Full Counting Statistics of energy transfers in $(1+1)D$ CFT

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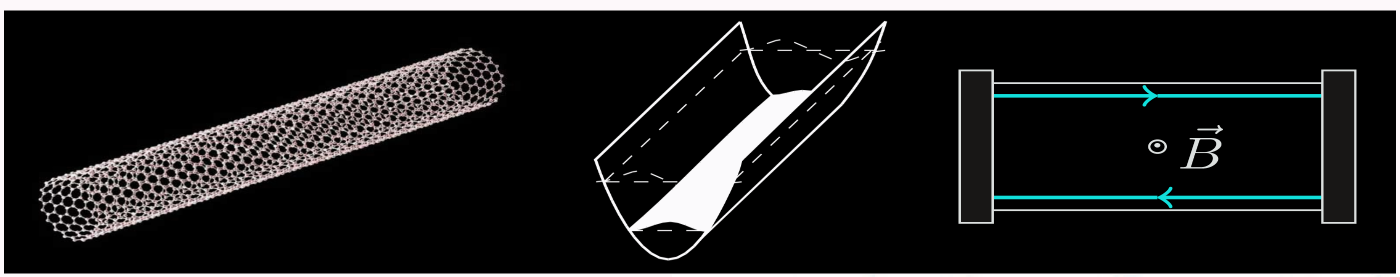
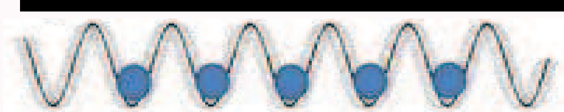
(based on **K. G.–E. Langmann–P. Moosavi**: J. Stat. Phys. **172**, 353 (2018)
and **K. G.–K. K. Kozlowski**: arXiv:1906.04276[math–ph])

- Many one-dimensional quantum systems have massless low-energy excitations described by **Conformal Field Theory**

  Examples: carbon nanotubes, electrons or cold atoms trapped in $1d$ potential wells, quantum Hall edge currents, XXZ spin chains

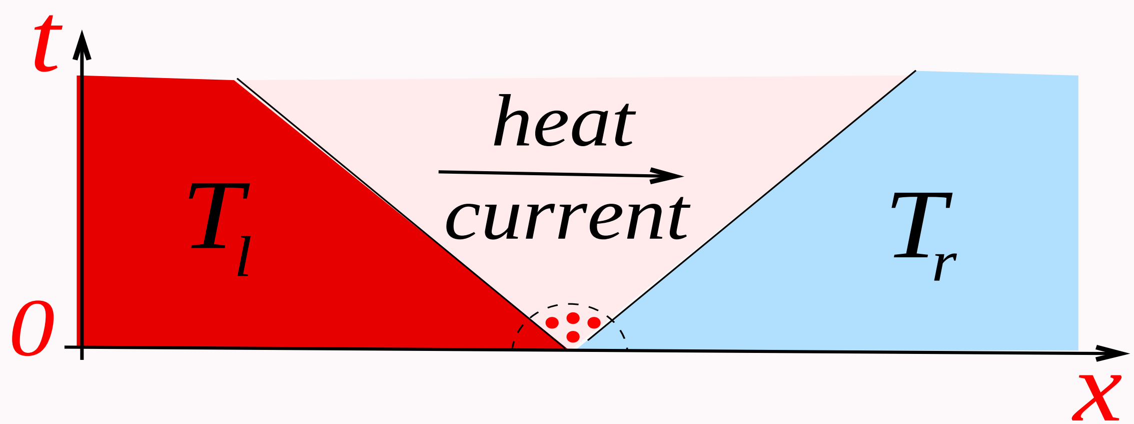




- **(1+1)**D **CFT** describes the low temperature equilibrium physics of such systems but also some of nonequilibrium situations as

  - evolution after **quantum quenches** to short-correlated pure states (see **Calabrese-Cardy**, J. Stat. Mech. 064003 (2016))

  - the **partitioning protocol** after two halves of a system in different equilibrium states are joined (see **Bernard-Doyon**, J. Stat. Mech. 064005 (2016))



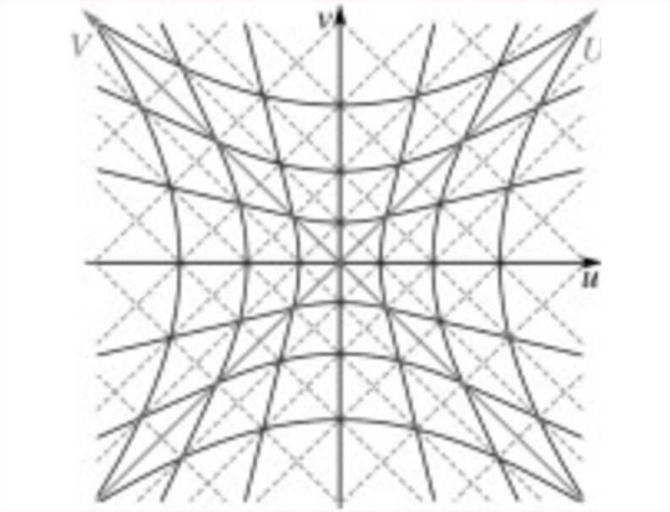
- **What is a $(\mathbf{1}+\mathbf{1})D$ CFT?**

It is a **QFT** in one space dimension with the projective representation in the **Hilbert** space of states of the symmetry group of $(\mathbf{1}+\mathbf{1})D$ **Minkowskian** conformal transformations

- Conformal transformations in $(\mathbf{1}+\mathbf{1})D$ spacetime are:

$$(x^-, x^+) \mapsto (f_+(x^-), f_-(x^+))$$

where $x^\pm \equiv x \pm t$



- Infinitesimally, the vector fields $\zeta_\pm(x^\mp)\,\partial_\mp$ are represented by the s. a. operators

$$\int \zeta_\pm(x^\mp)\,T_\pm(x^\mp)\,dx^\mp$$

where $T_\pm(x^\mp)$ are the right- and left-moving components of the **energy-momentum tensor**

