Gauge theory concepts in condensed matter physics

Workshop on:
"Quantum Transport, at the Intersection between Theoretical and Mathematical Physics"

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Credits and ...

I am indebted to – among others – the following colleagues:

- *R. Morf* – mentor in matters of the QHE.
- *G.-M. Graf* – who is able to fill in all the gaps I tend to leave open.
- Collaborators on matters related to this lecture: *A. Alekseev, V. Cheianov, B. Pedrini, A. Pizzo, U. M. Studer, E. Thiran, Ph. Werner.*
- I have profited from listening to lectures by *G.-M. Graf, V. Mastropietro* and *M. Porta*.
- I thank *Krzysztof Gawedzki* and *Paul Wiegmann* for discussions, encouragement and for their friendship.
- I thank the *organizers* for having invited me to present this talk.

... Contents:

1. Gauge Theory of States of Matter – General Ideas

- Goal: Use concepts and results from Gauge Theory, Current Algebra, and GR to develop a *"Gauge Theory of Phases/States of Matter"*, which complements the *Landau Theory of Phases and Phase Transitions* for systems of condensed matter **without** local order parameters, and which yields information on Green functions of currents, whence, via *Green-Kubo formulae*, on transport coefficients (conductivities).
- Use the Gauge Theory of States of Matter to classify ("topologically protected") correlated *bulk- and surface states* of *interacting* systems of condensed matter without local order parameters.
- Key tools to develop a Gauge Theory of States of Matter are:
  - *"Effective Actions"* = generating functionals of connected current Green functions $\leftrightarrow$ transport coeffs., in particular *conductivities*.
  - Implications of *Gauge Invariance* $\leftrightarrow$ current conservation $\leftrightarrow$ Ward ids., *Locality* & *Power Counting* on the form of *Effective Actions* $\rightarrow$ Classification of certain families of States of Matter.
  - *Gauge Anomalies* and their cancellations (anomaly inflow) $\leftrightarrow$ edge (surface) degrees of freedom $\leftrightarrow$ "holography"; etc.

Applications to Condensed-Matter Physics

- *Concrete Examples* (among others) of applications of the Gauge Theory of States of Matter" to the analysis of systems of condensed matter and of their transport properties:
  - Conductance quantization in ideal quantum wires
  - Theory of the Fractional Quantum Hall Effect and of "Chern insulators"
  - Theory of chiral states of light in wave guides
  - Time-reversal invariant planar "topological" insulators and superconductors; chiral edge spin currents, chiral spin liquids
  - Chiral magnetic effect[2]; higher-dimensional cousins of the QHE[3], 3D topological insulators, Weyl semi-metals, etc.

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- Applications in other areas of physics, such as aerodynamics, fluid dynamics, and *cosmology*, (work with Pedrini, Boyarsky, Ruchayskiy, and others)

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[2] Found in a preliminary form by A. Vilenkin; but see Alekseev-Cheianov-JF.
[3] Recently also studied by O. Zilberberg et al.

Digression on Effective Actions

Consider a physical system with matter degrees of freedom described by fields $\overline{\psi}, \psi, \ldots$ over a space-time, Λ, equipped with a metric $g_{\mu\nu}$ of signature $(-1, 1, 1, 1)$. Its dynamics is assumed to be derivable from an action functional $S(\overline{\psi}, \psi, ...; g_{\mu\nu})$. We assume that there is a conserved vector current (density) J^μ, with $\nabla_\mu J^\mu = 0$. If J^μ is carried by charged degs. of freedom, it couples to the em field, here described by its vector potential A_μ. The action of the system is then obtained by replacing ordinary derivatives by covariant ones ("minimal substitution"), and then

$$J^\mu(x) \equiv J^\mu_A(x) = \frac{\delta S(\overline{\psi}, \psi, ...; g_{\mu\nu}, A_\nu)}{\delta A_\mu(x)} \tag{1.1}$$

The *Effective Action* of the system on a space-time Λ with metric $g_{\mu\nu}$ and in an external electromagnetic field with vector potential A is then defined by, e.g., the functional integral[4]

$$\begin{aligned} S_{eff}(g_{\mu\nu}, A_\mu) \quad &:= \quad -i\hbar \; \ln\left(\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}\ldots \, exp\left[\frac{i}{\hbar} S(\overline{\psi}, \psi, \ldots; g_{\mu\nu}, A_\mu)\right]\right) \\ &+ \quad \text{(divergent) const.} \end{aligned} \tag{1.2}$$

[4] For an operator-definition see blackboard!

