FM430 - Esercizi proposti (17-4-2019)

1. Consider the 1D Ising model with nearest neighbour interactions in a box of side L with periodic boundary conditions, $H^{per}_{h,L}(\sigma)$. Compute

$$f_{\beta,h}(x-y) := \lim_{L\to\infty} \langle \sigma_x \sigma_y \rangle^{per}_{\beta,h,L},$$

where $\langle \cdot \rangle^{per}_{\beta,h,L}$ is the grand-canonical average with respect to $H^{per}_{h,L}(\sigma)$. Show that $f_{\beta,h}(x)$ converges exponentially to $[m(\beta,h)]^2$ as $|x| \to \infty$ (here $m(\beta,h) = \beta^{-1}\partial_h \psi(\beta,h)$ is the average magnetization), namely that

$$f_{\beta,h}(x) - [m(\beta,h)]^2 \sim e^{-\kappa|x|}.$$

Compute the rate κ.

2. Consider the Curie-Weiss model $H^{CW}_{h,N}(\sigma)$ with $h \geq 0$. Let $m^*(\beta,h)$ be the largest solution of $m = \tanh \beta(Jm+h)$.

   (a) Prove that, if $h > 0$, or if $h = 0$ and $\beta < \beta_c$, then, for all $\epsilon > 0$,

   $$\lim_{N\to\infty} \mathbb{P}_{\beta,h,N}(|m - m^*(\beta,h)| > N^{-1/2+\epsilon}) = 0,$$

   where $\mathbb{P}_{\beta,h,N}$ is the probability with respect to the grand-canonical distribution associated with $H^{CW}_{h,N}(\sigma)$ at inverse temperature β.

   (b) Similarly, prove that, if $h = 0$ and $\beta > \beta_c$, then, for all $\epsilon > 0$,

   $$\lim_{N\to\infty} \mathbb{P}_{\beta,0,N}\big(\big||m| - m^*(\beta,0)\big| > N^{-1/2+\epsilon}\big) = 0.$$

   (c) Finally, prove that, if $h = 0$ and $\beta = \beta_c$, then, for all $\epsilon > 0$,

   $$\lim_{N\to\infty} \mathbb{P}_{\beta_c,0,N}(|m| > N^{-1/4+\epsilon}) = 0.$$

3. Prove that the pressure of the Curie-Weiss model, $\psi^{CW}(\beta,h)$ with $h \geq 0$ can be written as follows:

$$\psi^{CW}(\beta,h) = \Big[-\beta J m^2/2 + \log\cosh[\beta(Jm+h)] + \log 2\Big]\Big|_{m=m^*(\beta,h)},$$

where $m^*(\beta,h)$ is the largest solution of $m = \tanh[\beta(Jm+h)]$.

4. Let $\psi(\beta,h)$ be the pressure of the d-dimensional Ising model with ferromagnetic interaction $J(x-y)$, and $\psi^{CW}(\beta,h)$ the pressure of the Curie-Weiss model with coupling $J = \hat{J}_0 := \sum_{x\neq 0} J(x)$. Prove that, if $h \geq 0$, then $\psi(\beta,h) \geq \psi^{CW}(\beta,h)$, via the following steps.

(a) Prove that, for any $m \in \mathbb{R}$, $H^{per}_{h,\Lambda}(\sigma)$ can be re-written as $H^{per}_{h,\Lambda}(\sigma) = H^{per,0}_{h,\Lambda}(\sigma) + H^{per,1}_{h,\Lambda}(\sigma)$, where

$$H^{per,0}_{h,\Lambda}(\sigma) = \frac{J}{2} m^2 |\Lambda| - (Jm + h) \sum_{x\in\Lambda} \sigma_x,$$
$$H^{per,1}_{h,\Lambda}(\sigma) = -\frac{1}{2} \sum_{\substack{x,y\in\Lambda: \\ x\neq y}} J(x-y)(\sigma_x - m)(\sigma_y - m).$$

(b) Using the previous rewriting, recognize that the grand-canonical partition function of $H^{per}_{h,\Lambda}(\sigma)$ can be rewritten as

$$Z^{per}_{\beta,h,\Lambda} = Z^{per,0}_{\beta,h,\Lambda} \langle e^{-\beta H^{per,1}_{h,\Lambda}} \rangle^{per,0}_{\beta,h,\Lambda},$$

where $Z^{per,0}_{\beta,h,\Lambda}$ is the grand-canonical partition function of $H^{per,0}_{h,\Lambda}(\sigma)$ and $\langle (\cdot) \rangle^{per,0}_{\beta,h,\Lambda}$ is the average with respect to the grand-canonical distribution associated with $H^{per,0}_{h,\Lambda}(\sigma)$ at inverse temperature β.

(c) Use Jensen's inequality (stating that $\int \mu(dx) f(x) \geq f(\int \mu(dx) x)$ for any probability measure μ and any **convex** function f), to conclude that

$$Z^{per}_{\beta,h,\Lambda} \geq Z^{per,0}_{\beta,h,\Lambda} e^{-\beta \langle H^{per,1}_{h,\Lambda} \rangle^{per,0}_{\beta,h,\Lambda}}.$$

Compute the right side explicitly as a function of m. Show that, by fixing m to be the largest solution of $m = \tanh[\beta(Jm+h)]$, one obtains, after having taken the thermodynamic limit,

$$\psi(\beta,h) \geq \psi^{CW}(\beta,h), \quad \forall h \geq 0,$$

as desired.

5. Let $\alpha_{CW}, \alpha'_{CW}, b_{CW}, \gamma_{CW}, \delta_{CW}$ be the critical exponents of the Curie-Weiss model with coupling constant J, defined by

$$\begin{aligned}
c(\beta,0) &\sim (\beta_c - \beta)^{-\alpha_{CW}}, && \text{as} \quad \beta \to \beta_c^-, \\
c(\beta,0) &\sim (\beta - \beta_c)^{-\alpha'_{CW}}, && \text{as} \quad \beta \to \beta_c^+, \\
m^*(\beta) &\sim (\beta - \beta_c)^{b_{CW}}, && \text{as} \quad \beta \to \beta_c^+, \\
m^*(\beta_c, h) &\sim h^{1/\delta_{CW}}, && \text{as} \quad h \to 0^+, \\
\chi(\beta) &\sim (\beta - \beta_c)^{-\gamma_{CW}}, && \text{as} \quad \beta \to \beta_c^-,
\end{aligned}$$

where: $\beta_c = J^{-1}$; $c(\beta,h) = -\beta^2 \partial^2_\beta \psi^{CW}(\beta,h)$ is the specific heat; for $h \neq 0$, $m^*(\beta,h) = \beta^{-1}\partial_h \psi^{CW}(\beta,h)$ is the average magnetization, and $m^*(\beta) = m^*(\beta,0^+)$ is the spontaneous magnetization; for $\beta < \beta_c$, $\chi(\beta) = \partial_h m^*(\beta,0)$ is the magnetic susceptibility. Prove that $\alpha_{CW} = \alpha'_{CW} = 0$, $b_{CW} = \frac{1}{2}$, $\gamma_{CW} = 1$ and $\delta_{CW} = 3$.