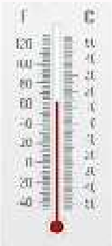
- The **energy density** and **energy current**

$$e(t,x) = T_+(x^-) + T_-(x^+) \qquad j(t,x) = T_+(x^-) - T_-(x^+)$$

satisfy the local conservation law $\partial_t e + \partial_x j = 0$

- The **Hamiltonian** is $H = \int e(t,x)\,dx$ and the **Gibbs equilibrium state** at inverse temperature β_0 is

$$\omega^{\mathrm{eq}}_{\beta_0}(A) = \frac{\mathrm{Tr}\Big(A\ \mathrm{e}^{-\beta_0 H}\Big)}{\mathrm{Tr}\Big(\mathrm{e}^{-\beta_0 H}\Big)}$$



where A are observables and the right-hand side requires passing through the thermodynamic limit

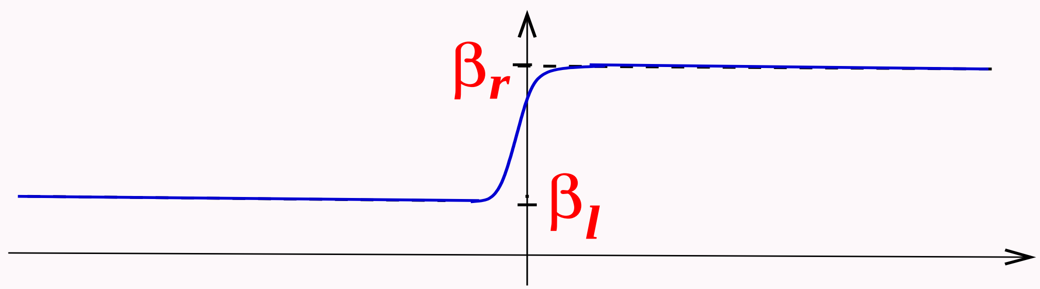
- **Examples of $(1+1)D$ CFT's**
  - Free massless bosonic field
  - Free massless fermionic field
  - (local) **Luttinger** model
  - **WZW** models
  - Coset models (e.g. unitary minimal models)

- **Smooth version of the partition protocol**

We shall consider nonequilibrium "**profile states**" defined by

$$\omega^{\mathrm{neq}}(A) = \frac{\mathrm{Tr}\big(A\, \mathrm{e}^{-G}\big)}{\mathrm{Tr}\big(\mathrm{e}^{-G}\big)} \qquad \text{for} \qquad G = \int \beta(x)\, e(0,x)\, dx$$

where $\beta(x)$ is a smooth inverse-temperature profile with the values β_ℓ and (β_r) far on the left (right)



- Again, such states have to be defined by taking the thermodynamic limit of their finite-box version

- They are not invariant under the dynamics generated by H

- **Finite-box CFT**

  - We shall work in a finite box $[-\frac{1}{4}L, \frac{1}{4}L]$ with the boundary conditions that guarantee that $T_+(x^-) = T_-(x^+)$ for $x = \pm\frac{1}{4}L$
  - Such b. c. assure conservation of energy within the box. E.g. for the bosonic free field one may take the **Neumann** or the **Dirichlet** b. c.
  - There is then only one independent component of the energy-moment. tensor $T_+(x^-) = T_+(x^- + L)$ with $T_-(x^+) = T_+(-x^+ \pm \frac{1}{2}L)$

  $$T_+(x) = \frac{2\pi}{L^2} \sum_{n=\infty}^{\infty} \mathrm{e}^{\frac{2\pi i n}{L}x}\left(L_n - \frac{c}{24}\delta_{n,0}\right) \equiv T(x)$$

  where L_n satisfy the **Virasoro algebra**:

  $$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}$$

  with the **central charge** c

Infinitesimal $\to$ global symmetry:

- Let $Diff_{+}^{\sim}S^1$ composed of smooth $f:\mathbb{R}\to\mathbb{R}$ s.t. $f'>0$ and $f(x+L)=f(x)+L$ be the covering group of the groups of orientation-preserving diffeos of the circle $S^1=\mathbb{R}/L\mathbb{Z}$

- $T(x)$ generates a unitary projective representation $f\mapsto U_f$ of $Diff_{+}^{\sim}S^1$ s. t.

$$U_f\,T(x)\,U_f^{-1}\;=\;f'(x)^2\,T(f(x))\;-\;\frac{c}{24\pi}(\mathsf{S}f)(x)$$

where $\mathsf{S}f=\frac{f'''}{f'}-\frac{3}{2}\big(\frac{f''}{f'}\big)^2$ is the **Schwarzian derivative** of f

- If f_s is the flow of a vector field $-\zeta(x)\partial_x$ with $\zeta(x+L)=\zeta(x)$, i.e.

$$\partial_s f_s(x)=-\zeta(f_s(x))\,,\qquad f_0(x)=x$$

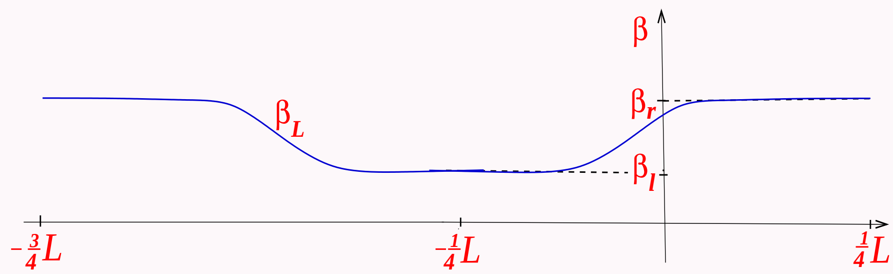
then

$$U_{f_s}\;=\;\exp\Big[\mathrm{i}s\int_0^L\zeta(x)\,T(x)\,dx\Big]$$

E.g. $U_{f_s}=\mathrm{e}^{2\pi\mathrm{i}s\,(L_0-\frac{c}{24})}$ for the translations $f_s(x)=x-sL$

- **Finite-box profile states**

  - For L big enough let $\beta_L(x) = \beta_L(x+L)$ be the profile extended from $[-\frac{1}{4}L, \frac{1}{4}L]$ to $[-\frac{3}{4}L, \frac{1}{4}L] \equiv \mathcal{I}_L$ by reflection:



  - For

$$G_L = \int_{-\frac{1}{4}L}^{\frac{1}{4}L} \beta(x)\, e(0,x)\, dx = \int_{-\frac{1}{4}L}^{\frac{1}{4}L} \beta(x)\, \big(T_+(x) + T_-(x)\big)\, dx = \int_{\mathcal{I}_L} \beta_L(x)\, T(x)\, dx$$

consider the finite-box nonequilibrium profile state

$$\omega_L^{\mathrm{neq}}(A) = \frac{\mathrm{Tr}\big(A\ \mathrm{e}^{-G_L}\big)}{\mathrm{Tr}\big(\mathrm{e}^{-G_L}\big)}$$

- **Reduction of profile states to equilibrium states**

  - Let $\varphi_L \in Diff^{\sim}_{+} S^1$ be s. t. $\varphi'_L(x) = \frac{\beta_{0,L}}{\beta_L(x)}$ with $\beta_{0,L}$ fixed by the requirement that $\varphi_L(x+L) = \varphi_L(x) + L$. Then

$$\boxed{U_{\varphi_L} G_L \, U_{\varphi_L}^{-1}} = \int_{\mathcal{I}_L} \beta_L(x) \, U_{\varphi_L} T(x) \, U_{\varphi_L}^{-1} \, dx$$

$$= \int_{\mathcal{I}_L} \beta_L(x) \, \varphi'_L(x)^2 \, T(\varphi_L(x)) \, dx \; - \; \underbrace{\frac{c}{24\pi} \int_{\mathcal{I}_L} \beta_L(x) \, (\mathsf{S}\varphi_L)(x) \, dx}_{\textbf{a number } C_{0,L}}$$

$$= \beta_{0,L} \int_{\mathcal{I}_L} \varphi'_L(x) \, T(\varphi_L(x)) \, dx \; - \; C_{0,L}$$

$$\overset{y=\varphi_L(x)}{=} \beta_{0,L} \int_{\mathcal{I}_L} T(y) \, dy \; - \; C_{0,L} \; = \; \boxed{\beta_{0,L} H_L \; - \; C_{0,L}}$$

$\Rightarrow$ **the conjugation by** U_{φ_L} **flattens the temperature profile !!!**

- This implies that the nonequilibrium profile state is related to the equilibrium state by the conformal symmetry:

$$\boxed{\omega_L^{\mathrm{neq}}(A) \,=\, \omega^{\mathrm{eq}}_{\beta_{0,L};L}\Big(U_{\varphi_L} A\, U_{\varphi_L}^{-1}\Big)}$$

- May be applied e.g. to $A = \prod_i T_+(x_i^-)\prod_j T_-(x_j^+)$ since we know how it transforms under the conjugation by U_{φ_L}

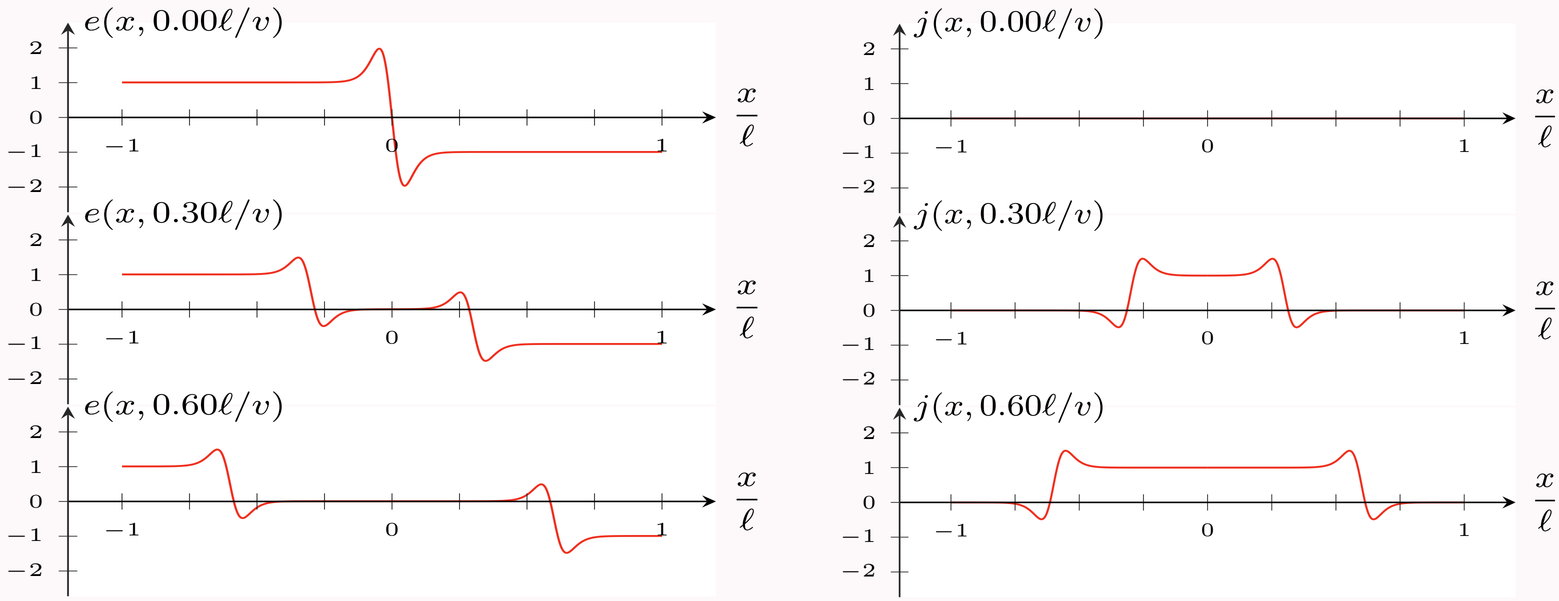
- In the thermodynamic limit $L \to \infty$ one obtains for the 1-point function of $T_\pm(x^\mp)$

$$\boxed{\omega^{\mathrm{neq}}\Big(T_\pm(x^\mp)\Big) \,=\, \frac{\pi\, c}{12\,\beta(x^\mp)^2} \,-\, \frac{c}{24\pi}(\mathsf{S}\varphi)(x^\mp)}$$

where $\varphi(x) = \int_0^x \frac{\beta_0}{\beta(x')}\,dx'$ with arbitrary β_0

- First derived for the local **Luttinger** model by **Lebowitz-Langmann-Mastropietro-Moosavi** in Phys. Rev. B **95**, 235142 (2017) by resumming a perturbative series