Properties of S_{eff}

A precise definition of R.S. in (1.2) requires specifying initial and final field configurations, e.g., corresponding to ground- or KMS-states.

1. The variational derivatives of S_{eff} with respect to A_μ are given by connected current Green functions:

$$\frac{\delta S_{eff}(g_{\mu\nu}, A_\mu)}{\delta A_\mu(x)} = \langle J^\mu(x) \rangle_{g,A}\,, \tag{1.3}$$

   and

$$\frac{\delta^2 S_{eff}(g_{\mu\nu}, A_\mu)}{\delta A_\mu(x)\, \delta A_\nu(y)} = \langle J^\mu(x) J^\nu(y) \rangle^c_{g,A}\,, \quad x \neq y\,, \tag{1.4}$$

   where $\langle (\cdot) \rangle_{g,A} = \ldots$, etc. ($\nearrow$ blackboard)

2. Consider effect of a gauge transformation, $A_\mu \mapsto A_\mu + \partial_\mu \chi$, where χ is an arbitrary smooth function on Λ, on the effective action, S_{eff}. After an integration by parts we find that

$$\frac{\delta S_{eff}(g_{\mu\nu}, A_\mu + \partial_\mu \chi)}{\delta \chi(x)} = \nabla_\mu \langle J^\mu(x) \rangle_{g,A} = 0 \tag{1.5}$$

   vanishes, because J^μ is conserved. Thus, S_{eff} is invariant under gauge transformations!

Properties of S_{eff} – ctd.

3. We may also vary S_{eff} with respect to the metric $g_{\mu\nu}$:

$$\frac{\delta S_{eff}(g_{\mu\nu}, A_\mu)}{\delta g_{\mu\nu}(x)} = \langle T^{\mu\nu}(x)\rangle_{g,A}\,,$$

   where $T^{\mu\nu}$ is the energy-momentum tensor of the system. Using local energy-momentum conservation, i.e., $\nabla_\mu T^{\mu\nu} = 0$, we find that $S_{eff}(g_{\mu\nu}, A_\mu)$ is invariant under coordinate transformations on Λ.

   A general (curved) metric g_{kj} on the sample space can be used to describe defects – disclinations – in a condensed-matter system.

   Invariance of S_{eff} under Weyl rescalings of the metric g_{kj} (i.e., under local variations of the density) implies that $\langle T^j_j(x)\rangle_{g,A} \equiv 0 \leftrightarrow$ scale-invariance (criticality) of the system.

4. If Hamiltonian of system exhibits positive energy gap above ground-state energy (insulator) the zero-temperature conn. current Green fcts. have good decay props. in space and time. In the scaling limit, i.e., in the limit of very large distances and very low frequencies, its eff. action then approaches a functional that is a space-time integral of *local, gauge-invariant polynomials* in A_μ and derivatives of A_μ.

Form of effective actions in the scaling limit

These terms can be organized according to their scaling dimensions, (power counting)

Properties 1 through 4 enable us to determine the general form of most relevant terms in effective actions, S_{eff}, (of insulators ...) in scaling limit!

Example: We consider an insulator with broken parity and time-reversal confined to a flat 2D region. Then $S_{eff}(A_\mu)$ tends to

$$\frac{\sigma_H}{2}\int_\Lambda A\wedge dA+\frac{1}{2}\int_\Lambda d^3x\sqrt{-g}\,[\underline{E}(x)\cdot\varepsilon\underline{E}(x)-\mu^{-1}B(x)^2]+\cdots,$$

as the scaling limit is approached, where σ_H is the Hall conductivity, ε is the tensor of dielectric constants, and μ is the magnetic permeability. –

Note: Chern-Simons term *not* gauge-invariant if $\partial\Lambda\neq\emptyset\to$ *holography*!

We also use extensions of these concepts to *non-abelian* gauge fields and currents only covariantly conserved. Such gauge fields may represent "real" external fields; but also "virtual" ones merely serving to develop the response theory needed to determine transport coefficients.

(For details, see my 1994 Les Houches lectures.)

