FM430 - Esercizi proposti (21-4-2021)

1. Consider the 1D Ising model with nearest neighbour interactions in a box of side L with periodic boundary conditions, $H^{per}_{h,L}(\sigma)$. Compute

$$f_{\beta,h}(x-y) := \lim_{L\to\infty} \langle \sigma_x \sigma_y \rangle^{per}_{\beta,h,L},$$

where $\langle \cdot \rangle^{per}_{\beta,h,L}$ is the grand-canonical average with respect to $H^{per}_{h,L}(\sigma)$. Show that $f_{\beta,h}(x)$ converges exponentially to $[m(\beta,h)]^2$ as $|x| \to \infty$ (here $m(\beta,h) = \beta^{-1}\partial_h \psi(\beta,h)$ is the average magnetization), namely that

$$f_{\beta,h}(x) - [m(\beta,h)]^2 \sim C e^{-\kappa|x|},$$

for suitable $C, \kappa > 0$. In particular, compute the rate $\kappa = \kappa(\beta,h)$.

2. **[The solution of the Curie-Weiss model via the Hubbard-Stratonovich transformation]** Using the Gaussian identity

$$e^{\frac{\alpha}{2}x^2} = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} e^{-\frac{m^2}{2\alpha}+mx}\, dm,$$

valid for all $\alpha > 0$, prove that the grandcanonical partition function of the Curie-Weiss model with $J > 0$ can be rewritten as:

$$\begin{aligned} Z^{CW}_{\beta,h,N} &= \sqrt{\frac{N\beta J}{2\pi}} \sum_{\sigma_1,\ldots,\sigma_N=\pm} \int_{-\infty}^{+\infty} e^{-N\beta J m^2/2 + \beta(Jm+h)\sum_{i=1}^N \sigma_i}\, dm \\ &= \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} e^{-N\beta J m^2/2} (2\cosh(\beta(Jm+h)))^N\, dm \end{aligned}$$

from which it follows that

$$\lim_{N\to\infty} \frac{1}{N} \log Z^{CW}_{\beta,h,N} = \max_{m\in\mathbb{R}} \Big[-\beta J m^2/2 + \log\cosh[\beta(Jm+h)] + \log 2 \Big].$$

Verify explicitly that this expression is the same as the expression $\psi^{CW}(\beta,h) = \max_{-1\le m\le 1}\{\beta h m - \beta f^{CW}(\beta,m)\}$ computed in class.

3. Let $\psi(\beta,h)$ be the pressure of the d-dimensional Ising model with ferromagnetic interaction $J(x-y)$, and $\psi^{CW}(\beta,h)$ the pressure of the Curie-Weiss model with coupling $J = \hat{J}_0 := \sum_{x\neq 0} J(x)$. Prove that, if $h \geq 0$, then $\psi(\beta,h) \geq \psi^{CW}(\beta,h)$, via the following steps.

(a) Prove that, for any $m \in \mathbb{R}$, $H^{per}_{h,\Lambda}(\sigma)$ can be re-written as $H^{per}_{h,\Lambda}(\sigma) = H^{per,0}_{h,\Lambda}(\sigma) + H^{per,1}_{h,\Lambda}(\sigma)$, where

$$H^{per,0}_{h,\Lambda}(\sigma) = \frac{J}{2}m^2|\Lambda| - (Jm+h)\sum_{x\in\Lambda}\sigma_x,$$

$$H^{per,1}_{h,\Lambda}(\sigma) = -\frac{1}{2}\sum_{\substack{x,y\in\Lambda:\\ x\neq y}} J(x-y)(\sigma_x - m)(\sigma_y - m).$$

(b) Using the previous rewriting, recognize that the grand-canonical partition function of $H^{per}_{h,\Lambda}(\sigma)$ can be rewritten as

$$Z^{per}_{\beta,h,\Lambda} = Z^{per,0}_{\beta,h,\Lambda}\langle e^{-\beta H^{per,1}_{h,\Lambda}}\rangle^{per,0}_{\beta,h,\Lambda},$$

where $Z^{per,0}_{\beta,h,\Lambda}$ is the grand-canonical partition function of $H^{per,0}_{h,\Lambda}(\sigma)$ and $\langle(\cdot)\rangle^{per,0}_{\beta,h,\Lambda}$ is the average with respect to the grand-canonical distribution associated with $H^{per,0}_{h,\Lambda}(\sigma)$ at inverse temperature β.

(c) Use Jensen's inequality (stating that $\int \mu(dx)f(x) \geq f(\int \mu(dx)x)$ for any probability measure μ and any **convex** function f), to conclude that

$$Z^{per}_{\beta,h,\Lambda} \geq Z^{per,0}_{\beta,h,\Lambda}e^{-\beta\langle H^{per,1}_{h,\Lambda}\rangle^{per,0}_{\beta,h,\Lambda}}.$$

Compute the right side explicitly as a function of m. Show that, by fixing m to be the largest solution of $m = \tanh[\beta(Jm+h)]$, one obtains, after having taken the thermodynamic limit,

$$\psi(\beta,h) \geq \psi^{CW}(\beta,h), \quad \forall h \geq 0,$$

as desired.

4. Consider the Curie-Weiss model $H^{CW}_{h,N}(\sigma)$ with $h \geq 0$. Let $m^*(\beta,h)$ be the largest solution of $m = \tanh\beta(Jm+h)$.

(a) Prove that, if $h > 0$, or if $h = 0$ and $\beta < \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N\to\infty}\mathbb{P}_{\beta,h,N}(|m - m^*(\beta,h)| > N^{-1/2+\epsilon}) = 0,$$

where $\mathbb{P}_{\beta,h,N}$ is the probability with respect to the grand-canonical distribution associated with $H^{CW}_{h,N}(\sigma)$ at inverse temperature β.

(b) Similarly, prove that, if $h = 0$ and $\beta > \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N\to\infty}\mathbb{P}_{\beta,0,N}\big(\big||m| - m^*(\beta,0)\big| > N^{-1/2+\epsilon}\big) = 0.$$

(c) Finally, prove that, if $h = 0$ and $\beta = \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N\to\infty}\mathbb{P}_{\beta_c,0,N}(|m| > N^{-1/4+\epsilon}) = 0.$$