- The sum and the difference of the last formulae unravels a nontrivial evolution of $\omega^{\mathrm{neq}}\big(e(t,x)\big)$ and $\omega^{\mathrm{neq}}\big(j(t,x)\big)$ with traveling heat waves:



Evolution of the mean energy density minus e_0 (left) and of the mean energy current (right)

before reaching the stationary values

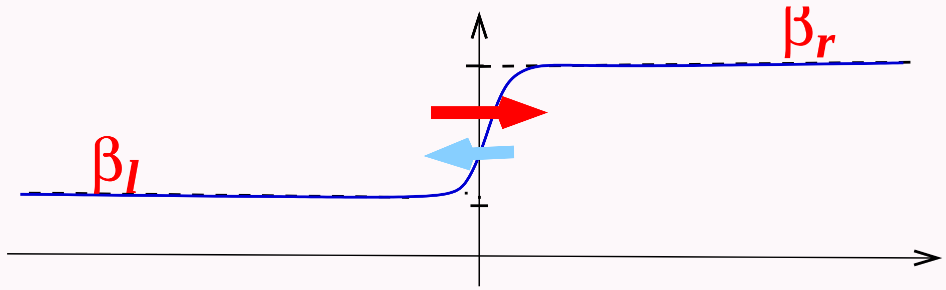
$$\lim_{t\to\infty} \omega^{\mathrm{neq}}(e(t,y)) = \frac{\pi}{12}\left(\beta_\ell^{-2} + \beta_r^{-2}\right) \equiv e_0$$

$$\lim_{t\to\infty} \omega^{\mathrm{neq}}(j(t,y)) = \frac{\pi}{12}\left(\beta_\ell^{-2} - \beta_r^{-2}\right) \equiv j_0 \neq 0$$

- **Full counting statistics of energy transfers**

  - Full counting statistics (**FCS**) captures fluctuations of charge or energy transfers in extended quantum systems (**Levitov-Lesovik**, JETP **58**, 230235 (1993))

  - For the profile states, one may obtain an exact expression for the **FCS** of energy transfers across the kink in a kink-like $\beta(x)$ -profile



  - Write

$$G_L = \int_{-\frac{1}{4}L}^{\frac{1}{4}L} \beta(x)\, e(0,x)\, dx \;=\; \beta_\ell E_\ell + \beta_r E_r$$

    where E_ℓ and E_r are the energies in the part of the box $[-\frac{1}{4}\,L, \frac{1}{4}\,L]$ to the left and to the right of the kink

- One gets access to the **FCS** of energy transfers from two measurements of $G_L = \beta_\ell E_\ell + \beta_r E_r$ separated by time t, starting in state ω_L^{neq}

- By spectral decomposition

$$G_L = \sum_i g_i P_i\,, \qquad G_L(t) \equiv \mathrm{e}^{\mathrm{i}tH_L} G_L \mathrm{e}^{-\mathrm{i}tH_L} = \sum_i g_i P_i(t)$$

- If the 1^{st} measurement gives the value g_i and the 2^{nd} one the value g_j then

$$g_j - g_i = \beta_\ell\big(E_\ell(t) - E_\ell(0)\big) + \beta_r\big(E_r(t) - E_r(0)\big) = (\Delta\beta)(\Delta E)$$

where $\Delta\beta = \beta_r - \beta_\ell$ and $\Delta E = E_r(t) - E_r(0) = -(E_\ell(t) - E_\ell(0))$ is the net transfer of the energy across the kink during time t

- By the **QM** rules the probability of the results (g_i, g_j) is

$$p_t(i,j) = \omega_L^{\rm neq}\Big(P_j(t)\,P_i\Big)$$

giving for the distribution of the energy transfers (called **FCS**):

$$p_{t,L}(\Delta E) = \sum_{i,j}\delta\Big(\Delta E - \frac{g_j-g_i}{\Delta\beta}\Big)\,\omega_L^{\rm neq}\Big(P_j(t)\,P_i\Big)$$

- The **Fourier** transform of the distribution of ΔE (called the generating function of **FCS**) is

$$\mathcal{F}_{t,L}(\lambda) \equiv \int \mathrm{e}^{\mathrm{i}\lambda\Delta E}\,p_{t,L}(\Delta E)\,d(\Delta E)$$

$$= \sum_{i,j}\mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}(g_j-g_i)}\,\omega_L^{\rm neq}\Big(P_j(t)\,P_i\Big) = \omega_L^{\rm neq}\Big(\mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}G_L(t)}\,\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}G_L}\Big)$$

$$= \omega^{\rm eq}_{\beta_{0,L};L}\Big(U_{\varphi_L}\,\mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}G_L(t)}\,\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}G_L}\,U_{\varphi_L}^{-1}\Big)$$

using the relation between the nonequilibrium and equilibrium states

- Upon lifting the conjugation by U_{φ_L} to the exponentials

$$\mathcal{F}_{t,L}(\lambda) \;=\; \omega^{\mathrm{eq}}_{\beta_{0,L};L}\Big(\mathrm{e}^{\,\frac{\mathrm{i}\lambda}{\Delta\beta}U_{\varphi_L}G_L(t)\,U_{\varphi_L}^{-1}}\;\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}U_{\varphi_L}G_L U_{\varphi_L}^{-1}}\Big)$$

- We have seen that

$$U_{\varphi_L}G_L\,U_{\varphi_L}^{-1} \;=\; \beta_{0,L}H_L \;-\; \underbrace{C_{0,L}}_{\textbf{a number}}$$

By the same manipulations

$$U_{\varphi_L}G_L(t)\,U_{\varphi_L}^{-1} \;= \int_{\mathcal{I}_L}\zeta_{t,L}(y)\,T(y)\,dy \;-\; \underbrace{C_{t,L}}_{\textbf{a number}}$$

where

$$\underbrace{\zeta_{t,L}(y)}_{\text{effective profile}} = \frac{\beta_{0,L}\,\beta_L\Big(\varphi_L^{-1}(y)+t\Big)}{\beta_L\Big(\varphi_L^{-1}(y)\Big)} \quad \text{and} \quad C_{t,L} = \tfrac{c}{24\pi}\int_{\mathcal{I}_L}\beta_L(x^+)(\mathsf{S}\varphi_L)(x)\,dx$$

