Many-body localization, LIOMs, and rare-region effects

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Outline

1. MBL and failure of thermalization (1d)
2. A concrete RG: "eliminate interactions up to range L," $L \to 2L$, etc.
3. The effect of the RG on the resonant regions (Griffiths regions)
4. The transition out of the MBL phase
   4.1 Avalanche effect
   4.2 Flow equations

Phenomenology of MBL

For a many-body quantum system with disorder, we may observe the following, which may be thought of as essential features of many-body localization (MBL):

1. Absence of transport
2. Anderson localization in configuration space (as in, e.g. IPR measures)
3. Area law entanglement
4. Violation of ETH (eigenstate thermalization hypothesis)
5. Absence of level repulsion
6. Logarithmic growth of entanglement for an initial product state

Typical example: disordered spin chain

Spin chain with random interactions and a weak transverse field on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^{K} h_i S_i^z + \sum_{i=-K}^{K} \gamma_i S_i^x + \sum_{i=-K-1}^{K} J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable.

Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

Ergodicity breaking and the emergence of an extensive set of local integrals of motion (LIOMs)

Loosely speaking, ergodicity should mean the spreading of wavepackets throughout the system. In an extreme case, there may be a complete set of of conserved quantities (quasilocal in nature) – a complete failure of ergodicity.

How do we know if a system has a complete set of quasilocal LIOMs? Can we construct them?

We seek a quasilocal unitary that diagonalizes H. That is, $D = U^\star H U$ is diagonal, and quasilocality means that the effect of U on a set of spins that span a distance L in the lattice should be (identity) + (exponentially small in L). There may be rare, nonpercolating regions where this property fails (resonant regions).

Then we may define LIOMs $\tau_i = U S_i^z U^\star$.

It is clear that $[H, \tau_i] = [D, S_i^z] = 0$.

Likewise $[\tau_i, \tau_j] = 0$.

Properties 1-6 listed above for MBL should follow if one can find a complete set of LIOMs[1]

[1] Huse, Nandkishore, Oganesyan, PRB '14; Serbyn, Papic, Abanin, PRL '13

One spin

For guidance, consider what happens for a single spin. Then

$$H = \begin{pmatrix} h & \gamma \\ \gamma & -h \end{pmatrix}$$

and for $\gamma \ll h$ the eigenfunctions are close to $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. The eigenfunctions resemble the basis vectors. This means the basis vectors can be used to label the eigenfunctions.

At the other extreme, if $\gamma \gg h$ the eigenfunctions are close to $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$. With complete hybridization, there is no meaningful way to associate eigenfunctions with basis vectors.

Perturbative and non-perturbative approaches

One may construct LIOMs perturbatively[2].

But rare regions where perturbation theory breaks down have the potential to spoil MBL. I will outline a nonperturbative construction[3] (which, however, depends on a physically reasonable assumption on eigenvalue statistics – essentially that the level spacings in a system of n spins are no smaller than some exponential in n.)

It is especially important to have a nonperturbative proof of an MBL phase, as some are questioning the numerical evidence for MBL[4].

[2] Integrals of motion in the many-body localized phase, Ros, Müller, Scardicchio NP '15

[3] Imbrie, *On many-body localization for quantum spin chains*, JSP '16

[4] *Quantum chaos challenges many-body localization*, Šuntajs, Bonča, Prosen, Vidmar arXiv:1905

What about the level spacing condition?

Assumption LLA(ν, C). *Consider the Hamiltonian H in boxes of size n. Its eigenvalues satisfy*

$$P\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n,$$

for all $\delta > 0$ and all n.

I have been developing tools for proving level-spacing conditions in simpler systems (noninteracting).

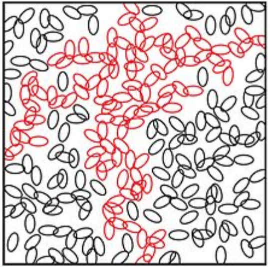
But in this talk I will focus on explaining the key mechanisms at work in the proof.

