MS410 - Esercizi proposti (17-6-2025)

1. [**Existence of non-translationally invariant Gibbs states in the 3D Ising model**] Consider the 3D nearest neighbor ferromagnetic Ising model with coupling $J > 0$ and zero external magnetic field in a cubic box $\Lambda = \Lambda_{2L+1} := \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : -L \le x_i \le L,\ i = 1, 2, 3\}$ of side $2L + 1$ centered at the origin, and denote by $\mu^{\mathrm{Dob}}_{\beta,0,\Lambda}$ the corresponding finite volume Gibbs measure with Dobrushin boundary conditions $\tau = \tau_{\mathrm{Dob}}$, where

$$(\tau_{\mathrm{Dob}})_x = \begin{cases} +1 & \text{if } x_3 \ge 0, \\ -1 & \text{if } x_3 < 0. \end{cases}$$

   - Prove that
$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\beta,0,\Lambda} \ge \langle \sigma_0 \rangle^{+}_{\beta,0,\Lambda_0} \tag{1}$$
where the average in the left side is with respect to (w.r.t.) $\mu^{\mathrm{Dob}}_{\beta,0,\Lambda}$ and the one in the right side is w.r.t. the finite volume Gibbs measure $\mu^{+}_{\beta,0,\Lambda_0}$ of the *two-dimensional* nearest neighbor Ising model with coupling $J > 0$ and zero external magnetic field in the square box $\Lambda_0 := \{(x_1, x_2) \in \mathbb{Z}^2 : -L \le x_i \le L,\ i = 1, 2\}$. [**Hint:** Denoting by $\sigma \in \{\pm 1\}^{\Lambda}$ an Ising spin random field distributed w.r.t. $\mu^{\mathrm{Dob}}_{\beta,0,\Lambda}$ and by $\omega \in \{\pm 1\}^{\Lambda_0}$ another Ising spin random field distributed w.r.t. $\mu^{+}_{\beta,0,\Lambda_0}$, let
$$s_x = \begin{cases} \frac{1}{2}(\sigma_x + \sigma_{rx}) & \text{if } x \in \Lambda_> \\ \frac{1}{2}(\sigma_x + \omega_x) & \text{if } x \in \Lambda_0 \end{cases} \qquad t_x = \begin{cases} \frac{1}{2}(\sigma_x - \sigma_{rx}) & \text{if } x \in \Lambda_> \\ \frac{1}{2}(\sigma_x - \omega_x) & \text{if } x \in \Lambda_0 \end{cases}$$
where we denoted $\Lambda_> := \{x = (x_1, x_2, x_3) \in \Lambda : x_3 > 0\}$ and, for any $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$, $rx := (x_1, x_2, -x_3)$. Note that, for any $x \in \Lambda_0 \cup \Lambda_>$, $s_x, t_x \in \{-1, 0, 1\}$ and $s_x = 0 \Leftrightarrow t_x \ne 0$. Observe that (1) is equivalent to $\langle\!\langle t_0 \rangle\!\rangle_\beta \ge 0$, where $\langle\!\langle \cdot \rangle\!\rangle_\beta$ is the average w.r.t. the product measure $\mu^{\mathrm{Dob}}_{\beta,0,\Lambda} \otimes \mu^{+}_{\beta,0,\Lambda_0}$. In order to prove that $\langle\!\langle t_0 \rangle\!\rangle_\beta \ge 0$, expand the numerator in the definition of $\langle\!\langle t_0 \rangle\!\rangle_\beta$ according to the realization of $A = \{x \in \Lambda_0 \cup \Lambda_> : s_x = 0\}$ and observe that, once A is fixed, there remains exactly one nontrivial Ising random variable (with values ± 1) at each vertex; verify that you can then apply the usual GKS inequalities to show that each term of the sum is non-negative.]
   - As a corollary of the previous item, show that for $\beta > \beta_c(2)$, where $\beta_c(2) = J^{-1}\mathrm{arctanh}(\sqrt{2} - 1)$ is the inverse critical temperature of the 2D nearest neighbor Ising model, $\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\beta,0} > 0 > \langle \sigma_{(0,0,-1)} \rangle^{\mathrm{Dob}}_{\beta,0}$, where $\langle \cdot \rangle^{\mathrm{Dob}}_{\beta,0} = \lim_{n\to\infty} \langle \cdot \rangle^{\mathrm{Dob}}_{\beta,0,B_{2L_n+1}}$ is an infinite volume Gibbs

state obtained along an appropriate increasing sequence of boxes $\{B_{2L_n+1}\}_{n\in\mathbb{N}}$, with

$$B_{2L+1} := \{(x_1,x_2,x_3)\in\mathbb{Z}^3 : -L \le x_1, x_2 \le L,\ -L \le x_3 \le L-1\}.$$