- Using $H_L = \frac{2\pi}{L}(L_0 - \frac{c}{24})$ and setting $s = \frac{\lambda}{\Delta\beta}$, $\tau_s = \frac{(\mathrm{i}-s)\beta_{0,L}}{L}$ one obtains

$$\mathcal{F}_{t,L}(\lambda)$$

$$= \boxed{\omega^{\mathrm{eq}}_{\beta_{0,L};L}\Big(\mathrm{e}^{\frac{\mathrm{i}\lambda}{\Delta\beta}\int_{\mathcal{I}_L}\zeta_{t,L}(y)T(y)\,dy}\,\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}\beta_{0,L}H_L}\Big)}\;\mathrm{e}^{-\frac{\mathrm{i}\lambda}{\Delta\beta}(C_{t,L}-C_{0,L})}$$

$$= \boxed{\frac{\mathrm{Tr}\Big(U_{f_s}\,\mathrm{e}^{2\pi\mathrm{i}\tau_s(L_0-\frac{c}{24})}\Big)}{\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_0(L_0-\frac{c}{24})}\Big)}}\;\mathrm{e}^{-\mathrm{i}s(C_{t,L}-C_{0,L})}$$

for $f_s \in Diff^{\sim}_{+}S^1$ denoting the flow of the vector field $-\zeta_{t,L}(y)\partial_y$

- The least explicit contribution is the boxed term

$$\frac{\mathrm{Tr}\Big(U_{f_s}\,\mathrm{e}^{2\pi\mathrm{i}\tau_s(L_0-\frac{c}{24})}\Big)}{\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau_0(L_0-\frac{c}{24})}\Big)}$$

- $\mathrm{Tr}\Big(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\Big)\equiv\chi(\tau)$ is the character of the **Virasoro** algebra representation in the space of states of our **CFT** in the box

- Similarly, $\mathrm{Tr}\Big(U_f\,\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{c}{24})}\Big)\equiv\Upsilon(f,\tau)$ may be viewed as the character of the corresponding representation of the group $Diff_+^{\sim}S^1$

- We may then write:

$$\boxed{\mathcal{F}_{t,L}(\lambda)\;=\;\frac{\Upsilon(f_s,\tau_s)}{\chi(\tau_0)}\;\;\mathrm{e}^{-\mathrm{i}s(C_{t,L}-C_{0,L})}}$$

- The **Virasoro** characters are well known for the unitary (positive-energy) representations but their $Diff_+^{\sim}S^1$ counterparts have not been studied

• **Characters of** $Diff^{\sim}_{+}S^1$

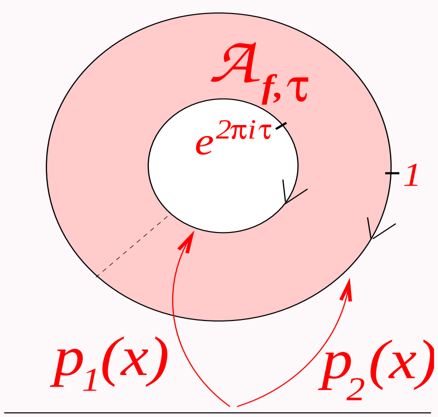
- Consider the complex annulus

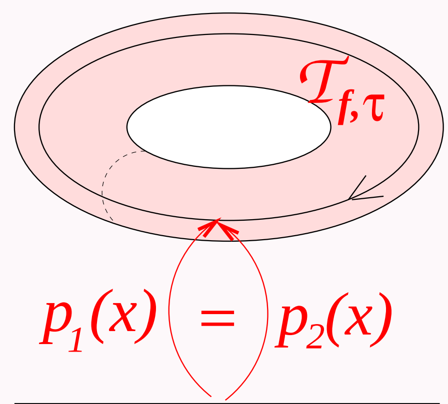
$$\mathcal{A}_{f,\tau} = \left\{ z \,\middle|\, |e^{2\pi i\tau}| \leq |z| \leq 1 \right\}$$

with the boundary components parameterized by

$$p_1(x) = e^{2\pi i\tau} e^{-\frac{2\pi i}{L} f(x)}\,, \qquad p_2(x) = e^{-\frac{2\pi i}{L} x}$$

and the complex torus $\mathcal{T}_{f,\tau}$ obtained from $\mathcal{A}_{f,\tau}$ by sewing ("**conformal welding**") its parameterized boundaries together





- According to **G. Segal** $\Upsilon(f,\tau) = \mathrm{Tr}\left(U_f\, \mathrm{e}^{2\pi \mathrm{i}\tau(L_0 - \frac{c}{24})}\right)$ is proportional to the **partition function** on the torus $\mathcal{T}_{f,\tau}$ of the chiral **CFT**

- The complex torus $\mathcal{T}_{f,\tau}$ is isomorphic to $\mathcal{T}_{f_0,\widehat{\tau}}$ for $f_0(x) \equiv x$ and some effective $\widehat{\tau}$ in the upper half plane

- The existence of such an isomorphism implies the relation

$$\Upsilon(f,\tau) \;=\; C_{f,\tau}\,\Upsilon(f_0,\widehat{\tau}) \;=\; C_{f,\tau}\,\chi(\widehat{\tau})$$

  where $C_{f,\tau}$ is a complex number due to the projective nature of the chiral **CFT** partition functions

- We shall need $C_{f,\tau}$ only for $f = f_s$ with f_s a flow of a vector field

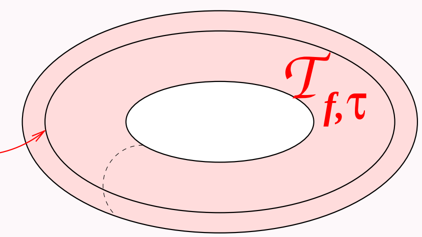
- $C_{f_s,\tau}$ may be found adapting the approach used by **Fewster-Hollands** in Lett. Math. Phys. **109** (2018), 747 to calculate $\big\langle 0\big|U(f_s)\big|0\big\rangle$

- **The isomorphism $\mathcal{T}_{f,\tau} \cong \mathcal{T}_{f_0,\widehat{\tau}}$ and an inhomogeneous Riemann-Hilbert problem on $\mathcal{T}_{f,\tau}$**

