MS410 - Esercizi proposti (10-6-2022)

1. [**Existence of non-translationally invariant Gibbs states in the 3D Ising model**] Consider the 3D nearest neighbor ferromagnetic Ising model with coupling $J > 0$ and zero external magnetic field in a cubic box $\Lambda = \Lambda_L$ of side L centered at the origin, and denote by $\mu^{\rm Dob}_{\beta,0,\Lambda}$ the corresponding finite volume Gibbs measure with Dobrushin boundary conditions $\tau = \tau_{\rm Dob}$, where
$$(\tau_{\rm Dob})_x = \begin{cases} +1 & \text{if } x \in \Lambda_0 \cup \Lambda_> \\ -1 & \text{if } x \in \Lambda_< \end{cases}$$
and we defined $\Lambda_> := \{x = (x_1, x_2, x_3) \in \Lambda : x_3 > 0\}$, $\Lambda_0 := \{x = (x_1, x_2, x_3) \in \Lambda : x_3 = 0\}$ and $\Lambda_< := \{x = (x_1, x_2, x_3) \in \Lambda : x_3 < 0\}$.

   - Prove that
   $$\langle \sigma_0 \rangle^{\rm Dob}_{\beta,0,\Lambda} \ge \langle \sigma_0 \rangle^{+}_{\beta,0,\Lambda_0} \tag{1}$$
   where the average in the left side is with respect to (w.r.t.) $\mu^{\rm Dob}_{\beta,0,\Lambda}$ and the one in the right side is w.r.t. the finite volume Gibbs measure $\mu^{+}_{\beta,0,\Lambda_0}$ of the *two-dimensional* nearest neighbor Ising model with coupling $J > 0$ and zero external magnetic field in the square box Λ_0 of side L. [**Hint:** Denoting by $\sigma \in \{\pm 1\}^{\Lambda}$ an Ising spin random field distributed w.r.t. $\mu^{\rm Dob}_{\beta,0,\Lambda}$ and by $\omega \in \{\pm 1\}^{\Lambda_0}$ another Ising spin random field distributed w.r.t. $\mu^{+}_{\beta,0,\Lambda_0}$, let
   $$s_x = \begin{cases} \frac{1}{2}(\sigma_x + \sigma_{rx}) & \text{if } x \in \Lambda_> \\ \frac{1}{2}(\sigma_x + \omega_x) & \text{if } x \in \Lambda_0 \end{cases} \qquad t_x = \begin{cases} \frac{1}{2}(\sigma_x - \sigma_{rx}) & \text{if } x \in \Lambda_> \\ \frac{1}{2}(\sigma_x - \omega_x) & \text{if } x \in \Lambda_0 \end{cases}$$
   where, for any $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$, $rx := (x_1, x_2, -x_3)$. Note that, for any $x \in \Lambda_0 \cup \Lambda_>$, $s_x, t_x \in \{-1, 0, 1\}$ and $s_x = 0 \Leftrightarrow t_x \neq 0$. Observe that (1) is equivalent to $\langle\!\langle t_0 \rangle\!\rangle_\beta \ge 0$, where $\langle\!\langle \cdot \rangle\!\rangle_\beta$ is the average w.r.t. the product measure $\mu^{\rm Dob}_{\beta,0,\Lambda} \otimes \mu^{+}_{\beta,0,\Lambda_0}$. In order to prove that $\langle\!\langle t_0 \rangle\!\rangle_\beta \ge 0$, expand the numerator in the definition of $\langle\!\langle t_0 \rangle\!\rangle_\beta$ according to the realization of $A = \{x \in \Lambda_0 \cup \Lambda_> : s_x = 0\}$ and observe that, once A is fixed, there remains exactly one nontrivial Ising random variable (with values ± 1) at each vertex; verify that you can then apply the usual GKS inequalities to show that each term of the sum is non-negative.]