Then, I will connect these to recent work by others on the nature of the transition out of the MBL phase.

Percolation picture validated for large disorder or weak interactions in 1d

Proof controls the probability of resonance for processes, and shows that the graph of resonances is non-percolating.

Then is possible to define quasilocal similarity transformations on H that diagonalize it, deforming the tensor product basis vectors into the exact eigenfunctions.



Resonances in the first step

Initially, the only off-diagonal term is $\gamma_i S_i^x$, which is local, so we may start by looking at single-flip resonances.

Let the spin configuration $\sigma^{(i)}$ be equal to σ with the spin at i flipped.
Let the associated change in energy be $\Delta E_i \equiv E_\sigma - E_{\sigma^{(i)}}$

We say that the site i is *resonant* if $|\Delta E_i| < \varepsilon \equiv \gamma^{1/20}$ for at least once choice of $\sigma_{i-1}, \sigma_{i+1}$. Then for nonresonant sites the ratio $\gamma_i / \Delta E_i$ is $\leq \gamma^{19/20}$.

A site is resonant with probability $\sim 4\varepsilon$. Hence resonant sites form a dilute set where perturbation theory breaks down.

Rotate away interaction terms $J(i) \equiv \gamma_i S_i^x$ for nonresonant sites i by defining

$$A \equiv \sum_{\text{nonresonant } i} A(i) \text{ with } A(i)_{\sigma\sigma^{(i)}} = \frac{J(i)_{\sigma\sigma^{(i)}}}{E_\sigma - E_{\sigma^{(i)}}}$$

and a renormalized Hamiltonian:

$$H^{(1)} = e^A H e^{-A} = H + [A, H] + \frac{[A,[A,H]]}{2!} + \ldots = H_0 + J^{\mathrm{res}} + J^{(1)}.$$

Properties of the new Hamiltonian:

The new interaction $J^{(1)}$ is quadratic and higher order in γ – the leading-order term has been eliminated.

Note that $A(i)$ commutes with $A(j)$ or $J(j)$ if $|i-j|>1$.

Thus we preserve quasi-locality of $J^{(1)}$; it can be written as $\sum_g J^{(1)}(g)$, where g is a sum of connected graphs involving spin flips $J(i)$ and associated energy denominators.

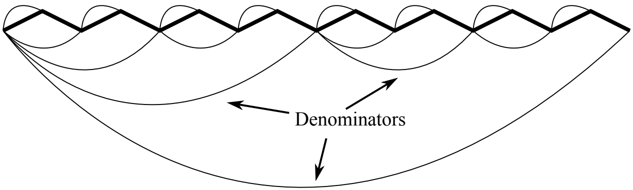
Define resonant blocks by taking connected components of the set of sites belonging to resonant graphs. We perform exact rotations O in small, isolated resonant blocks to diagonalize the Hamiltonian there.

Graph-based notion of resonance. Moment bounds control probability.

Use a sequence of length scales $L_k = (15/8)^k$, and continue rotating away interactions of lower order than γ^{L_k}.

$J^{(k)}$ is a sum of connected graphs $J^{(k)}_{\sigma\tilde{\sigma}}(g)$; quasilocality is preserved.

(Each graph g is a walk in spin-configuration space, whose trace in physical space is connected)



A graph of order L_k is resonant if $A^{(k)}_{\sigma\tilde{\sigma}}(g) \equiv \frac{J^{(k)}_{\sigma\tilde{\sigma}}(g)}{E^{(k)}_{\sigma} - E^{(k)}_{\tilde{\sigma}}} > (\gamma/\varepsilon)^{L_k}$.

Fractional moment bounds on graphs and the Markov inequality imply that the probability that g is resonant is $< \varepsilon^{L_k}$; then it is OK to sum over $\exp(O(L_k))$ graphs in the associated percolation problem.

Backtracking

The moment method breaks down for walks that return to previously visited sites in physical space. But backtracking sections can be handled with L^∞ bounds has they have a greater decay rate.

