MS410 - Primo esonero (28-4-2025)

1. Consider the 1D Ising model with nearest neighbour interactions in a box of side L with periodic boundary conditions, $H^{per}_{h,L}(\sigma)$. Compute
$$f_{\beta,h}(x-y) := \lim_{L\to\infty} \langle \sigma_x \sigma_y \rangle^{per}_{\beta,h,L},$$
where $\langle \cdot \rangle^{per}_{\beta,h,L}$ is the grand-canonical average with respect to $H^{per}_{h,L}(\sigma)$. Show that $f_{\beta,h}(x)$ converges exponentially to $[m(\beta,h)]^2$ as $|x| \to \infty$ (here $m(\beta,h) = \beta^{-1}\partial_h \psi(\beta,h)$ is the average magnetization), namely that
$$f_{\beta,h}(x) - [m(\beta,h)]^2 \sim Ce^{-\kappa|x|}, \tag{1}$$
for suitable $C, \kappa > 0$. In particular, compute the rate $\kappa = \kappa(\beta,h)$.

2. Consider the 1D Ising model with even, finite range, interaction $J(x-y)$ of range r (i.e., $J(x) = 0$ for $|x| > r$) in a box of side L with periodic boundary conditions. Prove that the grand-canonical partition function $Z^{per}_{\beta,h,L}$ can be written as $\mathrm{Tr}M^L$ for a suitable $2^r \times 2^r$ transfer matrix M, which in general is not symmetric. Note that the elements of M are non-negative and show that there exists a positive integer p such that M^p has positive elements: find the smallest value of p for which such property holds.

3. [**Perron-Frobenius theorem**] Consider a $n \times n$ real, not necessarily symmetric, matrix A with **positive** elements.
   - Using Brouwer fixed-point theorem (*Every continuous function from a convex compact subset K of a Euclidean space to K itself has a fixed point*) show that A admits a right eigenvector with positive components. [**Hint**: apply Brouwer fixed-point theorem to the map $\mathbf{v} \to [\sum_{i=1}^n (A\mathbf{v})_i]^{-1} A\mathbf{v}$ from the space of vectors $\mathbf{v}$ with non-negative components such that $\sum_{i=1}^n (\mathbf{v})_i = 1$ to itself.] Denote by $\mathbf{v}^*$ such eigenvector, normalized so that $\sum_i (\mathbf{v}^*)_i = 1$, and by $\lambda^* > 0$ its eigenvalue. Moreover, show that A admits a left eigenvector $\mathbf{u}^*$ with positive components and the same eigenvalue λ^*, normalized so that $\mathbf{u}^* \cdot \mathbf{v}^* = 1$. [**Hint**: apply again Brouwer fixed-point theorem to the map $\mathbf{u} \to (\lambda^*)^{-1} A^T \mathbf{u}$ from the space of vectors $\mathbf{u}$ with non-negative components such that $\mathbf{u} \cdot \mathbf{v}^* = 1$ to itself.]
   - Show that the eigenvalue λ^* is simple and that any other eigenvalue λ of A is smaller than λ^* in absolute value at least by a factor $1 - e^{-2c}$, i.e.,
   $$|\lambda| \le (1 - e^{-2c})\lambda^*,$$

with $e^{-c} := \min_{i,j,k} \frac{A_{ij}}{A_{kj}} \in (0,1]$ the *Perron-Frobenius gap*. [**Hint**: let $A^* := A/\lambda^*$ and show that for any non-zero vector $\mathbf{v}$ with non-negative components $e^{-c} \le \frac{(A^*\mathbf{v})_i}{(A^*\mathbf{v})_j} \le e^c$ for any $1 \le i,j \le n$ and any $k \ge 1$; in particular $(\mathbf{v}^*)_i \ge e^{-c}(\mathbf{v}^*)_j$ for any i,j, from which $\max_j(\mathbf{v}^*)_j \sum_i(\mathbf{u}^*_i) \le e^c$; using this one finds that, for any $\mathbf{v}$ with non-negative components and any i, $(A^*\mathbf{v})_i \ge e^{-2c}(\mathbf{u}^* \cdot \mathbf{v})(\mathbf{v}^*)_i$; therefore, if, given $\mathbf{v}$ a real vector, we denote by $|\mathbf{v}|$ the vector with components $|\mathbf{v}|_i := |(\mathbf{v})_i|$, letting $\mathbf{v}_\pm = (|\mathbf{v}| \pm \mathbf{v})/2$, we find that, for any real vector $\mathbf{g}$ such that $\mathbf{u}^* \cdot \mathbf{g} = 0$,

