Anomaly non-renormalization in Weyl semimetals (and related problems)

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I will focus mainly on the analogue of the chiral anomaly in interacting lattice Weyl semimetals; later I will briefly mention related results on 1d interacting fermions close to the quantum critical point or with quasi-periodic disorder.
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The anomaly is expressed by the celebrated triangle graph and in principle by a series of radiative corrections.
One of the main properties of the anomaly in QFT is the non-renormalization (Adler-Bardeen 1969): all the radiative interaction corrections cancel out and the anomaly is exactly determined by its lowest order contribution in perturbation theory (triangle graph).
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Is the anomaly non-renormalization valid with a finite lattice and without Lorentz invariance, that is in Weyl semimetals? In other words, the lattice corrections do exactly cancel, or produce a small but finite contribution?
In the positive case maybe one could get an example of universality like QHE or the optical conductivity on graphene.
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In the case of weak short range interactions non-renormalization can be established non-perturbatively (series uniformly convergent summing up to zero). The proof do not uses cancellations but a completely different and more robust mechanism.
We focus on the simple situation of a minimal number of Weyl nodes, i.e. two with opposite chirality, assuming broken time reversal symmetry. It is not restrictive to consider a simple model (Delplace Carpentier EPL 2010)

\[
h_0(\vec{k}) = \begin{pmatrix}
t_\perp \cos k_3 - \mu + t' + \alpha(\vec{k}) & t(\sin k_+ + i \sin k_-) \\
t(\sin k_+ - i \sin k_-) & -t_\perp \cos k_3 + \mu - t' - \alpha(\vec{k})
\end{pmatrix}
\]

with \( \alpha(k) = -t'(\cos k_+ \cos k_- - 1) \), where \( t, t' \) are planar hoppings, \( t_\perp \) is the perpendicular hopping and \( \mu \) describes the difference of densities. If \( |\mu - t'| \leq t_\perp \) there are 2 Fermi points \((0, 0, \pm p_F)\) with \( t_\perp \cos p_F = \mu - t \).
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We add an interaction so that the interacting model is

\[ H = \int d\vec{k} \psi_+^{\dagger}_{\vec{k}} h_0(\vec{k}) \psi_-^{\dagger}_{\vec{k}} + U \int d\vec{p} \hat{v}(\vec{p}) \rho_{\vec{p}} \rho_{-\vec{p}} + \nu N_3 \]

where \( \rho_{\vec{p}} = \int \frac{d\vec{k}}{(2\pi)^3} \psi_+^{\dagger}_{\vec{k}+\vec{p}} \psi_-^{\dagger}_{\vec{k}} \) is the local density, \( \hat{v}(\vec{p}) \) is a short range potential, and \( \psi^{\pm} = (a^{\pm}, b^{\pm}) \); \( N_3 = N_A - N_B \) is the staggered fermion number. \( \nu \) is a counterterm to fix the Weyl points.
In absence of interaction the 2-point function $g(x)$ close to the Weyl point have the form, if $k = k' \pm p_F$

$$\frac{\chi(k')}{Z_0} \left( \begin{array}{cc} -ik_0 \pm v_3^0 k'_3 & v_+(k_+ - ik_-) \\ v_0^0(k_+ + ik_-) & -ik_0 \mp v_3^0 k'_3 \end{array} \right)^{-1} (1 + R(k))$$

with $Z_0 = 1$, $v_3^0 = t_\perp \sin p_F$, $v_\pm^0 = t$. 
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with $Z_0 = 1$, $v_3^0 = t \sin p_F$, $v_\pm^0 = t$.

The model admits an effective description in term of massless Dirac particles with a local interaction. This QFT model is invariant under a global $\psi^\pm \to e^{\pm i\alpha} \psi^\pm$ and chiral $\psi^\pm \to e^{\pm i\gamma_5 \alpha} \psi^\pm$; the second symmetry is not true in the lattice model.
In absence of interaction the 2-point function $g(x)$ close to the Weyl point have the form, if $k = k' \pm p_F$

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In contrast to graphene, the velocity can be small and the Weyl points arbitrarily close. They are in general renormalized (their location is obtaining from $\nu$ by inversion).
Let us now couple the system to an external e.m. field $A_\mu$, $\mu = 0, 1, 2, 3$. We denote by $\langle \cdot \rangle_A$ the interacting Gibbs state of the system in the presence of the external field and $\langle \cdot \rangle = \langle \cdot \rangle_0$. The coupling is defined via the Peierls substitution.