2. Two-Dimensional Time-Reversal Invariant Topological Insulators and Chiral Edge Spin Current

As a first application, we consider *time-reversal invariant 2D topological insulators* (2D TRI TI) exhibiting *chiral edge spin currents*. We begin by recalling the *Pauli equation for a spinning electron*:

$$i\hbar D_0 \Psi_t = -\frac{\hbar^2}{2m} g^{-1/2} D_k \, g^{1/2} g^{kl} \, D_l \Psi_t \,, \tag{2.1}$$

where m is the (effective) mass of an electron, (g_{kl}) = metric on sample space $\Omega \subseteq \mathbb{R}^3$. Pauli spinors Ψ_t are (time-dependent) sections of spinor bundle over Ω:

$$\Psi_t(x) = \begin{pmatrix} \psi_t^{\uparrow}(x) \\ \psi_t^{\downarrow}(x) \end{pmatrix} \in L^2(\Omega, d\,vol.) \otimes \mathbb{C}^2 : \quad \text{2-component Pauli spinor}$$

Furthermore, the covariant derivatives are given by

$$i\hbar D_0 = i\hbar \partial_t + e\varphi - \underbrace{\vec{W}_0 \cdot \vec{\sigma}}_{\text{Zeeman coupling}} \,, \quad \vec{W}_0 = \mu c^2 \vec{B} + \cdots \,, \tag{2.2}$$

$U(1)_{em} \times SU(2)_{spin}$ - gauge invariance

where $\varphi =$ electrostatic potential, $\vec{B} =$ magnetic induction, $\mu =$ magnetic moment of electron;

$$\frac{\hbar}{i} D_k = \frac{\hbar}{i} \nabla_k + eA_k - \vec{W}_k \cdot \vec{\sigma} + \cdots \,, \tag{2.3}$$

$\vec{\nabla}$ is the covariant gradient, $\vec{A}$ is the vector potential, the dots stand for terms arising in a moving frame (ignored in the following), and

$$\vec{W}_k \cdot \vec{\sigma} := \underbrace{[(-\tilde{\mu}\vec{E} + \cdots) \wedge \vec{\sigma}]_k}_{\text{spin-orbit interactions}}, \quad \text{with } \vec{E} = \text{ electric field }, \tag{2.4}$$

with $\tilde{\mu} = \mu + \frac{e\hbar}{4mc^2}$, (the 2^{nd} term is due to Thomas precession).
Pauli eq. (2.1) displays perfect $U(1)_{em} \times SU(2)_{spin}$ - gauge invariance.

We now consider an interacting gas of electrons confined to a region Ω of a 2D plane, with $\vec{B} \perp \Omega$ and $\vec{E} \| \Omega$. Then the $SU(2)$ - connection, $\vec{W}_\mu$, is given by

$$W^3_\mu \cdot \sigma_3, \quad \text{with } W^K_\mu \equiv 0, \text{ for } K = 1, 2. \tag{2.5}$$

Effective action of a 2D T-invariant topological insulator

From (2.5) we conclude that parallel transport of Pauli spinors splits into parallel transport for spin $\uparrow$ and for spin $\downarrow$. The component $\psi^{\uparrow}$ of a Pauli spinor Ψ couples to the *abelian* connection $a + w$, while $\psi^{\downarrow}$ couples to $a - w$, where

$$a_\mu = -eA_\mu, \text{ and } w_\mu = W^3_\mu, \quad (\text{see } (2.2) - (2.4)).$$

Under time reversal, T,

$$a_0 \to a_0,\ a_k \to -a_k, \text{ but } w_0 \to -w_0,\ w_k \to w_k. \tag{2.6}$$

The dominant term in the effective action of a 2D TRI topological insulator, with $\vec{W}$ as in (2.5), is a Chern-Simons term. If either $w \equiv 0$ or $a \equiv 0$ a Chern-Simons term in a or in w alone would *not* be T-invariant. If $w \equiv 0$ the dominant term would thus be given by

$$S_\Lambda(A) = \int_\Lambda dt\, d^2x \{\varepsilon \underline{E}^2 - \mu^{-1} B^2\}, \tag{2.7}$$

which is the effective action of a conventional insulator.

