Accademia dei Lincei

Sept. 23, 2019

# Hydrodynamics of the Classical Toda Chain

Herbert Spohn

TU München

**Toda chain**

$$H_N = \sum_{j=1}^{N} \left\{ \tfrac{1}{2} p_j^2 + e^{-(q_{j+1} - q_j)} \right\}$$

positions $q_j \in \mathbb{R}$ — not ordered

momenta $p_j \in \mathbb{R}$

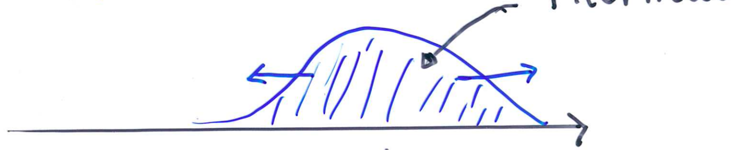
$\Rightarrow$ integrable (solitons)

$\Rightarrow$ random initial conditions ‖ energy $\mathcal{O}(N)$

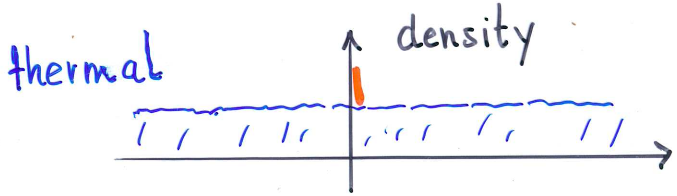
EXAMPLES:

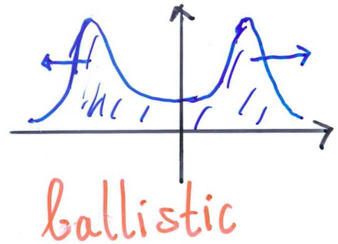
domain wall


expanding cloud




small deviations




$\Rightarrow$ hydrodynamic theory $\Leftarrow$

- many efforts since 2015
- mostly quantum

‖ Toda chain as road map ‖

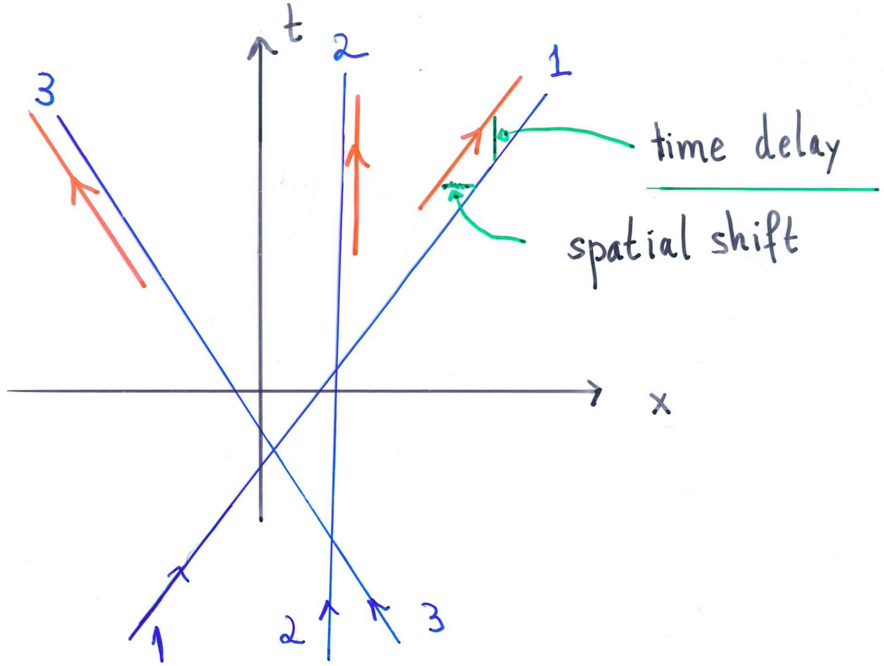
$\Rightarrow$ all integrable models have the "SAME" hydrodynamics