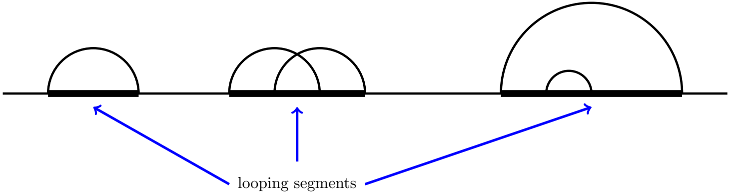
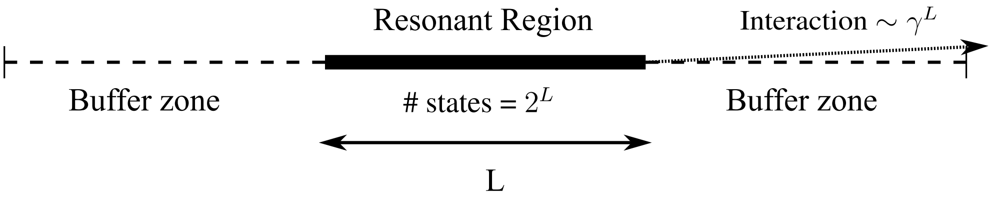


Figure 6: Timeline of the walk. Arches connect pairs of times where the walk is at the same site/block.

Resonant regions (= Griffiths regions) need buffer zones

These are regions where we have failure of the bounds needed to control the rotations.

Buffer zones are needed so that the smallness $\sim \gamma^L$ of a graph crossing the buffer is much smaller than the typical $\Delta E = 2^{-L}$ in the resonant region.



The buffer zone is expected to be thermalized by the resonant region.

In 1-d the buffer zone has volume comparable to that of the resonant block, so we can diagonalize H in the combined region, eliminating internal interactions while keeping the level-spacing larger than the interactions with spins outside.

Renormalization group picture

In RG terms, the rotations removing terms in the Hamiltonian up to order γ^L is analogous to "integrating out" short distance degrees of freedom in traditional RG.

At the same time, resonant regions up to some size R are "eliminated" once L is large enough so that the remaining interaction terms are smaller than the level spacing in the region (with its buffer zone, total size $R + 2L$). At that point, the region hosts a "metaspin" which takes 2^{R+2L} values, but the interactions are so small that there is little hybridization with spins elsewhere.

Deep in the localized region, this RG has the property that the density of remaining resonant regions (including their buffer zones with width given by the running RG length L) goes to zero with L.

Note two effects are in play:
(1) Elimination of smaller resonant regions reduces the density.
(2) Fattening of the buffer zones on the remaining regions increases the density.
My MBL proof shows that (1) dominates (2) deep in the weak coupling/strong disorder region, and the density goes to zero as $L \to \infty$.

Avalanche effect

For weaker disorder/stronger interactions, the decay rate can be reduced to the point where no buffer size can insulate the resonant region from the rest of the chain: the avalanche instability[5].

Matrix elements connecting the resonant region with spins outside the buffer zone should behave as $\gamma^L 2^{-(R+2L)/2}$. For this to be small compared with the level spacing $\sim 2^{-(R+2L)}$, we need $\gamma^L \leq 2^{-(R+2L)/2}$. This means that the buffer size must satisfy

$$L \geq \frac{\frac{1}{2}\log 2 \cdot R}{\log \gamma^{-1} - \log 2}.$$

This diverges when γ increases toward $1/2$.

At some point, then, increasing γ causes (2) to dominate (1); *i.e.* the fattening effect dominates the eliminations, and the density of resonant regions grows with L.