The Chern-Simons effective action

In the presence of both a and w, a combination of *two* Chern-Simons terms *is*, however, T-invariant:

$$\begin{aligned} S_\Lambda(a,w) &= \frac{\sigma}{2}\int_\Lambda \{(a+w)\wedge d(a+w)-(a-w)\wedge d(a-w)\} \\ &= \sigma\int_\Lambda \{a\wedge dw + w\wedge da\}\,, \end{aligned} \tag{2.8}$$

up to boundary terms.[5] The gauge fields a and w transform indep. under gauge transformations (preserving (2.5)), and the action (2.8) is anomalous under these gauge transformations on a 2D sample Ω with non-empty boundary. The anomalous chiral boundary action,

$$\sigma\left[\Gamma^{+}_{\partial\Omega\times\mathbb{R}}\big((a+w)_{\|}\big)-\Gamma^{-}_{\partial\Omega\times\mathbb{R}}\big((a-w)_{\|}\big)\right], \tag{2.9}$$

where

$$\Gamma^{(\pm)}_{\partial\Lambda}(a) := \tfrac{1}{2}\int_{\partial\Lambda}[a_+a_- - a_\pm \tfrac{\partial^2_\mp}{\square}a_\pm]du^+\,du^-$$

cancels the anomalies of the bulk action.

[5] The eff. action (2.8) first appeared in a paper w. U. M. Studer in 1993!

Chiral edge spin currents

The boundary action is the generating functional of connected Green functions of two counter-propagating chiral edge currents One of the two counter-propagating edge currents has spin $\uparrow$ (in $+3$-direction $\perp \Omega$), the other one has spin $\downarrow$. Thus, a net *chiral spin current*, s^3_{edge}, can be excited to propagate along the edge.

The bulk response equations (analogous to Hall's law) are given by

$$\boxed{j^k(x) = 2\sigma\varepsilon^{k\ell}\partial_\ell B(x), \quad s_3^\mu(x) = \frac{\delta S_\Lambda(a,w)}{\delta w_\mu(x)} = 2\sigma\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}(x)} \tag{2.10}$$

The second equation also implies that $\exists$ chiral edge spin-currents.

We should ask what kinds of quasi-particles in the bulk of such materials could produce the bulk Chern-Simons terms in (2.8): Knowing about the induced Chern-Simons term of QED_3, we argue that a 2D TRI topological insulator with bulk eff. action given in (2.8) must exhibit two species of charged quasi-particles in the bulk, with one species (spin $\uparrow$) related to the other one (spin $\downarrow$) by T.

Experimental situation

Each species has two degenerate states per wave vector mimicking a two-component Dirac fermion at small energies $\Rightarrow$ *quantization of* σ !

Materials of this kind have been produced and studied in the lab of L. Molenkamp in Würzburg.

The experimental data are not very clean, the likely reason being that, due to small magnetic impurities and/or electric fields in the direction $\perp \Omega$, condition (2.5) is violated, i.e., the $SU(2)$-gauge field $\vec{W}_\mu$ does not only have a non-vanishing 3-comp. in spin space and is genuinely non-abelian. In this situation, the spin current is not conserved, anymore, (but continues to be covariantly conserved), and T is broken.

The approach to 2D time-reversal invariant topological insulators outlined here can be generalized: Consider a state of matter with a bulk spectrum of two species of quasi-particles related to one another by T.

Generalizations

Want to study transport properties of such systems $\rightarrow$ study response of state when one species is coupled to a (real or virtual, abelian or *non-abelian*) ext. gauge field[6] W^+, the other one to a gauge field W^- related to each other by time-reversal, T, according to

$$(W_0^+)^T = W_0^-, \quad (W_k^+)^T = -W_k^-$$

Assuming again that the leading term in the effective action for the gauge fields W^+ and W^- is given by the sum of two identical Chern-Simons terms, but with opposite signs, time-reversal invariance is manifest, and one concludes that there are *two counter-propagating chiral edge currents* generating current (Kac-Moody) algebras (at level 1, for non-interacting electrons) based on the Lie group given by the gauge group of the gauge fields $W^\pm$. For non-interacting electrons, this group can usually be determined from band theory!

If one gives up the requirement of time-reversal invariance one arrives at a theory of <u>chiral states of matter</u>. In particular, if $\vec{W}$ is an $SU(2)$-gauge field coupling to the spin of electrons (see (2.2) and (2.4)) one finds a framework to describe *chiral spin liquids*; (Les Houches 1994).

[6] often dubbed "Berry connection" (!)