- two particle phase shift

$\phi(v,w)$ : $vt + \mathcal{O}(1)$

additive

NO definite sign



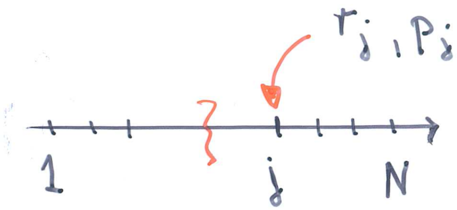

J. Moser 1975 Toda chain

$$\phi(v,w) = 2 \log|v-w|$$

will reappear

# Lax matrix, conservation laws

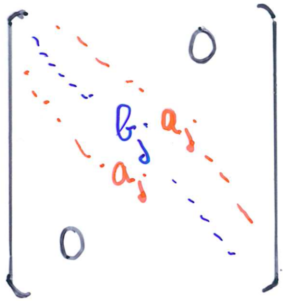
lattice field theory  stretch  $r_j = q_{j+1} - q_j$



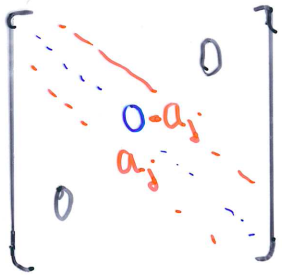

$$H_N = \sum_{j=1}^{N} \left( \tfrac{1}{2} p_j^2 + e^{-r_j} \right)$$

→ Henon, Flaschka 1974  $a_j = \frac{1}{2} e^{-r_j/2}$  $b_j = \frac{1}{2} p_j$

$N \times N$ Lax matrix  $L_N$ tridiagonal, $L_N = (L_N)^T$




$B_N$ matrix  $B_N = -(B_N)^T$




$$\frac{d}{dt} L_N = [B_N, L_N]$$

→ eigenvalues conserved

- conserved fields

$$Q^{(n)} = \mathrm{tr}[(L_N)^n]$$

local

density  $Q_j^{(n)} = ((L_N)^n)_{jj}$

range  $[j-n, j+n]$

$H_N = 2Q^{(2)}$

- conserved currents

$$\frac{d}{dt} Q_j^{(n)} = J_j^{(n)} - J_{j+1}^{(n)}$$

local

$$\rightsquigarrow \quad J_j^{(n)} = \left( L^n L^{\mathrm{off}} \right)_{jj}$$

$$L^{\mathrm{off}} = \begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & 0 & a_j & \ddots \\ & a_j & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

## generalized Gibbs ensemble (GGE)

in addition $\frac{d}{dt} r_j = p_{j+1} - p_j \rightsquigarrow$ stretch $\sum_{j=1}^{N} r_j$ is conserved

Gibbs

$$\frac{1}{Z_N} \prod_{j=1}^{N} dr_j \, dp_j \, e^{-P r_j} \, e^{-\mathrm{tr}[V(L_N)]}$$

pressure $P$ $\rightarrow$ ~~~~ $\leftarrow$

$$V(x) = \sum_{n=1}^{\bar{n}} \mu_n x^n, \qquad \bar{n} \text{ even}, \ \mu_{\bar{n}} > 0 \rightsquigarrow Z_N < \infty$$

thermal: $V(x) = x^2$

- $\mathrm{tr}[V(L_N)]$ is of finite range $\Rightarrow$ transfer matrix

Toda free energy:

$$F_{\mathrm{Toda}}(P, V) = \lim_{N \to \infty} -\frac{1}{N} \log Z_N(P, V)$$

¿ more general $V$ ?

