FM430 - Esercizi proposti (11-6-2019)

1. Consider the 2D Ising model with nearest neighbour interactions in a box Λ of side L with $+$ boundary conditions and zero external magnetic field and the corresponding truncated two point function,

$$f^+_{\beta,\Lambda}(x,y) := \langle \sigma_x \sigma_y \rangle^+_{\beta,0,\Lambda} - \langle \sigma_x \rangle^+_{\beta,0,\Lambda} \langle \sigma_y \rangle^+_{0,\Lambda}.$$

Let also $f^+_\beta(x,y) = \lim_{\Lambda \nearrow \mathbb{Z}^2} f^+_{\beta,\Lambda}(x,y)$.

(a) Write the low-temperature expansion for $f^+_{\beta,\Lambda}(x,y)$. By proceeding as in the Peierls' argument, prove that, for β sufficiently large, its thermodynamic limit decays exponentially to zero at large distances, i.e., prove that $f^+_\beta(x,y) \le C_\beta e^{-\kappa_\beta |x-y|}$ for some $C_\beta, \kappa_\beta > 0$.

(b) Write the high-temperature expansion for $f^+_{\beta,\Lambda}(x,y)$. By proceeding as in the proof that the spontaneous magnetization vanishes at high temperatures, prove that, for β sufficiently small, its thermodynamic limit decays exponentially to zero at large distances, i.e., prove that $f^+_\beta(x,y) \le C'_\beta e^{-\kappa'_\beta |x-y|}$ for some $C'_\beta, \kappa'_\beta > 0$.

2. Recall that, in the Gaussian approximation for the $O(2)$ spin model in dimension $d = 1, 2, 3$, the two point function

$$f^{Gauss}_\beta(x,y) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle e^{i(\theta_x - \theta_y)} \rangle^{Gauss}_{\beta,\Lambda}$$

admits the following explicit expression:

$$f^{Gauss}_\beta(x,y) = \exp\Big\{ -\frac{1}{\beta J} I_d(x-y) \Big\},$$

where

$$I_d(x) = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{1 - \cos(k \cdot x)}{2 \sum_{j=1}^d (1 - \cos k_j)}.$$

Prove that, asymptotically as $|x| \to \infty$,

$$\begin{aligned} I_3(x) &= C_3 - \frac{1}{4\pi |x|} + O(|x|^{-2}), \\ I_2(x) &= \frac{1}{2\pi} \log|x| + O(1), \\ I_1(x) &= \frac{|x|}{2} + O(1), \end{aligned}$$

where, in the first line, $C_3 = \int_{[-\pi,\pi]^3} \frac{dk}{(2\pi)^3} \frac{1}{2\sum_{j=1}^d (1-\cos k_j)}$. [**Hint**: For $d=3$, rewrite

$$C_3 - I_3(x) = \int\int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{\cos(k\cdot x)}{2\sum_{j=1}^d (1-\cos k_j)} [\chi(k) + (1-\chi(k))],$$

where $\chi(k)$ is a smooth version of the characteristic function of $B_\epsilon(0)$, i.e., it is a positive C^∞ radial function, equal to 1 for $|k| \le \epsilon$ and equal to 0 for $|k| \ge 2\epsilon$, for some sufficiently small $\epsilon > 0$. The contribution to $C_3 - I_3(x)$ from the term with $1-\chi(k)$ decays to zero as $|x| \to \infty$ faster than any power, because it is the Fourier transform of a C^∞ function. In the term with $\chi(k)$, we expand in Taylor series the denominator, $2\sum_{j=1}^d (1-\cos k_j) = |k|^2(1+r(k))$, where $r(k)$ is of order k^2. Correspondingly we rewrite $C_3 - I_3(x) = \int_{|k|\le 2\epsilon} \frac{dk}{(2\pi)^d} \frac{\cos(k\cdot x)}{k^2}$ plus a remainder, the remainder going to zero as $|x|^{-2}$ for $|x|$ large. Finally, the term we are left with can be computed by passing to spherical coordinates, and shown to be equal to $\frac{1}{4\pi|x|} + O(|x|^{-2})$. A similar decomposition of the Fourier integral defining $I_1(x)$ and $I_2(x)$ can be used in $d = 1, 2$: rewrite the Fourier integral as the sum of two terms, one proportional to $\chi(k)$ and one proportional to $1-\chi(k)$. The term with $1-\chi(k)$ is uniformly bounded as $|x| \to \infty$. The term with $\chi(k)$ can be further decomposed, by expanding the denominator in Taylor series, the contribution from the remainder being uniformly bounded as $|x| \to \infty$. We are left with $\int_{|k|\le 2\epsilon} (1-\cos(k\cdot x))\chi(k)/k^2$, which can be computed explicitly (in $d = 1$, it is equal to $\int_{\mathbb{R}} \frac{dk}{2\pi}(1-\cos(k\cdot x))\chi(k)/k^2 + O(1)$, and the first term can be computed using the residues' theorem; in $d = 2$, pass to radial coordinates and perform the integral over the angles explicitly, ...)]