3. 3D Topological Insulators and Weyl Semi-Metals

Next, study 3D systems representing topological insulators and Weyl semi-metals on a sample space-time $\Lambda := \Omega \times \mathbb{R}$, with $\partial\Omega \neq \emptyset$.
Eff. action describes response of systems to turning on external em field.
Until mid nineties, eff. action of *3D insulator* thought to be given by

$$S_\Lambda(A) = \frac{1}{2}\int_\Lambda dt\, d^3x \{\vec{E}\cdot\varepsilon\vec{E} - \vec{B}\cdot\mu^{-1}\vec{B}\} + \text{ “irrelevant” terms}\,, \quad (3.1)$$

where ε is the tensor of dielectric constants and μ is the magnetic permeability tensor. The action (3.1) is dimensionless. In $70's$, particle theorists taught us that one could add another dimensionless term:

$$S_\Lambda(A) \to S_\Lambda^{(\theta)}(A) := S_\Lambda(A) + \theta\, I_\Lambda(A)\,, \quad (3.2)$$

where I_Λ is a *“topological” term*, the “instanton number”, given by

$$\begin{aligned} I_\Lambda(A) &= \frac{1}{4\pi^2}\int_\Lambda dt\, d^3x\, \vec{E}(\vec{x},t)\cdot\vec{B}(\vec{x},t) = \\ &= \frac{1}{8\pi^2}\int_\Lambda F\wedge F \underset{Stokes}{=} \frac{1}{8\pi^2}\int_{\partial\Lambda} A\wedge dA, \qquad \left(\frac{e^2}{h}=1\right) \quad (3.3) \end{aligned}$$

"Vacuum angle" and surface degrees of freedom

In particle physics, the parameter θ is called "vacuum (or ground-state) angle". The "partition function" of an insulator (after having integrated over all matter degrees of freedom) is given by

$$\Xi_\Lambda^{(\theta)}(A) = \exp\left(iS_\lambda^{(\theta)}(A)\right),$$

with $S_\Lambda^{(\theta)}$ as in (3.2), (3.3). In the thermodynamic limit, $\Omega \nearrow \mathbb{R}^3$, $\Xi_\Lambda^{(\theta)}(A)$ is periodic in θ with period 2π and invariant under time reversal iff

$$\boxed{\theta = 0, \pi}$$

For $\theta = \pi\,, \partial\Lambda \neq \emptyset$, $\Xi_\Lambda^{(\theta)}(A)$ contains a factor only depending on $A|_{\partial\Lambda}$,

$$\exp\left(\pm\frac{i}{8\pi}\int_{\partial\Lambda} A \wedge dA\right), \tag{3.4}$$

breaking time reversal invariance: Must be cancelled by partition function of surface degs. of freedom[7] on $\partial\Lambda$ exhibiting a Hall conducivity of

$$\sigma_H = \mp\frac{1}{2}\cdot\frac{e^2}{h} \tag{3.5}$$

[7] I am indebted to H.-G. Zirnstein for instructive discussions of this point

Promoting the vacuum angle θ to an "axion"

As one learns from QED_3, the "boundary partition function" (3.4) is the partition function of one species of massless 2-component Dirac fermions coupled to $A|_{\partial\Lambda}$. *Gapless quasi-particles with spin $\frac{1}{2}$* propagating along $\partial\Lambda$ *could mimick such Dirac fermions* and cancel (3.4).

"Vacuum angle" θ could be ground-state expectation, $\theta = \langle\varphi\rangle$, of dynamical field, φ, called *"axion"*. $\rightarrow$ Replace topological term $\theta\, I_\Lambda(A)$ by

$$I_\Lambda(A,\varphi) := \frac{1}{8\pi^2}\int_\Lambda \varphi\, F\wedge F + S_0(\varphi)\,, \tag{3.6}$$

where $S_0(\varphi)$ is invariant under shifts $\varphi \mapsto \varphi + n\pi$, $n\in\mathbb{Z}$. $\rightarrow$ Realm of *axion-electrodynamics*. The *Maxwell-axion eqs.* are found to be

$$\begin{aligned}
\vec{\nabla}\cdot\vec{B} &= 0, \quad \vec{\nabla}\wedge\vec{E} + \dot{\vec{B}} = 0\,,\\
\vec{\nabla}\cdot\vec{E} &= \frac{e^2}{8\pi^2}(\vec{\nabla}\varphi)\cdot\vec{B}\,,\\
\vec{\nabla}\wedge\vec{B} &= \dot{\vec{E}} - \frac{e^2}{8\pi^2}\{\dot{\varphi}\vec{B} + \vec{\nabla}\varphi\wedge\vec{E}\}\,.
\end{aligned} \tag{3.7}$$