GGE average        fields

$\nu = \langle r_0 \rangle_{P,V}$ , $\langle (L^n)_{00} \rangle_{P,V} \cong \frac{1}{N} \langle \mathrm{tr}[(L_N)^n] \rangle_{P,V}$    n-th moment of DOS

$$\langle f(L)_{00} \rangle_{P,V} = \int dv\, \rho_Q(v) f(v)$$

average spectral measure for $e_0$ of Lax matrix under GGE

currents $$\langle (f(L) L^{\mathrm{off}})_{00} \rangle_{P,V} = \int dv\, \rho_J(v) f(v)$$

- Euler hydrodynamics

  slow variation    $\nu(x,t)$ , $\rho_Q(v; x,t)$ , $\rho_J(v; x,t)$    label

$$\rightsquigarrow \quad \partial_t \nu - \partial_x 2 \int dv\, \rho_Q(v) v = 0 \qquad Q_0^{(1)}$$

$$\partial_t \rho_Q + \partial_x \rho_J = 0$$

REQUIRED    GGE average $\langle Q^{(n)} \rangle$ and $\langle J^{(n)} \rangle$ as functional of $\rho_Q(v)$, $\nu$

fixed $P, V$

$\rightsquigarrow$ evolution equation for local DOS

DOS

• uses Dumitriu, Edelman 2002

$\beta$- ensembles of RMT

HS 2019

Doyon 2019

Bulchandani, Cao, Moore 2019

Theorem: $V$, $P > 0$

$$\mathcal{F}^{MF}(\rho) = \int dx\, V(x)\rho(x) - \int dx \int dy\, \log|x-y|\, \rho(x)\rho(y) + \int dx\, \rho(x) \log \rho(x)$$

minimize $\rho \geq 0$, $\int dx\, \rho(x) = P$ unique $\rho^*(x; P)$

Then Lax DOS under GGE

$$\rho_Q(v) = \partial_P \rho^*(v; P)$$

---

exact solution for $V(x) = x^2$

$P \to 0$ $\rho^*(v, P)$ Gauss

$P \to \infty$ $\rho^*(v, P)$ Wigner semicircle

Opper 1985

Allez, Bouchaud, Guionnet 2012

# Dyson's Brownian motion   mean field   Cépa, Lépingle 1997

$$dx_j(t) = -V'(x_j)\,dt + \frac{P}{N}\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{1}{x_i - x_j}\,dt + \sqrt{2}\,db_j(t) \qquad j = 1,\dots,N$$

- empirical measure $\frac{1}{N}\sum_{j=1}^{N} f(x_j(t)) = \int \rho_N(dx,t)\, f(x)$
- $\rho_N(dx,t) \to \rho(x,t)\,dx$

$$\partial_t \rho = \partial_x \left( V'(x) - P \int dy \frac{1}{x-y}\rho(y) + \partial_x \right) \rho \qquad \text{stationary } \rho_{stat}$$

- stationary measure

$$\frac{1}{Z}\prod_{j=1}^{N} dx_j\, e^{-V(x_j)}\, e^{\frac{P}{N}\sum_{i\neq j=1}^{N} \log|x_i - x_j|} = \mu_N^{MF}$$

$$\lim_{N\to\infty} \mu_N^{MF}\big|_{[1,\dots,m]} = (\rho_{stat})^{\otimes m} \qquad \Big\| \qquad P\,\rho_{stat} = \rho^*(P)$$

## collision rate assumption

What is $\rho_J$? Cao, Bulchandani, HS. 2019

$$\rho_J(v) = \frac{1}{\nu}\, v^{\text{eff}}(v)\, \rho_Q(v)$$

$$v^{\text{eff}}(v) = v + 2\int dw\, \underbrace{\log|v-w|}_{\text{phase shift}}\, \frac{1}{\nu}\, \rho_Q(w)\left(v^{\text{eff}}(w) - v^{\text{eff}}(v)\right)$$

### numerics

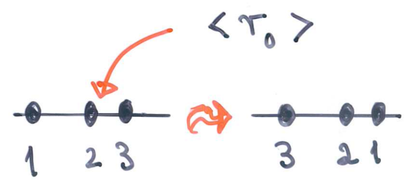
- sample Lax matrix under GGE $\rightsquigarrow$ $\rho_Q$, $\rho_J$ ( • • • )
- compute $\rho_{stat}$ from nonlinear Fokker Planck

  $\rightsquigarrow$ solve integral equations ( —— )