   - As a corollary of the previous item, show that for $\beta > \beta_c(2)$, where $\beta_c(2) = J^{-1}\text{arctanh}(\sqrt{2} - 1)$ is the inverse critical temperature of the 2D nearest neighbor Ising model, $\langle \sigma_0 \rangle^{\rm Dob}_{\beta,0} > 0 > \langle \sigma_{(0,0,-1)} \rangle^{\rm Dob}_{\beta,0}$, where $\langle \cdot \rangle^{\rm Dob}_{\beta,0} = \lim_{n\to\infty} \langle \cdot \rangle^{\rm Dob}_{\beta,0,\Lambda_{L_n}}$ is an infinite volume Gibbs state obtained along an appropriate increasing sequence of boxes $\{\Lambda_{L_n}\}_{n\in\mathbb{N}}$. In particular, $\langle \cdot \rangle^{\rm Dob}_{\beta,0}$ is not translationally invariant.

2. **[Logarithmic divergence of the specific heat of the 2D Ising model]** Recall that the free energy of the nearest neighbor 2D Ising model with coupling $J > 0$, inverse temperature $\beta > 0$ and no external magnetic field, $h = 0$, is given by Onsager's formula:

$$\psi(\beta, 0) = \log(2\cosh^2(\beta J)) - \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\log\varphi(k_1, k_2)$$

where, letting $t := \tanh(\beta J)$, we denoted $\varphi(k_1, k_2) := (t^2 + 2t - 1)^2 + 2t(1-t^2)(2-\cos k_1 - \cos k_2)$. Prove that $\psi(\beta, 0)$ is analytic in β for $\beta \neq \beta_c := J^{-1}\text{arctanh}(\sqrt{2}-1)$; moreover, prove that $\psi(\beta, 0)$ is continuously differentiable at β_c, while it is not twice differentiable at that point; in particular, show that the second derivative of $\psi(\beta, 0)$ with respect to β diverges logarithmically as $\beta \to \beta_c$.

3. **[Grassmann representation of the energy correlations of the 2D Ising model]** Consider the 2D ferromagnetic nearest neighbor Ising model in a square box $\Lambda = \Lambda_L \subset \mathbb{Z}^2$ of side L centered at 0, with $h = 0$ and *periodic* boundary conditions. Let $\mathcal{E}_\Lambda^{per}$ be the set of its nearest neighbor edges, and, for any $b \equiv \{x, y\} \in \mathcal{E}_\Lambda^{per}$, let $\tilde{\sigma}_b = \sigma_x\sigma_y$ be the 'bond spin' or 'energy observable'. Derive a Grassmann representation for its multipoint 'energy correlations'

$$\langle\tilde{\sigma}_{b_1}\cdots\tilde{\sigma}_{b_n}\rangle_{\beta,0,\Lambda}^{per},$$

where $b_1, \ldots, b_n$ are n distinct elements of $\mathcal{E}_\Lambda^{per}$, via the following steps:

   (a) Note that the partition function of the model with bond-dependent couplings,

$$Z_{\beta,0,\Lambda}^{per}(\boldsymbol{\epsilon}) := \sum_{\sigma\in\Omega_\Lambda}\exp\Big\{\sum_{b\in\mathcal{E}_\Lambda^{per}}(\beta J + \epsilon_b)\tilde{\sigma}_b\Big\},$$

   with $\Omega_\Lambda = \{\pm 1\}^\Lambda$, admits a Grassmann representation analogous to the one for $\boldsymbol{\epsilon} = \mathbf{0}$, namely:

$$Z_{\beta,0,\Lambda}^{per}(\boldsymbol{\epsilon}) = (-2)^{L^2}\Big(\prod_{b\in\mathcal{E}_\Lambda^{per}}\cosh(\beta J + \epsilon_b)\Big)\sum_{\boldsymbol{\theta}\in\{\pm\}^2}c_{\boldsymbol{\theta}}Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}), \qquad (2)$$

   with $c_{+,-} = c_{-,+} = c_{-,-} = -c_{+,+} = 1/2$ and, letting $\Phi = \{\bar{H}_x, H_x, \bar{V}_x, V_x\}_{x\in\Lambda}$ be a collection of $4L^2$ Grassmann variables,

$$Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}) = \int D\Phi\; e^{S_{\boldsymbol{\epsilon}}^{\boldsymbol{\theta}}(\Phi)},$$

where, letting $t(\epsilon_b) \equiv \tanh(\beta J + \epsilon_b)$, $H_{(L+1,x_2)} \equiv \theta_1 H_{(1,x_2)}$, and $V_{(x_1,L+1)} \equiv \theta_2 V_{(x_1,1)}$,

$$S_{\boldsymbol{\epsilon}}^{\boldsymbol{\theta}}(\Phi) = \sum_{x\in\Lambda} \Big[t(\epsilon_{\{x,x+\hat{e}_1\}})\bar{H}_x H_{x+\hat{e}_1} + t(\epsilon_{\{x,x+\hat{e}_2\}})\bar{V}_x V_{x+\hat{e}_2} \\ + \bar{H}_x H_x + \bar{V}_x V_x + i\bar{V}_x \bar{H}_x + iH_x V_x + H_x \bar{V}_x + V_x \bar{H}_x \Big]. \tag{3}$$