In particular, $\langle\cdot\rangle^{\mathrm{Dob}}_{\beta,0}$ is not translationally invariant. [**Hint:** By symmetry, $\langle\sigma_0\rangle^{\mathrm{Dob}}_{\beta,0,B_{2L+1}} = -\langle\sigma_{(0,0,-1)}\rangle^{\mathrm{Dob}}_{\beta,0,B_{2L+1}}$. Moreover, by FKG,

$$\langle\sigma_0\rangle^{\mathrm{Dob}}_{\beta,0,B_{2L+1}} \ge \langle\sigma_0\rangle^{\mathrm{Dob}}_{\beta,0,\Lambda_{2L+1}}.$$

Combining this with the result of the previous item and sending $L = L_n \to \infty$ implies the desired claim.]

2. [**Logarithmic divergence of the specific heat of the 2D Ising model**] Recall that the free energy of the nearest neighbor 2D Ising model with coupling $J>0$, inverse temperature $\beta>0$ and no external magnetic field, $h=0$, is given by Onsager's formula:

$$\psi(\beta,0) = \log(2\cosh^2(\beta J)) - \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\int_{-\pi}^{\pi}\frac{dk_2}{2\pi}\log\varphi(k_1,k_2)$$

where, letting $t := \tanh(\beta J)$, we denoted $\varphi(k_1,k_2) := (t^2+2t-1)^2 + 2t(1-t^2)(2-\cos k_1-\cos k_2)$. Prove that $\psi(\beta,0)$ is real-analytic in β for $\beta\neq\beta_c := J^{-1}\mathrm{arctanh}(\sqrt{2}-1)$; moreover, prove that $\psi(\beta,0)$ is continuously differentiable at β_c, while it is not twice differentiable at that point; in particular, show that the second derivative of $\psi(\beta,0)$ with respect to β diverges logarithmically as $\beta\to\beta_c$.

3. [**Grassmann representation of the energy correlations of the 2D Ising model**] Consider the 2D ferromagnetic nearest neighbor Ising model in a square box $\Lambda=\Lambda_L\subset\mathbb{Z}^2$ of side L, with $h=0$ and *periodic* boundary conditions. Let $\mathcal{E}^{per}_\Lambda$ be the set of its nearest neighbor edges, and, for any $b\equiv\{x,y\}\in\mathcal{E}^{per}_\Lambda$, let $\tilde\sigma_b=\sigma_x\sigma_y$ be the 'bond spin' or 'energy observable'. Derive a Grassmann representation for its multipoint 'energy correlations'

$$\langle\tilde\sigma_{b_1}\cdots\tilde\sigma_{b_n}\rangle^{per}_{\beta,0,\Lambda},$$

where $b_1,\dots,b_n$ are n distinct elements of $\mathcal{E}^{per}_\Lambda$, via the following steps:

(a) Note that the partition function of the model with bond-dependent couplings,

$$Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) := \sum_{\sigma\in\Omega_\Lambda}\exp\Big\{\sum_{b\in\mathcal{E}^{per}_\Lambda}(\beta J+\epsilon_b)\tilde\sigma_b\Big\},$$

with $\Omega_\Lambda = \{\pm 1\}^\Lambda$, admits a Grassmann representation analogous to the one for $\boldsymbol{\epsilon} = \mathbf{0}$, namely:

$$Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) = (-2)^{L^2} \Big(\prod_{b \in \mathcal{E}^{per}_\Lambda} \cosh(\beta J + \epsilon_b) \Big) \sum_{\boldsymbol{\theta} \in \{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}), \qquad (2)$$

with $c_{+,-} = c_{-,+} = c_{-,-} = -c_{+,+} = 1/2$ and, letting $\Phi = \{\bar{H}_x, H_x, \bar{V}_x, V_x\}_{x \in \Lambda}$ be a collection of $4L^2$ Grassmann variables,

$$Z^{\boldsymbol{\theta}}(\boldsymbol{\epsilon}) = \int D\Phi \; e^{S^{\boldsymbol{\theta}}_{\boldsymbol{\epsilon}}(\Phi)},$$

where, letting $t(\epsilon_b) \equiv \tanh(\beta J + \epsilon_b)$, $H_{(L+1,x_2)} \equiv \theta_1 H_{(1,x_2)}$, and $V_{(x_1,L+1)} \equiv \theta_2 V_{(x_1,1)}$,

$$\begin{aligned} S^{\boldsymbol{\theta}}_{\boldsymbol{\epsilon}}(\Phi) = \sum_{x \in \Lambda} \Big[& t(\epsilon_{\{x,x+\hat{e}_1\}}) \bar{H}_x H_{x+\hat{e}_1} + t(\epsilon_{\{x,x+\hat{e}_2\}}) \bar{V}_x V_{x+\hat{e}_2} \\ & + \bar{H}_x H_x + \bar{V}_x V_x + i \bar{V}_x \bar{H}_x + i H_x V_x + H_x \bar{V}_x + V_x \bar{H}_x \Big]. \end{aligned} \qquad (3)$$