Generalized chiral magnetic effect

From (3.7) we infer formula for the current $\vec{j}$ generated in an em field in the presence of an axion field – *"generalized chiral magnetic effect"* [8]:

$$\boxed{\vec{j} = -\frac{e^2}{4\pi h}\left(\dot{\varphi}\cdot\vec{B} + \vec{\nabla}\varphi\times\vec{E}\right)} \tag{3.8}$$

If φ only depends on time then eq. (3.8) describes the ordinary *chiral magnetic effect*, and $\dot{\varphi} = \mu_\ell - \mu_r \equiv \mu_5$ is the *chiral chemical potential* that tunes the asymmetry between left-handed and right-handed quasi-particles. The equation of motion for $\mu_5 \equiv \dot{\varphi}$ may take the form of a *reaction-diffusion equation* (BFR); see below.

Applications:

First consider a 3D spatially periodic (crystalline) system with a static axion φ, so that $\mu_5 = 0, \vec{E}\cdot\vec{B} \equiv 0$. Taking into account the periodicity of $\exp\big(iI_\Lambda(A,\varphi)\big)$ under shifts, $\varphi \mapsto \varphi + 2n\pi, n \in \mathbb{Z}$, invariance under lattice translations implies that

[8] ↗ACF, F-Pedrini ('98-2000), Hehl et al. ('08), S.-C. Zhang et al. ('10).

A 3D quantum Hall effect in axionic topological insulators

$$\varphi(\vec{x}) = 2\pi\left(\vec{K}\cdot\vec{x}\right) + \phi(\vec{x})\,, \tag{3.9}$$

where the vector $\vec{K}$ belongs to the *dual lattice*, and ϕ is invariant under lattice translations. Neglecting ϕ, we find that

$$\vec{\nabla}\varphi = 2\pi\vec{K} \text{ is “quantized”}.$$

which, with eq. (3.8), implies *Halperin's* 3*D* Hall effect with a quantized Hall conductivity! (I thank G. Moore for telling me about this effect.)

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But are there topological insulators with *dynamical* degrees of freedom described by an axion field? It has been argued that axions may emerge as effective degrees of freedom in:

- certain 3D topological insulators with anti-ferromagnetic short-range order, (magnetic fluctuations playing the role of a dyn. axion)[9] ; and in
- crystalline 3D Weyl semi-metals:

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[9] a conjecture proposed by S.-C- Zhang (inspired by our work in cosmology)

Weyl semi-metals

These are systems with two energy bands exhibiting two (or, more generally, an even number[10] of) double-cones in "frequency-quasi-momentum space". Assuming that the Fermi energy is close to the apices of those double-cones, such systems exhibit *chiral* quasi-particle states: At low frequencies, namely near the apices of those double-cones, the quasi-particles satisfy the *Weyl equation* of left- or right-handed Weyl fermions, respectively; (electron spin $\|$, or anti-$\|$ to momentum).
In such systems, the time-derivative, $\mu_5 \equiv \dot{\varphi}$ of the axion field really has the meaning of a (time-dependent) *difference of chemical potentials of left-handed and right-handed quasi-particles.* It satisfies a (reaction-diffusion) equation of motion of the kind

$$\dot{\mu}_5 + \tau^{-1}\mu_5 - D \triangle \mu_5 = L^2 \frac{e^2}{2\pi h} \vec{E} \cdot \vec{B}\,, \qquad (3.10)$$

where τ is a relaxation time, D a diffusion constant, L a constant with dimension of "length" related to the "axion decay constant" of particle physics; (see BFR for a discussion of (3.10) in the context of cosmology).

[10] This folllows from the celebrated Nielsen-Ninomiya theorem

How one might discover "axions" in Weyl semi-metals

As time $t \to \infty$ (assuming D is small and $\vec{E} \cdot \vec{B} \approx const.$), μ_5 approaches

$$\mu_5 \simeq \frac{\tau (Le)^2}{2\pi h} \vec{E} \cdot \vec{B} \, . \tag{3.11}$$