The axial density is chosen, following NN, as the flow between Weyl points.

\[
\hat{\rho}_5^\pm = \int \frac{d\vec{k}}{(2\pi)^3} \frac{\sin k_3}{Z} \hat{\psi}^\pm_{\vec{k} + \vec{p}} \hat{\psi}_{\vec{k}}^-
\]

and $\rho_5 \sim \rho_+ - \rho_-$. 
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\[ \hat{\rho}_5^\pm = \int \frac{d\vec{k}}{(2\pi)^3} \frac{\sin k_3}{\tilde{Z}} \hat{\psi}_k^+ \hat{\psi}_{\vec{k} + \vec{p}}^\pm \]

and $\rho_5 \sim \rho_+ - \rho_-$.  

\[ \tilde{Z} \] has to be chosen so that $\hat{\rho}_5^\pm$ is proportional to $\pm($total density$)$

\[ \langle \hat{\rho}_\vec{p}; \hat{\psi}_{\vec{k} + \vec{p}_F}^\pm \hat{\psi}_{\vec{k} + \vec{p}_F}^\pm \rangle = \pm \langle \hat{\rho}_\vec{p}; \hat{\psi}_{\vec{k} + \vec{p}_F}^\pm \hat{\psi}_{\vec{k} + \vec{p}_F}^\pm \rangle (1 + O(k, p)) \]
Let us now couple the system to an external e.m. field \( A_\mu, \mu = 0, 1, 2, 3 \). We denote by \( \langle \cdot \rangle_A \) the interacting Gibbs state of the system in the presence of the external field and \( \langle \cdot \rangle = \langle \cdot \rangle_0 \). The coupling is defined via the Peierls substitution.

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\hat{\rho}_5^f = \int \frac{d\vec{k}}{(2\pi)^3} \frac{\sin k_3}{\tilde{Z}} \hat{\psi}_k^+ \hat{\psi}_{k+p}^-
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and \( \rho_5 \sim \rho_+ - \rho_- \).

\( \tilde{Z} \) has to be chosen so that \( \hat{\rho}_5^f \) is proportional to \( \pm (\text{total density}) \)

\[
\langle \hat{\rho}_p^p; \hat{\psi}_{k+p_{F}^\pm}^- \hat{\psi}_{k+p_{F}^\pm}^+ \rangle = \pm \langle \hat{\rho}_p; \hat{\psi}_{k+p_{F}^\pm}^- \hat{\psi}_{k+p_{F}^\pm}^+ \rangle (1 + O(\mathbf{k}, \mathbf{p}))
\]

The lattice chiral density is not exactly local: therefore, one has to couple it to the \( A \) field via the Peierls substitution, in order to ensure full lattice gauge invariance, and denote by \( \rho_5^x(A) \) the gauge-invariant chiral density.
The generating function of correlations can be written as a Grassmann integral as

\[ e^W(A,A^5,\phi) = \int P(d\psi) e^{V(\psi)+B(A,\psi)+(A_\mu^5,j^{5}_\mu(A))+(\psi,\phi)}, \]

\[ V \] contains the interaction and \( \nu \) term. We call \( \Gamma^{5}_{\mu,\mu_1,..\mu_n} \) and \( \Gamma_{\mu,\mu_1,..\mu_n} \) the derivatives with respect to \( A^5_{\mu}, A_{\mu_1},.. \) and \( A_\mu, A_{\mu_1},.. \) at \( A = 0 \).
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By Gauge invariance

\[ \partial_\alpha W(A_\mu, +\partial_\mu \alpha, A^5_0, \phi e^{i\alpha}) = 0 \]

so that \( p_\mu \Gamma_{\mu,\mu_1,..\mu_n} = 0 \) implying the conservation of the current \( <p_\mu j_\mu>_A = 0 \) in presence of an e.m. field. Of course no conservation holds for the "chiral current"; there is no associated symmetry.
Response in $A$

$$\langle \hat{\rho}_p^5 \rangle_A = i \Gamma^5_{0, \nu}(p) \hat{A}_{\nu, p} + \frac{i}{2} \int dp_1 dp_2 \Gamma^5_{0, \nu, \sigma}(p_1, p_2) \delta \hat{A}_{\nu, p_1} \hat{A}_{\sigma, p_2} + \ldots$$
Background e.m. field