  - One searches for a holomorphic function Y on $\mathcal{A}_{f,\tau}$ s. t.

$$Y_1 - Y_2 = \frac{1}{L}(f - f_0) - \tau + \widehat{\tau} \qquad \text{for} \qquad Y_i = Y \circ p_i$$

jump of a holomorphic function Y
prescribed along the welding contour



  - Y is found from the solution Y_1 (that exists for a single value of $\widehat{\tau}$) of an explicit **Fredholm** equation in $L^2\big(\mathbb{R}/L\mathbb{Z}\big)$

  - The holomorphic function $W(z) = z\,\mathrm{e}^{2\pi\mathrm{i}Y(z)}$ on $\mathcal{A}_{f,\tau}$ has the boundary values $W_i = W \circ p_i = \mathrm{e}^{2\pi\mathrm{i}X_i}$ s.t. $X_1(x) = X_2(x) + \widehat{\tau}$ so that $W_1 = \mathrm{e}^{2\pi\mathrm{i}\widehat{\tau}}W_2$

  - The map W realizes the isomorphism $\mathcal{T}_{f,\tau} \cong \mathcal{T}_{f_0,\widehat{\tau}} = \mathbb{C}^\times/(w \sim \mathrm{e}^{2\pi\mathrm{i}\widehat{\tau}}w)$

- **Theorem.** For the flow f_s of $-\zeta(x)\partial_x$ let $W_s(z)$ be the holom. functions on $\mathcal{A}_{f_s,\tau}$ with bd. values $W_{s;i} = \mathrm{e}^{2\pi \mathrm{i} X_{s;i}}$ that realize the isomorphisms $\mathcal{T}_{f_s,\tau} \cong \mathcal{T}_{f_0,\widehat{\tau}_s}$.

  The proportionality constant between the $Diff_+^{\sim} S^1$-character $\Upsilon(f_s,\tau)$ and the **Virasoro** character $\chi(\widehat{\tau}_s)$ and the effective modular parameter satisfy the **ODE**s

$$\boxed{\begin{aligned} \partial_s \ln C_{f_s,\tau} \;&=\; -\mathrm{i}\,\frac{c}{24\pi}\int_{\mathcal{I}_L} \zeta(x)\,(\mathsf{S}X_{s;1})(x)\,dx \\ \partial_s \widehat{\tau}_s &= \int_{\mathcal{I}_L} \zeta(x)\,\big(X'_{s;1}(x)\big)^2 dx \end{aligned}}$$

  that determine them completely since $C_{f_0,\tau} = 1$ and $\widehat{\tau}_0 = \tau$

- The main tool in the proof is the transformation property of the **Euclidian** 1-point function

$$\big\langle\, T(z)\,\big\rangle_{\mathcal{T}_{f_s,\tau}} \,=\, W_s'(z)^2\,\big\langle\, T(W_s(z))\,\big\rangle_{\mathcal{T}_{f_0,\widehat{\tau}_s}} + \,\frac{c}{12}\,(\mathsf{S}W_s)(z)$$

of the holomorphic component of the energy momentum tensor

The integral of bd. value of the left-hand side against $\zeta(x)$ produces $\partial_s \ln \Upsilon(f_s,\tau)$ and of the $1^{\rm st}$ term on the right-hand side gives $\partial_s \ln \chi(\widehat{\tau}_s)$

$\square$

- **Corollary.** The result allows to control the finite-volume generating function for the **FCS** of energy transfers

$$\mathcal{F}_{t,L}(\lambda) \,=\, \frac{C_{f_s,\tau_s}\;\chi(\widehat{\tau}_s)}{\chi(\tau_0)}\,\mathrm{e}^{-\mathrm{i}s(C_{t,L}-C_{0,L})}$$

where $s=\frac{\lambda}{\Delta\beta}$, $\tau_s=\frac{(\mathrm{i}-s)\beta_{0,L}}{L}$, and f_s is the flow of $-\zeta_{t,L}(x)\partial_x$ if we replace τ by τ_s and $\zeta(x)$ by $\xi_{t,L}(x)\equiv\zeta_{t,L}(x)-\beta_{0,L}$

- **FCS for energy transfers in the thermodynamic limit**

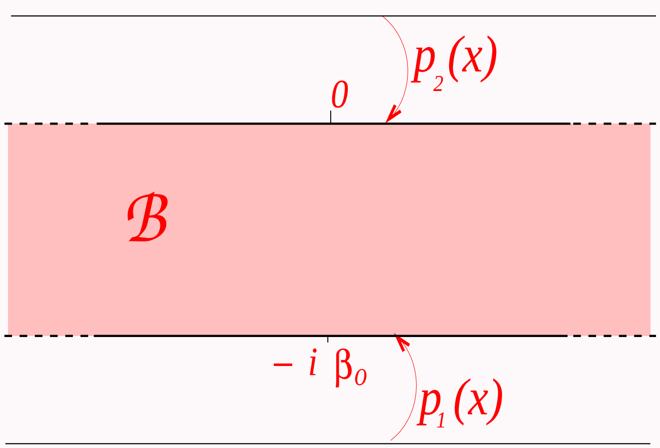
  - In the $L \to \infty$ limit, the conformal welding of tori becomes that of infinite cylinders $\mathcal{Z}_{g_s^\pm}$ obtained from the band