3. Consider the spin-spin two point function for the $O(N)$ model in $d = 2$ with periodic boundary conditions, $\langle \vec{S}_x \cdot \vec{S}_y \rangle^{per}_{\beta,\Lambda}$, with $\vec{S}_x \in \mathbb{R}^N$ and $|\vec{S}_x| = 1$. In class, we discussed the McBryan-Spencer argument, leading to a polynomially-decaying upper bound for the $N = 2$ case (rotator, or XY, model). Extend the argument to cover the $N = 3$ case (Heisenberg model); more precisely, prove that there exists $C > 0$ such that, for $N = 3$, any $\epsilon > 0$ and β sufficiently large,

$$\big|\langle \vec{S}_x \cdot \vec{S}_y \rangle_\beta\big| \le \frac{C}{|x-y|^{(1-\epsilon)/(2\pi\beta J)}},$$

where $\langle\cdot\rangle_\beta$ indicates any infinite volume Gibbs measure obtained as a limit of $\langle\cdot\rangle^{per}_{\beta,\Lambda_n}$ for $\Lambda_n \nearrow \mathbb{Z}^2$. [**Hint.** By symmetry, $\langle \vec{S}_x \cdot \vec{S}_y \rangle^{per}_{\beta,\Lambda} = \frac{3}{2}\sum_{i=1}^2 \langle S^i_x\, S^i_y \rangle^{per}_{\beta,\Lambda}$. Using spherical coordinates,

$$(S^1_x, S^2_x, S^3_x) = (\sin\theta_x \cos\phi_x, \sin\theta_x \sin\phi_x, \cos\theta_x), \quad \theta_x \in [0,\pi],\ \phi_x \in [0, 2\pi),$$

this can be rewritten as $\frac{3}{2}\langle \sin\theta_x \sin\theta_y e^{i(\phi_x - \phi_y)}\rangle^{per}_{\beta,\Lambda}$. After having represented the Hamiltonian in terms of the angles θ_x, ϕ_x, perform the complex deformation $\phi_x \to \phi_x + i\alpha_x$ at θ_x fixed, with α_x the same that was used for $N = 2$ and check that the upper bound on the spin-spin two point function obtained in this way is the same as for $N = 2$, up to an overall factor $3/2$.]

4. Use reflection positivity to prove that the ferromagnetic 1D Ising model with $J(x-y) = J|x-y|^{-\alpha}$ and $\alpha \in (1,2)$, has a phase transition, i.e.,

for β large enough and $h = 0$, it admits at least two distinct infinite volume Gibbs states. For this purpose, consider the model in a box Λ of side $2L$, $\Lambda = \{-L+1, \dots, 0, 1, \dots, L\}$, zero external field and periodic boundary conditions; let $H^{per}_{0,\Lambda}(\sigma)$ be the corresponding Hamiltonian,

$$H^{per}_{0,\Lambda}(\sigma) = \frac{1}{4} \sum_{x,y\in\Lambda} (\sigma_x - \sigma_y)^2 J_{per}(x-y),$$

with $J_{per}(x) = J \sum_{n\in\mathbb{Z}} |x + 2nL|^{-\alpha}$. Consider the two point function $\langle \sigma_x \sigma_y \rangle_\beta$, where $\langle \cdot \rangle_\beta$ indicates any infinite volume Gibbs measure obtained as a limit of $\langle \cdot \rangle^{per}_{\beta,\Lambda_n}$ for $\Lambda_n \nearrow \mathbb{Z}$, and prove that there exists $C > 0$ such that $\langle \sigma_x \sigma_y \rangle_\beta \geq 1 - C/\beta$, uniformly in x, y, which implies the existence of a phase transition for $\beta > C$. For this purpose, proceed as follows.