- Response in $A$

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\langle \hat{\rho}_{\mathbf{p}}^5 \rangle_A = i \Gamma_{0, \nu}^5 (\mathbf{p}) \hat{A}_{\nu, \mathbf{p}} + \frac{i}{2} \int d\mathbf{p}_1 d\mathbf{p}_2 \Gamma_{0, \nu, \sigma}^5 (\mathbf{p}_1, \mathbf{p}_2) \delta \hat{A}_{\nu, \mathbf{p}_1} \hat{A}_{\sigma, \mathbf{p}_2} + \ldots
\]

- $\Gamma_{\mu, \mu_1, \ldots, \mu_n}^5$ are derivatives of $W$, that is

\[
\langle \hat{J}_{\mu, \mathbf{p}}^5; \hat{J}_{\nu, + \mathbf{p}_1}; \hat{J}_{\sigma, \mathbf{p}_2} \rangle + \langle \hat{J}_{\mu, \mathbf{p}}^5; \hat{\Delta}_{\nu, \sigma, \mathbf{p}_1, \mathbf{p}_2} \rangle + \langle \hat{\Delta}_{\mu, \nu, \mathbf{p}, \mathbf{p}_1}^5; \hat{J}_{\sigma, \mathbf{p}_2} \rangle + \langle \hat{\Delta}_{\mu, \sigma, \mathbf{p}, \mathbf{p}_2}^5; \hat{J}_{\nu, \mathbf{p}_1} \rangle + \langle \hat{\Delta}_{\mu, \nu, \sigma, \mathbf{p}_1, \mathbf{p}_2}^5 \rangle
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where $\Delta, \Delta^5$ are derivatives of $B$ or $j^5$ (Schwinger terms).
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In the non interacting case $p_1 + p_2 = (p_0, 0)$, one has (Nielsen and Ninomiya (1983)):

$$p_0\Gamma_{0,\nu,\sigma}^5(p_1, p_2) = \frac{e^2}{\hbar^2} \frac{1}{2\pi^2} \varepsilon_{\alpha\beta\nu\sigma} p_1, \alpha p_2, \beta$$

This is same expression of the anomaly for $\bar{\psi}\gamma_\mu\gamma_5\psi$ for massless QED (but for Weyl semimetal strictly speaking is not an anomaly but a simulation).
Main Result

What happens in presence of interaction $U \neq 0$? The interaction produce non-universal modification in the physical quantities; the Fermi points are shifted and are given by $\pm p_F + b_\pm U + \ldots$, with $b_\pm \neq 0$, and the interacting Fermi velocities are given by $v_i = v_i^0 + a_i U \ldots$. Such quantities are expressed by series in $U$ with non trivial coefficients.
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\[ \text{triangle graph} \]
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- The anomaly is expressed by the celebrated triangle graph and in principle by a series of radiative corrections.

- The contributions of such terms is $\frac{e^2}{\hbar^2} \frac{1}{2\pi^2} p_1, p_2, p_3 + AU + CU^2 + \ldots$. Do such corrections cancel or not?

![Diagram](image-url)
Main Result

- **Theorem** (Giuliani Mastropietro Porta arXiv:1907.00682) There exists $U_0 > 0$, independent of the distance between the Fermi points, such that, if $|U| < U_0$, fixing $\nu = \nu(\lambda)$ and $Z^5 = Z^5(\lambda)$ we have

$$p_0 \Gamma^5_{0,\nu}(p_0, 0) = O(p_0^3 \log |p_0|)$$

and, if $p_1 + p_2 = (p_0, 0)$,

$$p_0 \Gamma^5_{0,\nu,\sigma}(p_1, p_2) = \frac{e^2}{\hbar^2} \frac{1}{2\pi^2} p_1,\alpha p_2,\beta \varepsilon_{\alpha\beta\nu\sigma},$$

up to an error $O(P^3 \log P)$, with $P = \max\{|p_1|, |p_2|\}$. 
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\]

*up to an error* $O(P^3 \log P)$, *with* $P = \max\{|p_1|, |p_2|\}$.