(b) Use the fact that

$$\langle \tilde{\sigma}_{b_1} \cdots \tilde{\sigma}_{b_n} \rangle^{per}_{\beta,0,\Lambda} = \frac{1}{Z^{per}_{\beta,0,\Lambda}} \frac{\partial^n}{\partial \epsilon_{b_1} \cdots \partial \epsilon_{b_n}} Z^{per}_{\beta,0,\Lambda}(\boldsymbol{\epsilon}) \Big|_{\boldsymbol{\epsilon}=\mathbf{0}}$$

to conclude, via (2), that, for any collection of distinct bonds $b_1, \ldots, b_n$,

$$\langle \prod_{i=1}^n \tilde{\sigma}_{b_i} \rangle^{per}_{\beta,0,\Lambda} = \frac{\sum_{\boldsymbol{\theta} \in \{\pm\}^2} c_{\boldsymbol{\theta}} \int D\Phi \, e^{S^{\boldsymbol{\theta}}_{\mathbf{0}}(\Phi)} \prod_{i=1}^n (t + (1-t^2) E_{b_i})}{\sum_{\boldsymbol{\theta} \in \{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0})}, \qquad (4)$$

where $t = \tanh \beta J$ and, for $b = \{x, x+\hat{e}_1\}$, $E_b = \bar{H}_x H_{x+\hat{e}_1}$, while, for $b = \{x, x+\hat{e}_2\}$, $E_b = \bar{V}_x V_{x+\hat{e}_2}$.

4. [**Asymptotic behavior of the energy-energy correlation of the 2D Ising model at** β_c] In the context of the previous problem, use the Grassmann representation of the energy correlations to compute the asymptotics of the truncated energy-energy correlation at the critical point. Namely, consider, e.g., two horizontal bonds, $b_1 = \{x, x+\hat{e}_1\}$ and $b_2 = \{y, y+\hat{e}_1\}$, with $x \neq y$, let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda} := \langle \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda} - \langle \tilde{\sigma}_{b_1} \rangle^{per}_{\beta,0,\Lambda} \langle \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda}$$

be the truncated energy-energy correlation, and let

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0} = \lim_{L \to \infty} \langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda}$$

be its thermodynamic limit.

(a) Starting from the Grassmann representation derived in item (b) of the previous problem, prove that, for all $\beta > 0$, the thermodynamic limit of the truncated energy-energy correlation can be written as

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0} = (1-t^2)^2 \Big[\langle \bar{H}_x H_{y+\hat{e}_1} \rangle \, \langle H_{x+\hat{e}_1} \bar{H}_y \rangle - \langle \bar{H}_x \bar{H}_y \rangle \, \langle H_x H_y \rangle \Big], \tag{5}$$

where, denoting $\Phi_{1,x} \equiv \bar{H}_x$ and $\Phi_{2,x} \equiv H_x$,

$$\langle \Phi_{a,x} \Phi_{b,y} \rangle := - \iint_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} (M_k^{-1})_{a,b} e^{-ik\cdot(x-y)},$$

for any $a, b \in \{1, 2\}$, where

$$M_k = \begin{pmatrix} 0 & 1+te^{-ik_1} & -i & -1 \\ -(1+te^{ik_1}) & 0 & 1 & i \\ i & -1 & 0 & 1+te^{-ik_2} \\ 1 & -i & -(1+te^{ik_2}) & 0 \end{pmatrix}$$

[**Hint:** Recall that, as discussed in class, $Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0}) > 0$ for all $\boldsymbol{\theta} \in \{(+,-),(-,+),(-,-)\}$, and $Z^{(+,+)}_{\beta,0,\Lambda} \neq 0$ for all $\beta \neq \beta_c$. Therefore, for any $\beta \neq \beta_c$, we can divide and multiply by $Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0})$ the term with label $\boldsymbol{\theta}$ appearing in the numerator in the right hand side of (4) (for $n = 1, 2$), thus finding:

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta,0,\Lambda} = (1-t^2)^2 \sum_{\boldsymbol{\theta} \in \{\pm\}^2} c_{\boldsymbol{\theta}} Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0}) \langle E_{b_1}; E_{b_2} \rangle_{\boldsymbol{\theta}},$$

where $\langle E_{b_1}; E_{b_2} \rangle_{\boldsymbol{\theta}} = \langle E_{b_1} E_{b_2} \rangle_{\boldsymbol{\theta}} - \langle E_{b_1} \rangle_{\boldsymbol{\theta}} \langle E_{b_2} \rangle_{\boldsymbol{\theta}}$ and

$$\langle A \rangle_{\boldsymbol{\theta}} := \frac{1}{Z^{\boldsymbol{\theta}}_{\beta,0,\Lambda}(\mathbf{0})} \int D\Phi e^{S^{\boldsymbol{\theta}}_{\mathbf{0}}(\Phi)} A(\Phi).$$

Now, for any $\beta \neq \beta_c$ and any $\boldsymbol{\theta}$, the following 'fermionic Wick rule' for the expectations of monomials of order 4 holds, namely:

$$\begin{aligned} \langle \bar{H}_x H_{x+\hat{e}_1} \bar{H}_y H_{y+\hat{e}_1} \rangle_{\boldsymbol{\theta}} &= \langle \bar{H}_x H_{x+\hat{e}_1} \rangle_{\boldsymbol{\theta}} \langle \bar{H}_y H_{y+\hat{e}_1} \rangle_{\boldsymbol{\theta}} \\ &- \langle \bar{H}_x \bar{H}_y \rangle_{\boldsymbol{\theta}} \langle H_{x+\hat{e}_1} H_{y+\hat{e}_1} \rangle_{\boldsymbol{\theta}} + \langle \bar{H}_x H_{y+\hat{e}_1} \rangle_{\boldsymbol{\theta}} \langle H_{x+\hat{e}_1} \bar{H}_y \rangle_{\boldsymbol{\theta}} \end{aligned}$$

Moreover, for any pair of Grassmann variables $\Phi_{a,x}, \Phi_{b,y}$ with $a, b \in \{1, 2\}$, the expectation $\langle \Phi_{a,x} \Phi_{b,y} \rangle_{\boldsymbol{\theta}}$ converges to $\langle \Phi_{a,x} \Phi_{b,y} \rangle$ as $L \to \infty$, uniformly in $\boldsymbol{\theta}$. Therefore, eq.(5) holds for all $\beta \neq \beta_c$. Finally, from the fact that: (1) $\langle \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \rangle^{per,T}_{\beta,0}$ and $\langle \tilde{\sigma}_{b_i} \rangle^{per,T}_{\beta,0}$ are monotone increasing in β (by GKS), and (2) $\langle \Phi_{a,x} \Phi_{b,y} \rangle$, is continuous in $t = \tanh(\beta J)$, a posteriori eq.(5) holds for $\beta = \beta_c$ as well (as one can prove by letting $\beta \to \beta_c^-$ first and then $\beta \to \beta_c^+$).]

(b) Using the explicit formula derived in the previous item, prove that, at $\beta = \beta_c \Leftrightarrow t = \sqrt{2} - 1$,

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta_c, 0} \sim \frac{1}{\pi^2} \frac{1}{|x-y|^2},$$

asymptotically as $|x-y| \to \infty$. [**Hint:** An explicit computation of the right hand side of (5) shows that, at $\beta = \beta_c$,

$$\langle \tilde{\sigma}_{b_1}; \tilde{\sigma}_{b_2} \rangle^{per}_{\beta_c, 0} = -\Bigg(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot(x-y)}}{2 - \cos k_1 - \cos k_2} \sin k_2 \Bigg)^2$$
$$- \Bigg(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot(x-y)}}{2 - \cos k_1 - \cos k_2} \big[1 - e^{ik_1} - (\sqrt{2}-1)(1 - \cos k_2)\big] \Bigg) \cdot$$
$$\cdot \Bigg(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot(x-y)}}{2 - \cos k_1 - \cos k_2} \big[1 - e^{-ik_1} - (\sqrt{2}-1)(1 - \cos k_2)\big] \Bigg).$$