(a) Show that the Hamiltonian is reflection positive with respect to the reflection $r\sigma_x = \sigma_{1-x}$; i.e., it can be written as $H^{per}_{0,\Lambda}(\sigma) = A_L(\sigma) + A_R(\sigma) - \int_0^\infty d\mu\, C_\mu(\sigma)\, rC_\mu(\sigma)$, where: $A_L(\sigma)$ only depends on the spins in $\Lambda_L = \{-L+1, \dots, 0\}$, $A_R(\sigma)$ only depends on the spins in $\Lambda_R = \{1, \dots, L\}$, $A_L(\sigma) = rA_R(\sigma)$, $C_\mu(\sigma)$ only depends on the spins in Λ_R and admits the following explicit representation:

$$C_\mu(\sigma) = \sqrt{\frac{J}{\Gamma(\alpha)}} \sum_{x=1}^{L} \sigma_x \big[e^{-\mu(x-\frac{1}{2})} + 2 \sum_{n\geq 1} e^{-\mu 2nL} \cosh\big(\mu(x - \tfrac{1}{2})\big)\big],$$

where $\Gamma(\alpha)$ is Euler's gamma function. [**Hint.** In order to obtain the representation above, including the formula for $C_\mu(\sigma)$, consider the expression for the left-right interaction, $-J \sum_{y\in\Lambda_L} \sum_{x\in\Lambda_R} \sum_{n\in\mathbb{Z}} \sigma_x \sigma_y |y - x + 2nL|^{-\alpha}$, and rewrite $|y - x + 2nL|^{-\alpha}$ by using

$$|z|^{-\alpha} = (1/\Gamma(\alpha)) \int_0^\infty d\mu\, \mu^{\alpha-1} e^{-\mu|z|}.$$

Then, perform the summations over x, y, n explicitly.]

(b) Using reflection positivity, derive the Gaussian domination bound, $Z_{\beta,\Lambda}(h) \leq Z_{\beta,\Lambda}(0)$, where $h \in \mathbb{R}^{2L}$ and

$$Z_{\beta,\Lambda}(h) = \sum_{\sigma\in\Omega_\Lambda} \exp\Big\{ -\frac{1}{4} \sum_{x,y\in\Lambda} (\sigma_x - \sigma_y + h_x - h_y)^2 J_{per}(x-y) \Big\}.$$

From this derive the infrared bound, i.e., letting

$$\hat{\sigma}_k = |\Lambda|^{-1/2} \sum_{x\in\Lambda} \sigma_x e^{-ikx}$$

for $k \in \frac{\pi}{L}\mathbb{Z}$ mod 2π, then, for any $k \neq 0$ mod 2π,

$$\langle |\hat{\sigma}_k|^2 \rangle_{\beta,\Lambda}^{per} \leq \frac{1}{2\beta J\, E_k},$$

where $E_k = \sum_{n\geq 1} \frac{1-\cos(kn)}{n^\alpha}$.

(c) Show that, for $\alpha \in (1,2)$, $\int_0^\pi dk(1/E_k) < +\infty$. From this, conclude that $\langle \sigma_x \sigma_y \rangle_\beta \geq 1 - C/\beta$, as desired. [**Hint.** In order to prove the integrability of $1/E_k$, note that $1-\cos(kn) \geq \frac{2}{\pi^2}(kn)^2$, for $|kn| \leq \pi$, so that $E_k \geq \frac{2k^2}{\pi^2}\sum_{n=1}^{P_k} n^{2-\alpha}$, where $P_k = [\pi/|k|]$. This can be further bounded from below by (const.)$|k|^{\alpha-1}$, from which $\int_0^\pi dk(1/E_k) \leq$ (const.) $\int_0^\pi k^{1-\alpha} dk < +\infty$.]