- $A_0 = A_1 \equiv 0$, $A_2(t, x) = Bx_1$, $A_3(t, x) = -Et$ *we get, at quadratic order,*

\[
\partial_t \langle N^5_t(A) \rangle_A = \frac{e^2}{\hbar^2 c} \frac{1}{2\pi^2} EB \text{ where } N^5 = \sum_x \rho^5_x
\]
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Remarks

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Related to universality of optical conductivity in graphene; all interaction corrections cancel out in the Hubbard model on the honeycomb lattice (Giuliani Mastropietro Porta CMP 2012; PRB 2012)
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Important to understand if is true with Coulomb interactions or strong coupling.

One can see QFT as an effective theory emerging from some deeper unknown description free of uv divergences and with less symmetries. Weyl semimetals provide an example of emerging QED from a non Lorentz invariant background. The correlations have small corrections (proportional to the ratio between the momentum scale and the lattice ) but the anomaly non-renormalization is a robust property: no corrections are present.
The proof of universality combines two main ingredients: (a) invariance under local gauge transformation and Ward Identities; (b) regularity properties of the correlations $\Gamma_{\mu,\nu,\sigma}^5$.

We perform an exact RG analysis integrating momentum scales of decreasing size $\sim \gamma^h$, $\gamma > 1$. We get a sequence of effective potentials $V^h$ at scale $h = 0, -1, -2, ...$. $V^h$ is given by a local part and an irrelevant part, expressed by by sum of non-local monomials $\int d\mathbf{x} d\mathbf{y} W_{n,m}^h \psi^n A^m$.
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The kernels $W^h_{n,m}$ are expressed in terms of a convergent power series in $U$ and rcc $\nu^h, Z^h_{\mu}, Z^5_{\mu}$. Technical part; cluster expansion and Gram bound for fermionic determinants ($M_{\alpha,\beta} = (f_{\alpha}, g_{\beta})$, $|\det M| \leq \prod_\alpha ||f_{\alpha}|| ||g_{\alpha}||$). In order to achieve convergence Feynman graphs cannot be used; one needs cancellations by anticommutativity (classical trick in constructive QFT: Caianello 1956, Gawedski-Kupianenen 1985).
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Finiteness of radius of convergence (uniformly in $\sin p_F$) is in (Mastropietro JSP 2015; JPA 2015). The RG has two regimes, the first with dimension $7/2 - 5/4n$ the second with dimension $4 - 3/2n$.
It is convenient to separate

\[ W^{(h)}_{n,m} = W^{(h)}_{n,m;0} + W^{(h)}_{n,m;1} \]

where \( W^{(h)}_{n,m;0} \) depends only on the rcc, while \( W^{(h)}_{n,m;1} \) depends on the irrelevant terms.
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Convergence allows the following non-perturbative bound holds:

\[
\int d\mathbf{x} d\mathbf{y} |W_{n,m;i}^{(h)}(\mathbf{x}, \mathbf{y})| e^{\kappa \sqrt{2h d(\mathbf{x}, \mathbf{y})}} \leq C_{n,m} 2^{(4-\frac{3}{2} n-m)h} 2^{\theta_i h}.
\]

with \( \theta = 0 \) if \( i = a \), \( \theta = 1 \) if \( i = 2 \).

Note the improvement when some irrelevant quartic term contribute. This improvement plays a key role in the proof.
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The above bound has immediate implications on the regularity properties of the \( \hat{\Gamma}_{n,m} \).
The anomaly is expressed by the derivative of $\hat{\Gamma}^{5}_{\mu,\nu,\sigma}$ which is bounded by $\sum_{h \leq 0} 2^{-h} 2^h$, which is infinite; $\hat{\Gamma}^{5}_{\mu,\nu,\sigma}$ is continuous but not differentiable.
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However one can decompose as $\hat{\Gamma}^5_{1,\mu,\nu,\sigma} + \hat{\Gamma}^5_{2,\mu,\nu,\sigma}$; the second term has at least a $U$ vertex so by the previous bound is differentiable. The same is true for the Schwinger terms which are differentiable.
Regularity properties and RG

- The anomaly is expressed by the derivative of $\hat{\Gamma}_{\mu,\nu,\sigma}^5$ which is bounded by $\sum_{h \leq 0} 2^{-h} 2^h$, which is infinite; $\hat{\Gamma}_{\mu,\nu,\sigma}^5$ is continuous but not differentiable.