(b) Use the fact that

$$\langle \tilde{\sigma}_{b_1} \cdots \tilde{\sigma}_{b_n} \rangle^{per}_{\beta,0,\Lambda} = \frac{1}{Z^{per}_{\beta,0,\Lambda}} \frac{\partial^n}{\partial \epsilon_{b_1} \cdots \partial \epsilon_{b_n}} Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) \Big|_{\boldsymbol{\epsilon}=\mathbf{0}}$$

to conclude, via (2), that, for any collection of distinct bonds $b_1, \ldots, b_n$,

$$\langle \prod_{i=1}^n \tilde{\sigma}_{b_i} \rangle^{per}_{\beta,0,\Lambda} = \frac{\sum_{\boldsymbol{\theta}\in\{\pm\}^2} c_{\boldsymbol{\theta}} \int D\Phi \, e^{S_{\mathbf{0}}^{\boldsymbol{\theta}(\Phi)}(\Phi)} \prod_{i=1}^n (t + (1-t^2)E_{b_i})}{\sum_{\boldsymbol{\theta}\in\{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0})},$$

where $t = \tanh \beta J$ and, for $b = \{x, x+\hat{e}_1\}$, $E_b = \bar{H}_x H_{x+\hat{e}_1}$, while, for $b = \{x, x+\hat{e}_2\}$, $E_b = \bar{V}_x V_{x+\hat{e}_2}$.

4. [**Asymptotic behavior of the energy-energy correlation of the 2D Ising model at** β_c] In the context of the previous problem, use the Grassmann representation of the energy correlations to compute the asymptotics of the truncated energy-energy correlation at the critical point. Namely, consider, e.g., two horizontal bonds, $b_1 = \{x, x+\hat{e}_1\}$ and $b_2 = \{y, y+\hat{e}_1\}$, with $x \neq y$, let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0,\Lambda} := \langle \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda} - \langle \tilde{\sigma}_{b_1} \rangle^{per}_{\beta,0,\Lambda} \langle \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda}$$

be the truncated energy-energy correlation, and let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0} = \lim_{L\to\infty} \langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0,\Lambda}$$

be its thermodynamic limit.

(a) Starting from the Grassmann representation derived in item (b) of the previous problem, prove that, for all $\beta > 0$, the thermodynamic limit of the truncated energy-energy correlation can be written as

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0} = (1-t^2)^2 \Big[\langle \bar{H}_x H_{y+\hat{e}_1} \rangle \, \langle H_{x+\hat{e}_1} \bar{H}_y \rangle - \langle \bar{H}_x \bar{H}_y \rangle \, \langle H_x H_y \rangle \Big],$$

where, denoting $\Phi_{1,x} \equiv \bar{H}_x$ and $\Phi_{2,x} \equiv H_x$,

$$\langle \Phi_{a,x} \Phi_{b,y} \rangle := - \iint\limits_{[0,2\pi)^2} \frac{d^2k}{(2\pi)^2} (M_k^{-1})_{a,b} e^{-ik(x-y)},$$

for any $a, b \in \{1, 2\}$, where

$$M_k = \begin{pmatrix} 0 & 1 + te^{-ik_1} & -i & -1 \\ -(1 + te^{ik_1}) & 0 & 1 & i \\ i & -1 & 0 & 1 + te^{-ik_2} \\ 1 & -i & -(1 + te^{ik_2}) & 0 \end{pmatrix}$$

(b) Using the explicit formula derived in the previous item, prove that, at $\beta = \beta_c \Leftrightarrow t = \sqrt{2} - 1$,

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta_c,0} \sim \frac{1}{\pi^2} \frac{1}{|x-y|^2},$$

asymptotically as $|x - y| \to \infty$.