Now, letting $\frac{\sin k_2}{2 - \cos k_1 - \cos k_2} \equiv f_1(k)$ and $\frac{1 - e^{ik_1} - (\sqrt{2}-1)(1-\cos k_2)}{2 - \cos k_1 - \cos k_2} \equiv f_2(k)$, we rewrite

$$\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} f_i(k) = \int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} f_i(k) \big[\chi_\epsilon(k) + (1 - \chi_\epsilon(k))\big],$$

where $\chi_\epsilon(k)$ is a C^∞ compactly supported function, supported in the ball of radius ϵ centered at the origin, and identically equal to 1 on the ball of radius $\epsilon/2$ centered at the origin, with $\epsilon > 0$ sufficiently small. Now, since the Fourier transform of a C^∞ function decays faster than any power at large distances, we have that, for $|x| \geq 1$,

$$\Big| \int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} f_i(k)(1 - \chi_\epsilon(k)) \Big| \leq \frac{C_N}{|x|^N}$$

for all $N \geq 1$ and some $C_N > 0$. Next, in the term whose integrand is proportional to $\chi_\epsilon(k)$, we Taylor expand the numerator and denominator in the definition of $f_i(k)$ around the origin, and rewrite it as:

$$\int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{|k|^2} (l_i(k) + r_i(k)) \chi_\epsilon(k),$$

where $l_1(k) = 2k_2$, $l_2(k) = -2ik_1$ and $r_i(k) = O(|k|^2)$. The term whose integrand is proportional to $r_i(k)$ can be bounded as

$$\Big| \int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{|k|^2} r_i(k) \chi_\epsilon(k) \Big| \leq C \frac{\log|x|}{|x|^2}$$

for all $|x| \geq \epsilon^{-1}$ and a suitable $C > 0$ [To prove it, write $\chi_\epsilon(k) = \chi_{|x|^{-1}}(k) + (\chi_\epsilon(k) - \chi_{|x|^{-1}}(k))$: the term whose integrand is proportional to $\chi_{|x|^{-1}}(k)$ is bounded by (const.)$|x|^{-2}$, while the other by (const.)$\frac{\log|x|}{|x|^2}$, as one can realize by rewriting

$$|x|^2 \int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{|k|^2} r_i(k)(\chi_\epsilon(k) - \chi_{|x|^{-1}}(k)) =$$

$$= \int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} (\partial_{k_1}^2 + \partial_{k_2}^2) \frac{e^{-ik\cdot x}}{|k|^2} r_i(k)(\chi_\epsilon(k) - \chi_{|x|^{-1}}(k))$$

and then integrating twice by parts with respect to k_1 and k_2.] We are finally left with

$$\int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{|k|^2} l_i(k)\chi_\epsilon(k) = \int_{\mathbb{R}^2} d^2z L_i(x-z)\Xi_\epsilon(z) \equiv L_i * \Xi_\epsilon(x),$$

where $\Xi_\epsilon(z) := \int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot z}\chi_\epsilon(k)$ is a function decaying to zero at infinity faster than any power (i.e., for $|z| \geq 1$, $|\Xi_\epsilon(z)| \leq C_N|x|^{-N}$ for all $N \geq 1$) and of total integral 1: $\int_{\mathbb{R}^2} d^2z\, \Xi_\epsilon(z) = 1$; moreover, $L_i(z) := \int\limits_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{|k|^2} l_i(k)$ with $l_1(k) = 2k_2$ and $l_2(k) = -2ik_1$, which must be interpreted as a tempered distribution, as any Fourier transform of L^1_{loc} functions. The explicit expression of $L_i(x)$ can be expressed using the residues theorem and gives:

$$L_1(x) = \frac{-i}{\pi} \frac{x_2}{|x|^2} \quad \text{and} \quad L_2(x) = -\frac{1}{\pi} \frac{x_1}{|x|^2}.$$

Finally, using the properties of $\Xi_\epsilon(z)$ we find that $L_i * \Xi_\epsilon(x) = L_i(x) + R_i(x)$, with $|R_i(x)| \leq C/|x|^2$ for $|x| \geq 1$ and some $C > 0$. In conclusion,

$$\begin{aligned} \langle \tilde\sigma_{b_1}; \tilde\sigma_{b_2} \rangle^{per}_{\beta_c,0} &= -\Big(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(x-y)} f_1(k) \Big)^2 \\ &\quad - \Big(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(x-y)} f_2(k) \Big) \Big(\int\limits_{[-\pi,\pi)^2} \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(x-y)} f_2(-k) \Big) \\ &= -(L_1(x-y))^2 - L_2(x-y)\cdot L_2(y-x) + R(x-y), \end{aligned}$$

where $|R(x)| \leq C\frac{\log|x|}{|x|^3}$ for $|x| \geq 1$. Using the explicit expressions of $L_1(x), L_2(x)$, we get the desired result.]