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- The only non differentiable term is a renormalized triangle graph with vertices and velocities at scale $h$; again we can write the propagators as the relativistic part and a rest, and the vertices and velocities as their value at $h = -\infty$ plus a rest. In conclusion the renormalized triangle is a relativistic triangle (non differentiable) and a rest which is differentiable.
In conclusion the following decomposition is found

\[ \Gamma_{\mu,\nu,\sigma}^5(p_1, p_2) = \Gamma_{\mu,\nu,\sigma}^{5,\text{rel}}(p_1, p_2) + H_{\mu,\nu,\sigma}^5(p_1, p_2), \]

where \( \Gamma_{\mu,\nu,\sigma}^{5,\text{rel}}(p_1, p_2) = \frac{Z_{\mu} Z_{\nu} Z_{\sigma}}{Z^3 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(\bar{p}_1, \bar{p}_2) \) where \( I_{\mu,\nu,\sigma}(p_1, p_2) \) is the undressed relativistic chiral triangle graph with momentum cutoff and \( \bar{p} = (p_0, v_1 p_1, v_2 p_2, v_3 p_3) \).
In conclusion the following decomposition is found

$$\Gamma^5_{\mu,\nu,\sigma}(p_1, p_2) = \Gamma^5_{\mu,\nu,\sigma,\text{rel}}(p_1, p_2) + H^5_{\mu,\nu,\sigma}(p_1, p_2);$$

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The fact that $$H^5_{\mu,\nu,\sigma}$$ depend on the irrelevant term implies that is more regular, that is differentiable.
Renormalization Group Analysis

- We now combine the use of Ward Identities (WI) with the regularity properties of $\Gamma^{5,\text{rel}}_{\mu,\nu,\sigma}$ and $H^{5}_{\mu,\nu,\sigma}$.
- The WI implies that the vertex renormalizations are proportional to the velocities:
  
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$$p_\mu I_{\mu,\nu,\sigma}(p_1, p_2) = \frac{1}{6\pi^2} p_1,\alpha p_2,\beta \varepsilon_{\alpha\beta\nu\sigma},$$

where $p = p_1 + p_2$, and $p_1,\nu I_{\mu,\nu,\sigma}(p_1, p_2) = \frac{1}{6\pi^2} p_1,\alpha p_2,\beta \varepsilon_{\alpha\beta\mu\sigma}$ (non conservation of current in QED with momentum cut-off).
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\]

\( \bar{p}_\alpha = v_\alpha p_\alpha \) and note that

\[
v_\alpha v_\beta v_\nu v_\sigma \varepsilon_{\alpha\beta\nu\sigma} = v_1 v_2 v_3 \varepsilon_{\alpha\beta\nu\sigma}.
\]
The second term $H_{\mu,\nu,\sigma}(p_1, p_2)$ is essentially impossible to evaluate directly, being an infinite series. However, we know by the WI that $p_{1,\nu}\Gamma_{\mu,\nu,\sigma} = 0$. In contrast with $\Gamma_{\mu,\nu,\sigma}$, $H_{\mu,\nu,\sigma}(p_1, p_2)$ has continuous derivatives hence we can write in Taylor up to order 1. By the WI we get the condition $H_{\mu,\nu,\sigma}(0, 0) = 0$, $\partial H_{\mu,\nu,\sigma}/\partial p_2, \beta(0, 0) = -\frac{1}{6}\pi^2 \epsilon_{\nu\beta\mu\sigma}$. The contribution of $H$ depends only on the derivatives fixed above so that $p_{\mu}\Gamma_{\mu,\nu,\sigma}(p_1, p_2) = p_{\mu}\Gamma_{\mu,\nu,\sigma}(p_1, p_2) + p_{\mu}\sum_{j=1,2}p_{j,\alpha}\partial H_{\mu,\nu,\sigma}/\partial p_{j,\alpha}(0, 0) = \frac{1}{2}\pi^2 p_{1,\alpha}p_{2,\beta}\epsilon_{\alpha\beta\nu\sigma}$. This ends the proof.