  $\partial_P \Rightarrow$ integral operator

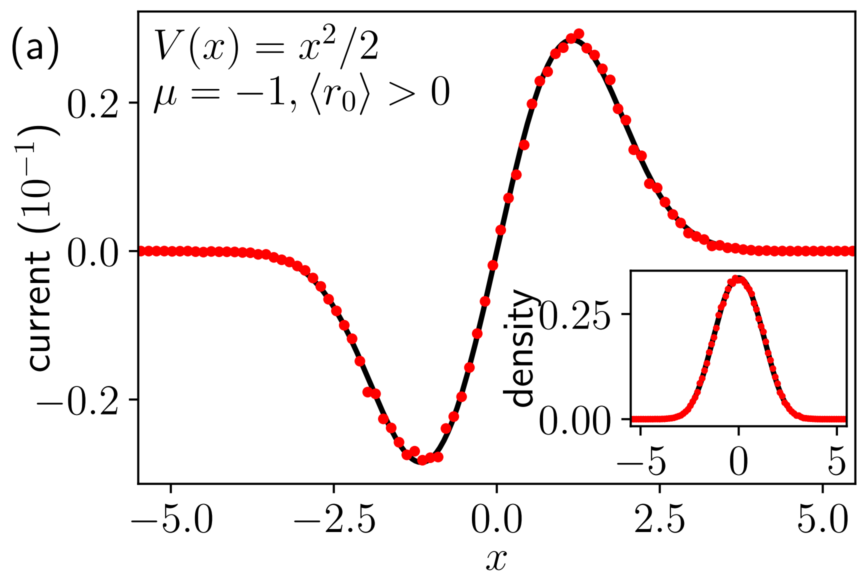
---

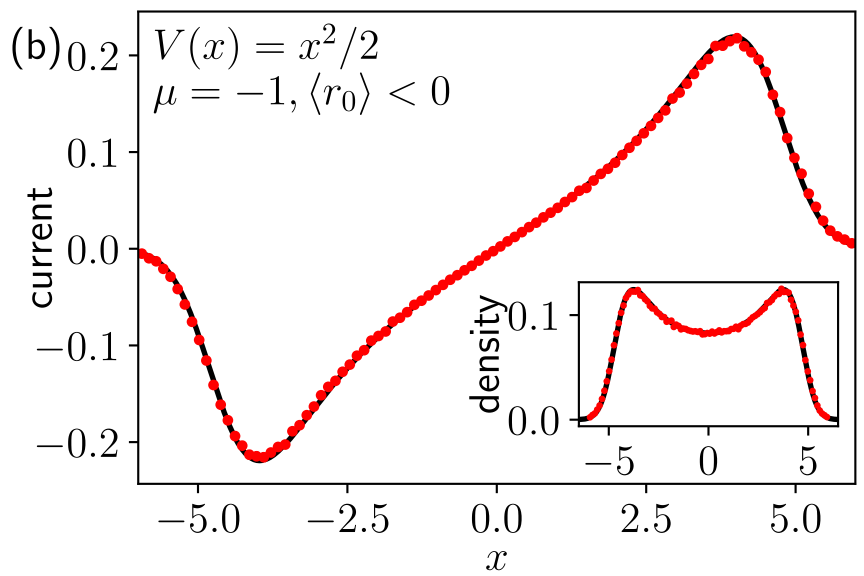
$V(x) = \frac{1}{2}x^2$ (thermal) $\langle r_0 \rangle = 7.64,\ 0,\ -1.28$



