FM430 - Esercizi proposti (4-6-2021)

1. Consider the 2D ferromagnetic nearest neighbor Ising model in a square box $\Lambda = B(L) \subset \mathbb{Z}^2$ of side L centered at 0 (i.e., we take $\Lambda = \{[-L/2]+1, \dots, [L/2]\}^2$) with $h = 0$ and $+$ boundary conditions. Recall that the low temperature expansion for the log of its partition function is:

$$\log Z^+_{\beta,0,\Lambda} = 2\beta J L(L+1) + \sum_{n\geq 1} \frac{1}{n!} \sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda} \zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma}),$$

where $\mathcal{C}_\Lambda$ is the set of allowed Peierls contours in Λ, $\mathcal{C}^n_\Lambda$ is the set of n-ples $(\gamma_1, \dots, \gamma_n)$ with elements γ_i in $\mathcal{C}_\Lambda$. Moreover, for any $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathcal{C}^n_\Lambda$, $\zeta(\boldsymbol{\gamma}) := \prod_{i=1}^n \zeta(\gamma_i)$, with $\zeta(\gamma_i) = e^{-2\beta J|\gamma_i|}$, and $\varphi_c(\boldsymbol{\gamma}) = \varphi_c(\gamma_1, \dots, \gamma_n)$ is the Ursell's (or Mayer's) function.

   (a) Prove that, for β large enough, letting $I(\gamma)$ be the *interior* of γ (i.e., the set of points of $\mathbb{Z}^2$ enclosed by the Peierls contour γ), the pressure $\psi(\beta, 0)$ is analytic in β and can be written as

$$\psi(\beta,0) = 2\beta J + \sum_{n\geq 1}\frac{1}{n!} \sum_{\substack{\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_n)\in\mathcal{C}^n_{\mathbb{Z}^2}:\\ I(\gamma_1)\ni 0}} \frac{\zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma})}{|I(\gamma_1)|}. \qquad (1)$$

   (b) Use (1) to prove that, for β large enough,

$$\psi(\beta, 0) = 2\beta J + e^{-4\beta J} + 2e^{-6\beta J} + O(e^{-8\beta J}).$$

2. In the context of the previous problem, show that

$$\langle\sigma_0\rangle^+_{\beta,0,\Lambda} = \frac{\sum_{n\geq 0}(1/n!)\sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda} \zeta_0(\boldsymbol{\gamma})\varphi(\boldsymbol{\gamma})}{\sum_{n\geq 0}(1/n!)\sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda} \zeta(\boldsymbol{\gamma})\varphi(\boldsymbol{\gamma})}, \qquad (2)$$

where $\zeta_0(\gamma_1, \dots, \gamma_n) = \prod_{i=1}^n \zeta_0(\gamma_i)$, with $\zeta_0(\gamma) = (-1)^{\mathbb{1}(I(\gamma)\ni 0)} e^{-2\beta J|\gamma|}$, and $\varphi(\gamma_1, \dots, \gamma_n) = \prod_{1\leq i<j\leq n} \mathbb{1}(\gamma_i \sim \gamma_j)$. Writing the right side of (2) as the exp of its log, and rewriting both the log of the numerator and that of the denominator as the sums of their cluster expansions, prove that, for β large enough,

$$\langle\sigma_0\rangle^+_{\beta,0,\Lambda} = \exp\Big\{\sum_{n\geq 1}\frac{1}{n!}\sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda}(\zeta_0(\boldsymbol{\gamma}) - \zeta(\boldsymbol{\gamma}))\varphi_c(\boldsymbol{\gamma})\Big\}, \qquad (3)$$

from which it follows that

$$\begin{aligned} m^*(\beta) := \lim_{L\to\infty}\langle\sigma_0\rangle^+_{\beta,0,\Lambda} &= \exp\Big\{-2\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\boldsymbol{\gamma}\in\mathcal{C}^n_{\mathbb{Z}^2}:\\ N_{\boldsymbol{\gamma}}(0) \text{ is odd}}} \zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma})\Big\} \\ &= 1 - 2e^{-4\beta J} - 8e^{-6\beta J} + O(e^{-8\beta J}). \end{aligned} \qquad (4)$$

where, if $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, we let $N_{\boldsymbol{\gamma}}(x) := \sum_{i=1}^n \mathbb{1}(I(\gamma_i) \ni x)$.