A non-vanishing initial value of μ_5 may be triggered by strain applied to the system, leading to a slightly $\ell \leftrightarrow r$ - asymmetric population of the Fermi sea. Due to *"inter-valley" scattering processes*, a non-vanishing μ_5 will then relax towards 0, with a relaxation time given by τ, unless an electric field $\vec{E}$ and a magnetic induction $\vec{B}$ are applied to the system, with $\vec{E} \cdot \vec{B} \neq 0$, in which case μ_5 relaxes towards the R.S. of (3.11). Recalling Eq. (3.8) for the current density in the presence of an axion, we conclude that the *conductivity tensor*, $\sigma = (\sigma_{k\ell})_{k,\ell=1,2,3}$, is given by

$$\sigma_{k\ell} = \sigma_{k\ell}^{(0)} + \frac{\tau (L\alpha)^2}{4\pi^2} B_k B_\ell \, ,$$

the first term on R.S. being the Ohmic conductivity (due to phonon- and impurity scattering), and the second term a manifestation of the *chiral magnetic effect*; (perhaps, too small to be detected in actual measnts.)

And how one might discover "axionic insulators"

People[11] have described various other Gedanken experiments serving to discover effects due to axions in Weyl semi-metals; but we won't review their ideas here. Instead, we describe some axionic effects in topological insulators with an effective action given by – see (3.1) and (3.6) –

$$S_\Lambda(A,\varphi) = S_\Lambda(A) + \frac{1}{8\pi^2}\int_\Lambda \varphi F \wedge F + S_0(\varphi)\,, \tag{3.12}$$

where $S_0(\varphi)$ is invariant under shifts $\varphi \mapsto \varphi + n\pi$, $n \in \mathbb{Z}$. It is compatible with time-reversal invariance that $S_0(\varphi)$ has minima at $\varphi = n\pi$. Then the material described by (9.9) is *not* an ordinary insulator, but may exhibit a *Mott transition* to a conducting state at a positive temperature: The bulk of such a material will be filled with *domain walls* across which φ jumps by (an integer multiple of) π. Applying the insight described after (3.4) and (3.5), one predicts that such domain walls may carry gapless two-component Dirac-type fermions. At sufficiently high temperatures, domain walls can be expected to become macroscopic, and this would then give rise to a *non-vanishing conductivity*.[12]

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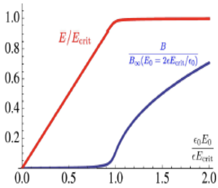
[11] e.g., theorists in Würzburg including J. Erdmenger

[12] ↗ F-Werner (2014)

Instabilities in axionic topological insulators

It has been pointed out (F-Pedrini, 2000) that a dynamical axion φ with $\mu_5 \equiv \dot{\varphi} =$ const., or $=$ a periodic function of time, t, will give rise to the growth of a *helical em field*; modes of the magnetic induction $\vec{B}$ at wave vectors of size $\leq const.\mu_5$ will be unstable and exhibit unlimited growth. This growth is stopped by the relaxation of μ_5 to 0. (Our mechanism has first been applied in cosmology.)

Another, albeit related instability has been pointed out by Ooguri and Oshikawa: If $\vec{E}$ and $\vec{B}$ are time-indep., an external electric field $\vec{E}$ applied to an axionic magnetic material is screened once its strength $|\vec{E}|$ exceeds a certain critical value E_c, the excess energy giving rise to a magnetic field – see Phys. Rev. Lett. **108**, 161803 (2012):



Courtesy Ooguri & Oshikawa

10. Summary, Open Problems

1. Apparently, concepts and methods from gauge theory can be used to study general features of strongly correlated systems with non-trivial *interactions* in cond-mat physics; e.g., to characterize certain *"topological states of matter"* that cannot be characterized by local order parameters. This has been illustrated in this lecture by showing how ideas and results from gauge theory, in particular, $3D$ Chern-Simons theory, the chiral magnetic effect and axion electrodynamics in $(3+1)-D$ systems, yield rather surprizing insights into properties of such states of matter.
2. What has been missing is an account of the *bare-hands analysis* of spectral properties of many-body Hamiltonians describing "topol. states of matter" at energies close to the ground-state energy and to derive properties of quasi-particles, using multi-scale analysis. I recommend the work of our distinguished colleagues in Rome and elsewhere, who have addressed such problems, to the attention of the audience! Of course, many questions remain open. ...

I thank you for your attention, and, well, ...

Sincerely, JF

“Survivre et Vivre” – almost half a Century later

Here is something more important to think about and to discuss with you:

“... depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l’espèce humaine, et même de la vie tout court menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d’armements. ...”

Alexandre Grothendieck

Réveillez-vous, indignez-vous!

(*Stéphane Hessel*)