Perfect universality even without Lorentz invariance. Convergence is independent from the distance between Fermi points (anomaly does not depend on the size of infrared regime). Essentially impossible to see by direct graph cancellation (in contrast to QFT). Open problem (at least on a rigorous side); long range interaction or disorder. The decomposition used above cannot be used.
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Natural question: for which properties is the behavior found in Bethe ansatz solvable models 1d generic even when the Luttinger description breaks down and the physics is dominated by irrelevant terms?
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We perform a two-regime RG analysis (Bonetto Mastropietro AHP 2018, EPL 2018); in the infrared linear regime the scaling dimension is $2 - n/2$ (quartic terms marginal), in the ultraviolet quadratic one is $3/2 - n/4$ (quartic terms relevant)
In the first quadratic regime the interaction is relevant; however there is only one singularity so in the spiness case there is no local quartic term, and terms with derivatives are irrelevant.
**One dimensional case**

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- In the second regime there are 2 fermi points: equal to Thirring model if the bare interaction is suitably tuned up to irrelevant terms.
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In order to compute the transport coefficients one separates the correlation expression in a dominant relativistic part (which is now non continuous) and a rest (depending on the irrelevant terms) which is continuous.
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- However the cut-off breaks the local phase symmetry and produces an extra term in the WI for this interacting QFT theory (the relativistic anomaly).
The extra term in the WI for the relativistic emerging theory can be decomposed in a dominant part plus terms which are negligible.
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One get a closed expression for the relativistic dominant part correlations (non differentiable).
One dimensional case: spinless

- The transport coefficients are expressed by the relativistic part (known) while the contribution from the irrelevant terms are fixed by the lattice WI.

\[ D = \frac{Kv}{\pi} \quad \kappa = \frac{K}{\pi v} \]

where (\( r \) is the distance from criticality)

\[ K = 1 - \tau \]
\[ \tau = \lambda v (0) - v (2) p F (O(\lambda r)) \]

\[ v = \sin (p F (1 + O(\lambda r))) \]

\[ \mu = \mu R + \nu, \quad \nu = \lambda v (0) p F / \pi + O(\lambda) \]
\[ \mu_R = - \cos (p F) = \pm 1 - r. \]

\( \tau \) is the anomaly of the emerging theory.

Moreover at criticality one gets the non-interacting values

\[ K \rightarrow 1 \quad D/D_0 \rightarrow 1 \]

as \( r \rightarrow 0 \).

\( \mu_c \) is shifted by the interaction (see e.g. Zotos et al (2016)).

Main point: the \( O(\lambda r) \) are bounds on convergent series.
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- One gets the remarkably simple Luttinger liquid relations uniformly up to the quantum critical point; $D$ is the Drude weight and $\kappa$ is the susceptibility:

$$D = K v / \pi \quad \kappa = K / \pi v$$

where ($r$ is the distance from criticality)

$$K = \frac{1 - \tau}{1 + \tau}, \quad \tau = \lambda \frac{v(0) - v(2p_F)}{2\pi v} + O(\lambda^2 r) \quad v = \sin p_F (1 + O(\lambda r))$$

$$\mu = \mu_R + \nu, \quad \nu = \lambda v(o) p_F / \pi + O(\lambda), \quad \mu_R = -\cos p_F = \pm 1 \mp r. \quad \tau \text{ is the anomaly of the emerging theory.}$$
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Main point: the $O(\lambda r)$ are bounds on convergent series.
$D$ and $K$ as function of density (or magnetic field), both in Heisenberg or non solvable cases. $D/D_0$ and $K$ tend to 1: Features found in the solvable case persists up to the critical point.
One dimensional case: spinful

In the spinful case in the first quadratic regime the interaction is relevant and there are no cancellations; the estimated radius of convergence decreases with $r$. 

The linear dominant term is the one of the model with several interactions; there are several anomalies in the emerging WI. 

The LL relations are still true in the convergence regime 

$$D = \frac{K}{\pi v} \kappa = \frac{K}{\pi v}$$

$$K = \sqrt{(1 - 2 \nu \rho)^2 - \nu^2 \frac{4}{1 + 2 \nu \rho} - \nu^2 \frac{4}{v} \nu \rho}$$

$$\frac{\lambda v(0)}{2 \pi \sin pF} + \ldots$$

$\nu \rho, \nu^4$ are the anomalies of the emerging theory.