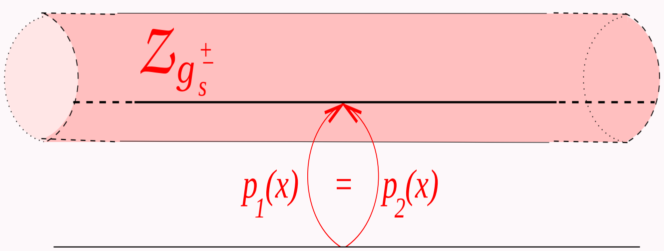
$$\mathcal{B} = \{z \,|\, -\beta_0 \leq \mathrm{Im}(z) \leq 0\}$$

with the boundary parameterizations

$$\mathbb{R} \ni x \mapsto p_1(x) = -\mathrm{i}\beta_0 + g_s^\pm(x)\,, \qquad \mathbb{R} \ni x \mapsto p_2(x) = x$$

by setting $p_1(x) = p_2(x)$



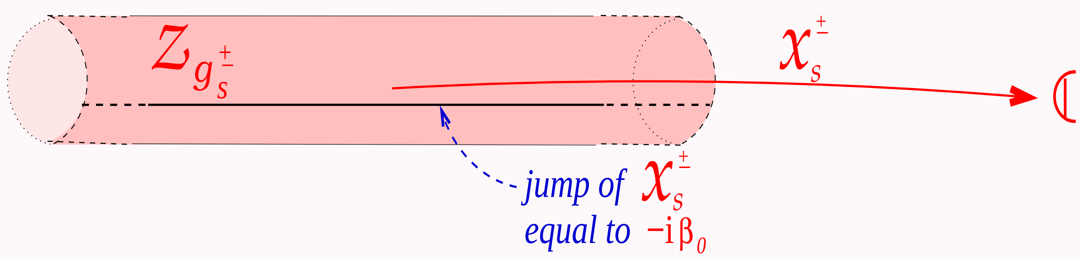


- $g_s^{\pm}(x) = f_s^{\pm}(x) + \beta_0 s$ for the flow $f_s^{\pm}$ of the vector field $-\zeta_t^{\pm}(x)\partial_x$ where

$$\zeta_t^{\pm}(x) = \beta_0 \frac{\beta\big(\varphi^{-1}(\pm x) \pm t\big)}{\beta\big(\varphi^{-1}(\pm x)\big)} \qquad \text{for} \qquad \varphi(x) = \int_0^x \frac{\beta_0}{\beta(x')}\, dx'$$

- The complex cylinder $\mathcal{Z}_{g_s^{\pm}}$ is isomorphic to $\mathcal{Z}_{\mathrm{id}}$ with the isomorphism given by a holomorphic functions $\mathcal{X}_s^{\pm}$ on $\mathcal{B}$ with boundary values $\mathcal{X}_{s;i}^{\pm} = \mathcal{X}_s^{\pm} \circ p_i$ s.t. $\mathcal{X}_{s;1}^{\pm} = \mathcal{X}_{s;2}^{\pm} - \mathrm{i}\beta_0$

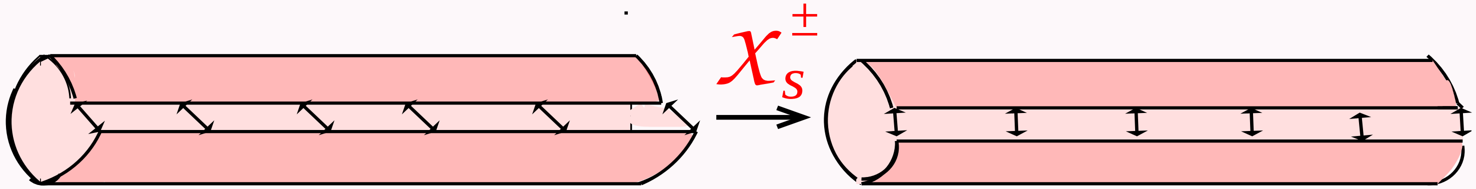
- Finding $\mathcal{X}_s^{\pm}$ is a **Riemann-Hilbert** problem on $\mathcal{Z}_{g_s^{\pm}}$ that reduces to solving an explicit **Fredholm** equation in $L^2(\mathbb{R})$ for $\mathcal{X}_{s;1}^{\pm\,\prime} - 1$



Example. If $g_s^{\pm}(x) = x - D^{\pm} s$ for constant $D^{\pm}$ (a uniform shift) then

$$\mathcal{X}_s^{\pm}(z) = \left(1 - \mathrm{i}\,\frac{D^{\pm}}{\beta_0}\,s\right)^{-1} z$$

and $\quad {\mathcal{X}_{s;1}^{\pm}}'(x) = \left(1 - \mathrm{i}\,\frac{D^{\pm}}{\beta_0}\,s\right)^{-1}$



- The main problem was to prove the uniform convergence on compacts of the derivatives of functions $X_{s;1}$ solving the finite-volume **Fredholm** eqn to the derivatives of functions $\mathcal{X}_{s;1}^{\pm}$ solving the infinite-volume one
- This was done by a detailed analysis of particular classes of **Fredholm** operators in $L^2(\mathbb{R})$ and it formed the technical core of our work

- **Theorem.** $\mathcal{F}_t(\lambda) = \lim_{L\to\infty} \mathcal{F}_{t,L}(\lambda) = \prod_{\pm} \mathcal{F}_t^{\pm}(\lambda)$ where $\mathcal{F}_t^{\pm}(\lambda)$ are the contributions of the right- and left-movers and

$$\boxed{\begin{aligned}\partial_\lambda \ln \mathcal{F}_t^{\pm}(\lambda) \;&=\; \tfrac{1}{\Delta\beta}\Big(S_{\rm Sch}(\mathcal{X}_{s;1}^{\pm}) + \mathrm{i}\,\tfrac{c}{24\pi}\int \big(\beta(x) - \beta(x^{\pm})\big)(\mathsf{S}\varphi)(x)\,dx\Big)\\ S_{\rm Sch}(\mathcal{X}_{s;1}^{\pm}) \;&=\; -\mathrm{i}\,\tfrac{c}{24\pi}\int \xi_t^{\pm}(x)\Big((\mathsf{S}\mathcal{X}_{s.1}^{\pm})(x) \;-\; \tfrac{2\pi^2}{\beta_0^2}\big({\mathcal{X}_{s;1}^{\pm}}'(x)\big)^2\Big)\,dx\end{aligned}}$$

  for $\xi_t^{\pm}(x) = \zeta_t^{\pm}(x) - \beta_0$ and $\mathcal{X}_{s;1}^{\pm}(x)$ given by a conformal welding of the infinite cylinder $\mathcal{Z}_{g_s^{\pm}}$

- **Corollary.** $\mathcal{F}_t(\lambda)$ is universal depending only on the profile $\beta(x)$ and the central charge c of the **CFT** (entering as a power)

- **Remark.** The action $S_{\rm Sch}(\mathcal{X})$ of diffeomorphisms of $\mathbb{R}$ controls the regime of the **Sachdev-Ye-Kitaev** model dominated by the **Goldstone** boson of the conformal symmetry breaking