3. In the context of the previous two problems, show that, $\forall x_1, x_2 \in \Lambda$,

$$\langle \sigma_{x_1}\sigma_{x_2}\rangle^+_{\beta,0,\Lambda} = \frac{\sum_{n\geq 0}(1/n!)\sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda}\zeta_{x_1x_2}(\boldsymbol{\gamma})\varphi(\boldsymbol{\gamma})}{\sum_{n\geq 0}(1/n!)\sum_{\boldsymbol{\gamma}\in\mathcal{C}^n_\Lambda}\zeta(\boldsymbol{\gamma})\varphi(\boldsymbol{\gamma})}, \tag{5}$$

where $\zeta_{x_1x_2}(\gamma_1, \ldots, \gamma_n) = \prod_{i=1}^n \zeta_{x_1x_2}(\gamma_i)$, with

$$\zeta_{x_1x_2}(\gamma) = (-1)^{\sum_{i=1}^2 \mathbb{1}(I(\gamma)\ni x_i)} e^{-2\beta J|\gamma|}.$$

Proceeding as in the proof of (3)-(4), prove that, for β large enough, the thermodynamic limit of the *truncated* two-point function,

$$\langle \sigma_{x_1}; \sigma_{x_2}\rangle^{+,T}_{\beta,0} := \langle \sigma_{x_1}\sigma_{x_2}\rangle^+_{\beta,0} - (m^*(\beta))^2,$$

with $\langle \sigma_{x_1}\sigma_{x_2}\rangle^+_{\beta,0} = \lim_{L\to\infty}\langle \sigma_{x_1}\sigma_{x_2}\rangle^+_{\beta,0,\Lambda}$, can be written as:

$$\langle \sigma_{x_1}; \sigma_{x_2}\rangle^{+,T}_{\beta,0} = e^{-2\sum_{n\geq 1}\frac{1}{n!}\sum^{(1\vee 2)}_{\boldsymbol{\gamma}}\zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma})} - e^{-2\sum_{n\geq 1}\frac{1}{n!}\sum_{i=1}^2\sum^{(i)}_{\boldsymbol{\gamma}}\zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma})}, \tag{6}$$

where $\sum^{(1\vee 2)}_{\boldsymbol{\gamma}}$ (resp. $\sum^{(i)}_{\boldsymbol{\gamma}}$) indicates the sum over $\boldsymbol{\gamma} \in \mathcal{C}^n_{\mathbb{Z}^2}$ with the constraint that $\sum_{i=1}^2 N_{\boldsymbol{\gamma}}(x_i)$ is odd (resp. that $N_{\boldsymbol{\gamma}}(x_i)$ is odd). Note that (6) can be equivalently rewritten as

$$\langle \sigma_{x_1}; \sigma_{x_2}\rangle^{+,T}_{\beta,0} = \langle \sigma_{x_1}\sigma_{x_2}\rangle^+_{\beta,0}\Big(1 - e^{-4\sum_{n\geq 1}\frac{1}{n!}\sum^{(1\wedge 2)}_{\boldsymbol{\gamma}}\zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma})}\Big), \tag{7}$$

where $\sum^{(1\wedge 2)}_{\boldsymbol{\gamma}}$ indicates the sum over $\boldsymbol{\gamma} \in \mathcal{C}^n_{\mathbb{Z}^2}$ with the constraint that both $N_{\boldsymbol{\gamma}}(x_1)$ and $N_{\boldsymbol{\gamma}}(x_2)$ are odd. From this, conclude that, for β large enough,

$$0 \leq \langle \sigma_{x_1}; \sigma_{x_2}\rangle^{+,T}_{\beta,0} \leq 4\sum_{n\geq 1}\frac{1}{n!}\sum^{(1\wedge 2)}_{\boldsymbol{\gamma}}\zeta(\boldsymbol{\gamma})\varphi_c(\boldsymbol{\gamma}) \leq Ce^{-\kappa|x_1-x_2|},$$

for suitable $C, \kappa > 0$. Find a lower bound on κ as a function of β.

4. Consider the 2D ferromagnetic nearest neighbor Ising model in a square box $\Lambda = B(L) \subset \mathbb{Z}^2$ of side L centered at 0, with $h = 0$ and *periodic* boundary conditions. Let $\mathcal{E}^{per}_\Lambda$ be the set of its nearest neighbor edges, and, for any $b \equiv \{x, y\} \in \mathcal{E}^{per}_\Lambda$, let $\tilde{\sigma}_b = \sigma_x\sigma_y$ be the 'bond spin' or 'energy observable'. Derive a Grassmann representation for its multipoint 'energy correlations'

$$\langle \tilde{\sigma}_{b_1}\cdots\tilde{\sigma}_{b_n}\rangle^{per}_{\beta,0,\Lambda},$$

where $b_1, \ldots, b_n$ are n distinct elements of $\mathcal{E}^{per}_\Lambda$, via the following steps:

(a) Note that the partition function of the model with bond-dependent couplings,

$$Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) := \sum_{\sigma\in\Omega_\Lambda} \exp\Big\{\sum_{b\in\mathcal{E}^{per}_\Lambda} (\beta J + \epsilon_b)\tilde{\sigma}_b\Big\},$$

with $\Omega_\Lambda = \{\pm 1\}^\Lambda$, admits a Grassmann representation analogous to the one for $\boldsymbol{\epsilon} = \mathbf{0}$, namely:

$$Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) = (-2)^{L^2}\Big(\prod_{b\in\mathcal{E}^{per}_\Lambda} \cosh(\beta J + \epsilon_b)\Big) \sum_{\boldsymbol{\theta}\in\{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}), \qquad (8)$$

with $c_{+,-} = c_{-,+} = c_{-,-} = -c_{+,+} = 1/2$ and, letting $\Phi = \{\bar{H}_x, H_x, \bar{V}_x, V_x\}_{x\in\Lambda}$ be a collection of $4L^2$ Grassmann variables,

$$Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}) = \int D\Phi\ e^{S^{\boldsymbol{\theta}}_{\boldsymbol{\epsilon}}(\Phi)},$$

where, letting $t(\epsilon_b) \equiv \tanh(\beta J + \epsilon_b)$, $H_{(L+1,x_2)} \equiv \theta_1 H_{(1,x_2)}$, and $V_{(x_1,L+1)} \equiv \theta_2 V_{(x_1,1)}$,

$$\begin{aligned} S^{\boldsymbol{\theta}}_{\boldsymbol{\epsilon}}(\Phi) = \sum_{x\in\Lambda} \Big[& t(\epsilon_{\{x,x+\hat{e}_1\}})\bar{H}_x H_{x+\hat{e}_1} + t(\epsilon_{\{x,x+\hat{e}_2\}})\bar{V}_x V_{x+\hat{e}_2} \\ & + \bar{H}_x H_x + \bar{V}_x V_x + i\bar{V}_x \bar{H}_x + iH_x V_x + H_x \bar{V}_x + V_x \bar{H}_x\Big]. \end{aligned} \qquad (9)$$

(b) Use the fact that

$$\langle \tilde{\sigma}_{b_1}\cdots\tilde{\sigma}_{b_n}\rangle^{per}_{\beta,0,\Lambda} = \frac{1}{Z^{per}_{\beta,0,\Lambda}} \frac{\partial^n}{\partial\epsilon_{b_1}\cdots\partial\epsilon_{b_n}} Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon})\Big|_{\boldsymbol{\epsilon}=\mathbf{0}}$$

to conclude, via (8), that, for any collection of distinct bonds $b_1, \ldots, b_n$,

$$\langle \prod_{i=1}^n \tilde{\sigma}_{b_i}\rangle^{per}_{\beta,0,\Lambda} = \frac{\sum_{\boldsymbol{\theta}\in\{\pm\}^2} c_{\boldsymbol{\theta}} \int D\Phi\, e^{S_{\mathbf{0}}^{\boldsymbol{\theta}(\Phi)}(\Phi)} \prod_{i=1}^n (t + (1-t^2)E_{b_i})}{\sum_{\boldsymbol{\theta}\in\{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0})},$$

where $t = \tanh\beta J$ and, for $b = \{x, x+\hat{e}_1\}$, $E_b = \bar{H}_x H_{x+\hat{e}_1}$, while, for $b = \{x, x+\hat{e}_2\}$, $E_b = \bar{V}_x V_{x+\hat{e}_2}$.

5. In the context of the previous problem, use the Grassmann representation of the energy correlations to compute the asymptotics of the

truncated energy-energy correlation at the critical point. Namely, consider, e.g., two horizontal bonds, $b_1 = \{x, x + \hat{e}_1\}$ and $b_2 = \{y, y + \hat{e}_1\}$, with $x \neq y$, let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0,\Lambda} := \langle \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda} - \langle \tilde{\sigma}_{b_1} \rangle^{per}_{\beta,0,\Lambda} \langle \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda}$$

be the truncated energy-energy correlation, and let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0} = \lim_{L \to \infty} \langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0,\Lambda}$$

be its thermodynamic limit.

(a) Starting from the Grassmann representation derived in item (b) of the previous problem, prove that, for all $\beta > 0$, the thermodynamic limit of the truncated energy-energy correlation can be written as

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0} = (1-t^2)\Big[\langle \bar{H}_x H_{y+\hat{e}_1} \rangle \, \langle H_{x+\hat{e}_1} \bar{H}_y \rangle - \langle \bar{H}_x \bar{H}_y \rangle \, \langle H_x H_y \rangle \Big],$$

where, denoting $\Phi_{1,x} \equiv \bar{H}_x$ and $\Phi_{2,x} \equiv H_x$,

$$\langle \Phi_{a,x} \Phi_{b,y} \rangle := - \iint_{[0,2\pi)^2} \frac{d^2k}{(2\pi)^2} (M_k^{-1})_{a,b} e^{-ik(x-y)},$$

for any $a, b \in \{1, 2\}$, where

$$M_k = \begin{pmatrix} 0 & 1 + te^{-ik_1} & -i & -1 \\ -(1 + te^{ik_1}) & 0 & 1 & i \\ i & -1 & 0 & 1 + te^{-ik_2} \\ 1 & -i & -(1 + te^{ik_2}) & 0 \end{pmatrix}$$

(b) Using the explicit formula derived in the previous item, prove that, at $\beta = \beta_c \Leftrightarrow t = \sqrt{2} - 1$,

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta_c,0} \sim \frac{1}{\pi^2} \frac{1}{|x - y|^2},$$

asymptotically as $|x - y| \to \infty$.