One cannot take the $r \to 0$ limit; however for $\lambda$ small one can see that $K$ does not tend to the non-interacting value 1 but $D$ becomes close to $D_0$. 

Vieri Mastropietro (Università di Milano)

Anomaly non-renormalization in Weyl semimetals (and related problems)

September 23, 2019 25 / 33
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$$D = K v / \pi \quad \kappa = K / \pi v \quad \text{with}$$

$$K = \sqrt{\frac{(1 - 2\nu_\rho) - \nu_4^2}{(1 + 2\nu_\rho) - \nu_4^2}} \quad v = \sin p_F \frac{(1 + \nu_4)^2 - 4\nu_\rho^2}{(1 - \nu_4)^2 - 4\nu_\rho^2}$$

$$\nu_4 = \lambda v(0) / 2\pi \sin p_F + \ldots \quad \nu_\rho = \lambda (v(0) - v(2\pi_F)/2)/2\pi \sin p_F + \ldots$$

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One cannot take the $r \to 0$ limit; however for $\lambda$ small one can see that $K$ does not tend to the non-interacting value 1 but $D$ becomes close to $D_0$. 
Contrary to the spinless case, we cannot get $p_F = 0$. $K$ shows the tendency to a strongly interacting fixed point while $D$ is close to the non-interacting value. Cfr the behavior of the Hubbard model by Bethe ansatz (e.g. Schultz 1993)
Another case when irrelevant terms are crucial; interacting Aubry-Andre’ model (XXZ chain with quasi random disorder)

\[
H = -\varepsilon \left( \sum_{x \in \Lambda} (a_{x+1}^+ a_x + a_{x-1}^+ a_x^-) + \right)
\]

\[
\sum_{x \in \Lambda} u \cos(2\pi(\omega x + \theta))a_x^+ a_x^- + U \sum_{x,y} v(x - y) a_x^+ a_x^- a_y^+ a_y^-
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with \( v(x - y) = \delta_{y-x,1} + \delta_{x-y,1} \).
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Umklapp terms (marginal or relevant) are effectively irrelevant except at half filling, as they do not fill momentum conservation from the Fermi points; Due to incommensurability however momenta are almost conserved, hence it is not clear if this improvement with respect to scaling dimension holds.
The Aubry-Andre’ model

- In the non interacting case $U = 0$ the states are obtained by the antisymmetrization (fermions) of the eigenfunctions of almost Mathieu equation

\[-\varepsilon \psi(x + 1) - \varepsilon \psi(x - 1) + u \cos(2\pi(\omega x + \theta)) \psi(x) = E \psi(x)\]
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- For almost every $\omega, \theta$ the almost Mathieu operator has
  a) for $\varepsilon/u < \frac{1}{2}$ only pps with exponentially decaying eigenfunctions (Anderson localization);
  b) for $\varepsilon/u > \frac{1}{2}$ purely absolutely continuous spectrum (extended quasi-Bloch waves)
\[ \varepsilon = U = 0 \text{ molecular limit } H = \sum_x (\cos 2\pi(\omega x) - \mu) a_x^+ a_x^- \]

\[ < T a_x^- a_y^+ > |_0 = \delta_{x,y} \tilde{g}(x, x_0 - y_0) \]

\[ \tilde{g}(x, x_0 - y_0) = \frac{1}{\beta} \sum_{k_0} e^{-i k_0 (x_0 - y_0)} \]

\[ -i k_0 + \cos 2\pi(\omega x) - \cos 2\pi(\omega \bar{x}) \]

GS occupation number \( \chi(\cos 2\pi(\omega x) \leq \mu) \).
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  GS occupation number $\chi(\cos 2\pi(\omega x) \leq \mu)$.