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[5] *Many-body delocalization as a quantum avalanche*. Thiery, Huveneers, Müller, De Roeck, PRL '18

Flow equations

To capture this physics, define a 2-parameter RG flow[6] via the following parameters:

$\zeta^{-1} = \log \gamma^{-1}$, so that ζ is the localization length governing decay of matrix elements.
$\zeta_c^{-1} = \log 2$, the critical decay rate for the avalanche instability.
$\ell = \log L$ is the RG flow parameter.
$\rho(\ell)$ is the density of resonant regions after running the RG up to scale $L = e^{\ell}$.

Then ρ should increase if $\zeta > \zeta_c$, decrease if $\zeta < \zeta_c$. Simplest equation:

$$\frac{d\rho}{d\ell} = b\rho(\zeta - \zeta_c).$$

[6] *Kosterlitz-Thouless scaling at many-body localization phase transitions*, Dumitrescu, Goremykina, Parameswaran, Serbyn, Vasseur, PRB '19

Back-reaction of resonant regions on decay rate

Even though small resonant regions are eliminated in this RG, they renormalize the decay rate by providing shortcuts where decay of matrix elements will cease.

Decay can be described in terms of an "Agmon metric," where resonant regions are contracted to points.

In the MBL proof at weak coupling, I showed that the sum over scales of the density of resonant regions at that scale is small, so exponential decay is preserved (at a reduced rate)[7].

Effectively, a decay $e^{-\zeta^{-1}|x-y|}$ becomes $e^{-\zeta^{-1}(1-\rho)|x-y|}$, so we obtain the second flow equation:

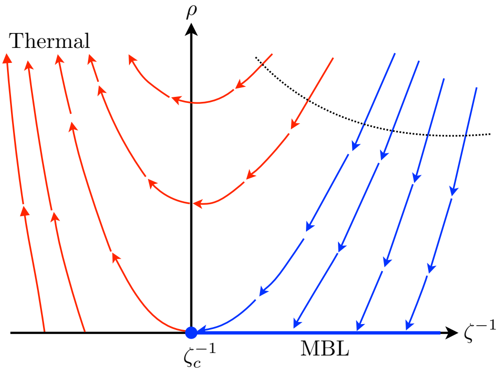
$$\frac{d\zeta^{-1}}{d\ell} = -c\rho\zeta^{-1}.$$

[7] The idea goes back to Fröhlich, Spencer, CMP 1983.

KT flow

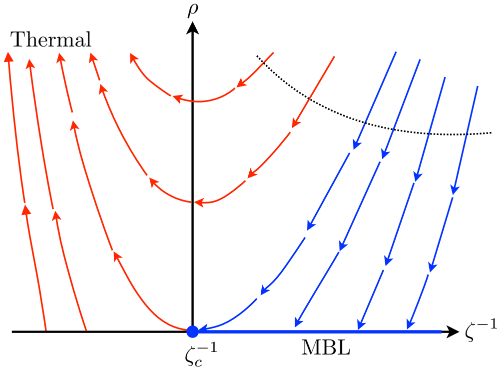
Thus we arrive at the following system of flow equations, proposed by Dumitrescu *et al* as a way to describe the transition out of the MBL phase:

$$\frac{d\rho}{d\ell} = b\rho(\zeta - \zeta_c), \qquad \frac{d\zeta^{-1}}{d\ell} = -c\rho\zeta^{-1}.$$



These equations should be familiar as they also describe the Kosterlitz-Thouless transition. There is a line of semi-stable fixed points along the ζ^{-1}-axis, which terminates at ζ_c^{-1}. To the left, all flows are driven to full thermalization with $\rho = 1$.

Parallels with the KT transition



Like the vortices, Griffiths regions represent nonperturbative effects, and the tendency of these effects to grow or shrink with the flow determines the phase reached from any starting point in the diagram. Vortex binding is analogous to resonant region elimination discussed above.

When bound, vortices renormalize the stiffness. Likewise, when eliminated, Griffiths regions renormalize the decay rate.

Note: the basic mechanisms of elimination and short-cuts have been incorporated in many phenomenological RG models of the MBL transition. The KT picture may be a kind of mean-field approximation, and it remains to be seen whether it will become the definitive theory of the transition.