pm\infty)$ are odd functions of λ
3. $v_{\alpha,\lambda}(\pm\infty)$ have a single maximum in a period
4. $\mathrm{sgn}(v_{\alpha,\lambda}(+\infty)) = \mathrm{sgn}(v_{\alpha,\lambda}(-\infty)) = \mathrm{sgn}(\lambda)$

$$S_{[0,\infty]}(t) = t\sum_{\alpha}\int \mathrm{d}\lambda\,\mathrm{sgn}(\lambda)v_{\alpha,\lambda}(0)S^{YY}_{\alpha,\lambda}(0)$$

Half-chain entropy
interaction

Alba, Bertini, MF, Scipost **7**, 005 (2019)



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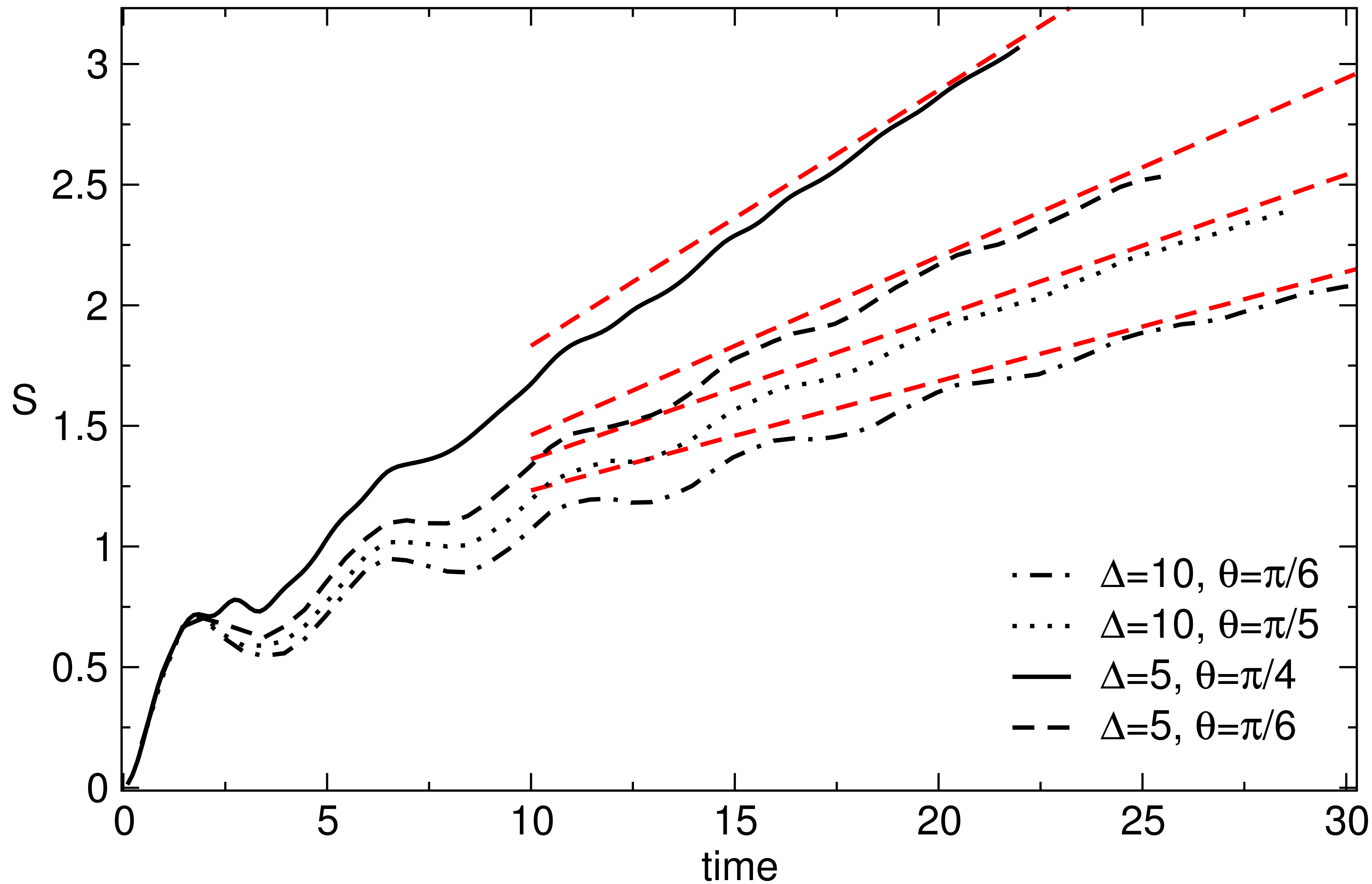
$$S_{[0,\infty]}(t) = t\sum_\alpha \int d\lambda \, \mathrm{sgn}(\lambda) v_{\alpha,\lambda}(0) S^{YY}_{\alpha,\lambda}(0)$$

$$\neq \quad S_{[0,\infty]}(t) = t\sum_\alpha \int d\lambda \, | v_{\alpha,\lambda}(0) | S^{YY}_{\alpha,\lambda}(0)$$

the previously conjectured equivalence to the rate at which the two parts exchange thermodynamic entropy is WRONG!

$$H = \sum_{\ell} s^x_\ell s^x_{\ell+1} + s^y_\ell s^y_{\ell+1} + \Delta s^z_\ell s^z_{\ell+1}$$

$$\left|\Psi_L\right\rangle = \left|\ldots \uparrow \downarrow \ldots\right\rangle \qquad \left|\Psi_R\right\rangle = \left|\ldots \nearrow \nearrow \ldots\right\rangle$$



Conclusions

- Entropy of the time averaged state
- Thermodynamic entropy
- Entanglement entropy of subsystems from inhomogeneous states

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Thank you for your attention