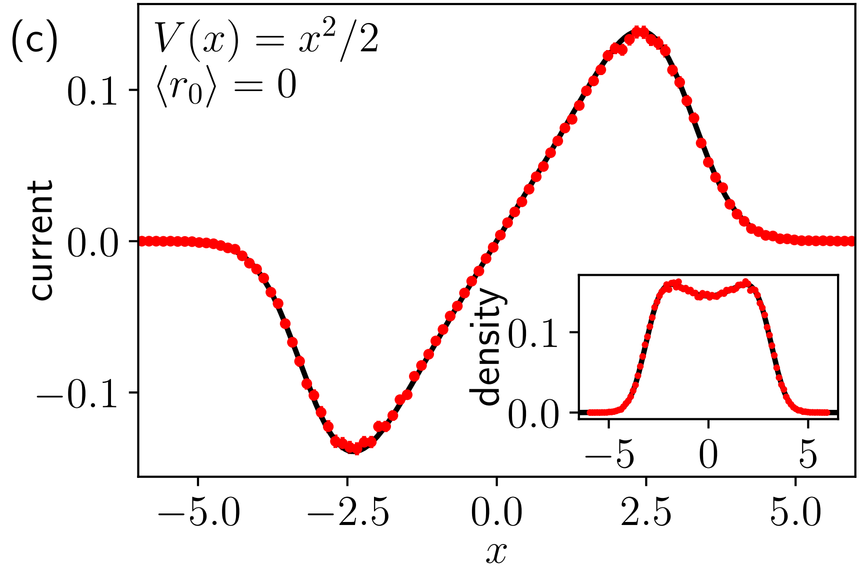

---

$V(x) = -x^2 + x^4$, $P = 2$

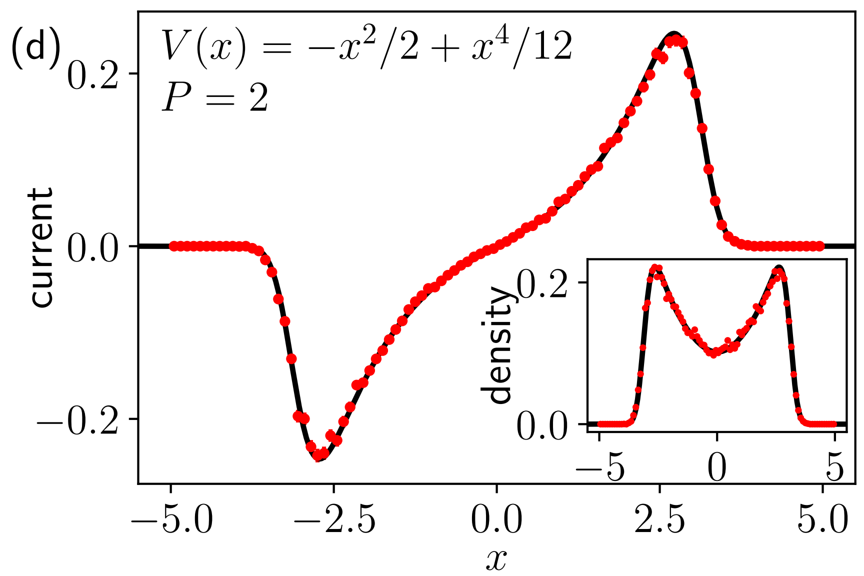

(a)
$V(x) = x^2/2$
$\mu = -1, \langle r_0 \rangle > 0$
current $(10^{-1})$
0.2
0.0
−0.2
−5.0
−2.5
0.0
2.5
5.0
$x$
density
0.25
0.00
−5
0
5



(b)
$V(x) = x^2/2$
$\mu = -1, \langle r_0 \rangle < 0$
current
0.2
0.1
0.0
−0.1
−0.2
−5.0
−2.5
0.0
2.5
5.0
$x$
density
0.1
0.0
−5
0
5



(c)
$V(x) = x^2/2$
$\langle r_0 \rangle = 0$
current
0.1
0.0
−0.1
−5.0
−2.5
0.0
2.5
5.0
$x$
density
0.1
0.0
−5
0
5



(d)
$V(x) = -x^2/2 + x^4/12$
$P = 2$
current
0.2
0.0
−0.2
−5.0
−2.5
0.0
2.5
5.0
$x$
density
0.2
0.0
−5
0
5

# conclusions / future

- general scheme

  how general?

  specifics are phase shift

  classical / quantum

- collision rate assumption

  see Pa. Ferrari 2018

  ball in box

  see hard rods $\phi(v, w) = 1$

  Boldrighini, Dobrushin, Suhov 1983