- Under the map

$$z \longmapsto \frac{1 - \mathrm{i}e^{\frac{2\pi}{v\beta_0}z}}{1 + \mathrm{i}e^{\frac{2\pi}{v\beta_0}z}} \equiv u(z)$$

  the conformal welding of the edges of the band $\mathcal{B}$ becomes the conformal welding of the boundaries of the unit discs $\Delta_\pm$ composing $\mathbb{C}P^1$

- The latter welding may be studied numericallly as discussed by **Sharon-Mumford** in Int. J. Computer Vision **70**, 55 (2006) who used it to code $2D$ shapes by elements of $Diff_+ S^1/SL_2(\mathbb{R})$

- The numerical algorithms give a direct access to the functions $\mathcal{X}^{\pm\,\prime}_{s;1}(x)$ and permit to simulate $\mathcal{F}_t(\lambda)$ (work in progress with **L. Chevillard**)

- For long times, the leading contribution to $\mathcal{F}_t^{\pm}(\lambda)$ comes from the term

$$\mathrm{i}\,\frac{\pi c}{12\,\beta_0^2}\int \xi_t^{\pm}(x)\big(\mathcal{X}_{s;1}^{\pm\,\prime}(x)\big)^2 dx$$

in $S_{\mathrm{Sch}}(\mathcal{X}_{s;1}^{\pm})$ with $\xi_t^{\pm}(x)$ approaching $D^{\pm}\mathbf{1}_{\mathcal{I}_t^{\pm}}(x)$ where

$$D^{\pm} = \begin{cases} \frac{\beta_0\,\Delta\beta}{\beta_\ell} \\ -\frac{\beta_0\,\Delta\beta}{\beta_r} \end{cases} \qquad |\mathcal{I}_t^{\pm}| = \begin{cases} \frac{\beta_0}{\beta_\ell}\,t \\ \frac{\beta_0}{\beta_r}\,t \end{cases}$$

On $\mathcal{I}_t^{\pm}$ the functions $\mathcal{X}_{s;1}^{\pm\,\prime}$ should approach the constants $\Big(1-\mathrm{i}\,\frac{D^{\pm}}{\beta_0}\,s\Big)^{-1}$ resulting from uniform-shift welding of $\mathcal{Z}_{g_s^{\pm}}$ for $g_s^{\pm}(x) = x - D^{\pm}s$

- This gives the **large-deviations** asymptotics of **Bernard-Doyon** J. Phys. A: Math. Theor. **45**, 362001 (2012)

$$\lim_{t\to\infty}\frac{1}{t}\,\ln\mathcal{F}_t(\lambda) \;=\; \frac{\pi c}{12}\left(\frac{1}{\beta_\ell-\mathrm{i}\lambda}-\frac{1}{\beta_\ell}+\frac{1}{\beta_r+\mathrm{i}\lambda}-\frac{1}{\beta_r}\right) \;\equiv\; \Phi(\lambda)$$

- The formula for $\Phi(\lambda)$ also agrees with the large-volume long-time limit of the **Levitov-Lesovik** formula for free fermions

- For long times $p_t(\Delta E) \asymp \mathrm{e}^{-tI(\frac{\Delta E}{t})}$ with the **rate function**

$$I(\sigma) = \max_{\nu\in[-\beta_r,\beta_\ell]}\Big(\nu\sigma - \Phi(-\mathrm{i}\nu)\Big) = \begin{cases} \beta_\ell\sigma + o(\sigma) & \text{for} \quad \sigma\to\infty \\ -\beta_r\sigma + o(\sigma) & \text{for} \quad \sigma\to-\infty \end{cases}$$

possessing the **Gallavotti-Cohen** symmetry $I(-\sigma) = I(\sigma) + \sigma\Delta\beta$

- $\Delta E(t)$ is distributed at large t as a **Lévy process** with the jump rates

$$w(x,y) = \frac{\pi c}{12}\Big(\mathrm{e}^{-\beta_\ell(y-x)}\theta(y-x) + \mathrm{e}^{-\beta_r(x-y)}\theta(x-y)\Big)$$

where $\theta(\,\cdot\,)$ is the **Heaviside** step function

Conclusions

- In a **CFT** conformal symmetries may be used to map inhomogeneous situations to homogeneous ones allowing to express the nonequilibrium states with temperature profile in terms of equilibrium ones
- As an example of a quantum probability question in a nonequilibrium **CFT** we analyzed the **FCS** of energy transfers in such profile states
- The finite volume **FCS** was expressed by characters of $\widetilde{Diff_+} S^1$ that were reduced to **Virasoro** characters using conformal welding of tori
- In the thermodynamic limit the **FCS** was rephrased in terms of the **Schwarzian** action of fields obtained from conformal welding of cylinders and it showed a universal form with a large deviations regime
- In **CFT**'s with the current-algebra symmetries, the **FCS** of charge transfers in states with chemical potential profile may be treated similarly using gauge transformations

- 1^{st} moment $\left\langle (\Delta E)(t) \right\rangle = \frac{1}{\mathrm{i}} \partial_\lambda \big|_{\lambda=0} \mathcal{F}_t(\lambda)$

$$= \frac{\pi c}{12\, \beta_0^2 \Delta\beta} \sum_{\pm} \widehat{\xi}_t^{\pm}(0) + \frac{c}{24\pi\Delta\beta} \sum_{\pm} \int (\beta(x) - \beta(x^{\pm}))(\mathsf{S}\varphi)(x)\, dx$$

where $\widehat{\xi}_t^{\pm}(p) = \int \mathrm{e}^{\mathrm{i}py} \xi_t^{\pm}(y)\, dy$

- 2^{nd} moment $\left\langle (\Delta E)(t)\,; (\Delta E)(t) \right\rangle^c = -\partial_\lambda^2 \big|_{\lambda=0} \ln \mathcal{F}_t(\lambda)$

$$= \frac{c}{48\pi^2 (\Delta\beta)^2} \sum_{\pm} \int \frac{p(p^2 + \frac{4\pi^2}{\beta_0^2})}{1 - \mathrm{e}^{-\beta_0 p}}\, \widehat{\xi}_t^{\pm}(p)\, \widehat{\xi}_t^{\pm}(-p)\, dp$$