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- If we set $x = x' + \bar{x}_\rho$, $\rho = \pm$, for small $(\omega x')_{\text{mod.1}}$

  $\hat{g}(x' + \bar{x}_\rho, k_0) \sim \frac{1}{-i k_0 \pm v_0(\omega x')_{\text{mod.1}}}$
Molecular limit

- \( \varepsilon = U = 0 \) molecular limit
  \[ H = \sum_x (\cos 2\pi(\omega x) - \mu) a_x^+ a_x^- \]
  \[ < T a_x^- a_y^+ > |_0 = \delta_{x,y} \bar{g}(x, x_0 - y_0) \]
  \[ \bar{g}(x, x_0 - y_0) = \frac{1}{\beta} \sum_{k_0} \frac{e^{-i k_0 (x_0 - y_0)}}{-i k_0 + \cos 2\pi (\omega x) - \cos 2\pi (\omega \bar{x})} \]

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- Let us introduce \( x_\pm = \pm \bar{x}, x_\pm \) Fermi coordinates.

- If we set \( x = x' + \bar{x} \rho, \rho = \pm, \) for small \( (\omega x') \mod 1 \)
  \[ \hat{g}(x' + \bar{x} \rho, k_0) \sim \frac{1}{-i k_0 \pm v_0 (\omega x') \mod 1} \]

- We assume Diophantine conditions; KAM (Kolmogorv Arnol Moser) methods.

  \[ ||\omega x|| \geq C_0 |x|^{-\tau} \quad (*) \]
  \[ ||\omega x \pm 2\omega \bar{x}|| \geq C_0 |x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (**) \]
Localized regime

**Theorem 1.**

*In the spinless interacting Aubry-Andre’ model, assuming (*) and \( \tilde{x} \) verifying (**) if \( u = 1, \mu = \cos 2\pi(\omega \tilde{x}) + \nu \) for small \( \varepsilon, U \) and suitable \( \nu \), for any \( N, L = 1/T = \infty \)*

\[
| \langle T a_x^- a_y^+ \rangle | \leq C e^{-\xi |x-y|} \log(1 + \min(|x||y|))^{\tau} \frac{1}{1 + (\Delta |x_0 - y_0|))} N^{(***)}
\]

*with \( \Delta = (1 + \min(|x|,|y|))^{-\tau}, \xi = |\log(\max(|\varepsilon|,|U|))|.*

Assuming (*) and \( \tilde{x} \) half integer the same holds with \( \Delta \) replaced by \( \sigma = O(\varepsilon^{2\tilde{x}}) \)

Exponential decay in coordinates signal persistence of localization in presence of interactions.

Some idea of the proof

- We integrate higher scales getting a sequence of effective potentials. According to power counting, the theory is non renormalizable; all effective interactions have positive dimension, $D = 1$. 

- One has to distinguish among the monomials $\prod_{i} \psi_{\epsilon i} x'_{i}, x_{0}, i, \rho_{i}$ in the effective potential between resonant and non resonant terms. Resonant terms; $x'_{i} = x'_{j}$. Non Resonant terms $x'_{i} \neq x'_{j}$ for some $i, j$. (In the non interacting case only two external lines are present). Non resonant terms almost connect the Fermi coordinates.

- Methods coming from direct proof of convergence of Lindstedt series for tori of quasi integrable systems. The relevance of all terms suggests that localization (the unperturbed case) is broken.
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- The relevance of all terms suggest that localization (the unperturbed case) is broken.
The non resonant terms are irrelevant. The idea is that if two propagators have similar (not equal) small size (non resonant subgraphs), then the difference of their coordinates is large and this produces a "gain" as passing from $x$ to $x + n$ one needs $n$ vertices.

That is if $(\omega x'_1)_{\text{mod}1} \sim (\omega x'_2)_{\text{mod}1} \sim \Lambda^{-1}$ then by the Diophantine condition

$$2\Lambda^{-1} \geq ||\omega(x'_1 - x'_2)|| \geq C_0|x'_1 - x'_2|^{-\tau}$$

that is $|x'_1 - x'_2| \geq \bar{C}\Lambda^{\tau-1}$. This implies that such subgraphs have associated an high power of $t, U$ which change the dimensions.
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- The resonant terms are vanishing by Pauli principle

- Therefore perturbation theory is convergent for small $t$. In contrast delocalized behavior is found for large $\varepsilon$. 
Conclusions

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In presence of quasi-random disorder, again the localizationn at $T = 0$ persists due to irrelevance of terms almost connecting Fermi points; number theoretical (Diophantine) properties ensure such irrelevance. Again major problem understand the role of such irrelevant terms at $T \neq 0$. 