$$A^*\mathbf{g} = A^*(\mathbf{g}_+ - e^{-2c}(\mathbf{u}^* \cdot \mathbf{g}_+)\mathbf{v}^*) - A^*(\mathbf{g}_- - e^{-2c}(\mathbf{u}^* \cdot \mathbf{g}_-)\mathbf{v}^*),$$

so that, for any $1 \le i \le n$,

$$|A^*\mathbf{g}|_i \le \Big(A^*(\mathbf{g}_+ - e^{-2c}(\mathbf{u}^* \cdot \mathbf{g}_+)\mathbf{v}^*) + A^*(\mathbf{g}_- - e^{-2c}(\mathbf{u}^* \cdot \mathbf{g}_-)\mathbf{v}^*)\Big)_i.$$

Hence, $\mathbf{u}^* \cdot |A^*\mathbf{g}| \le (1 - e^{-2c})(\mathbf{u}^* \cdot |\mathbf{g}|)$ so that, iterating, $\mathbf{u}^* \cdot |(A^*)^k\mathbf{g}| \le (1 - e^{-2c})^k(\mathbf{u}^* \cdot |\mathbf{g}|)$. Therefore, given any real vector $\mathbf{v}$, writing $\mathbf{v} = (\mathbf{u}^* \cdot \mathbf{v})\mathbf{v}^* + \mathbf{g}$ with $\mathbf{g} = \mathbf{v} - (\mathbf{u}^* \cdot \mathbf{v})\mathbf{v}^*$, we find $\mathbf{u}^* \cdot |(A^*)^k\mathbf{v} - (\mathbf{u}^* \cdot \mathbf{v})\mathbf{v}^*| \le 2(1 - e^{-2c})^k \mathbf{u}^* \cdot |\mathbf{v}|$, from which the result follows.]

Consider now a matrix A with non-negative elements, such that there exists a positive integer $p \ge 1$ such that A^p has positive components. Note that by applying the previous results to A^p, one finds that also A admits a unique maximal eigenvalue (such that all other eigenvalues of A are strictly smaller in absolute value) with left and right eigenvectors whose components are all positive.

4. In the context of Problem 2, apply Perron-Frobenius theorem (in the form discussed at the end of the previous problem) to M to conclude that $\psi(\beta, h) = \log \lambda(\beta, h)$ with $\lambda(\beta, h)$ the unique maximal eigenvalue of M. Using the theorem of analytic dependence of the simple eigenvalues of analytic matrices (*Let $A(\varepsilon)$ be a $n \times n$ matrix whose elements are analytic in ε in a neighborhood $U \subset \mathbb{C}$ of $\varepsilon_0 \in \mathbb{C}$; if λ_0 is a simple eigenvalue of $A(\varepsilon_0)$, then there exists a neighborhood $V \subset U$ of ε_0 and an analytic function $\lambda : V \to \mathbb{C}$ such that $\lambda(\varepsilon)$ is a simple eigenvalue of $A(\varepsilon)$ for all $\varepsilon \in V$ and $\lambda(\varepsilon_0) = \lambda_0$*) to conclude that $\psi(\beta, h)$ is real-analytic in β, h for all $\beta > 0$ and $h \in \mathbb{R}$.
   Next, define $f_{\beta,h}(x - y)$ as in Problem 1, and prove that Eq.(1) holds. Give a lower bound on κ in terms of the Perron-Frobenius gap of the transfer matrix.

5. **[The solution of the Curie-Weiss model via the Hubbard-Stratonovich transformation]** Using the Gaussian identity

$$e^{\frac{\alpha}{2}x^2} = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} e^{-\frac{m^2}{2\alpha}+mx}\, dm,$$

valid for all $\alpha > 0$, prove that the grandcanonical partition function of the Curie-Weiss model with coupling $J_0 > 0$ can be rewritten as:

$$\begin{aligned} Z^{CW}_{\beta,h,N} &= \sqrt{\frac{N\beta J_0}{2\pi}} \sum_{\sigma_1,\dots,\sigma_N=\pm} \int_{-\infty}^{+\infty} e^{-N\beta J_0 m^2/2+\beta(J_0 m+h)\sum_{i=1}^N \sigma_i}\, dm \\ &= \sqrt{\frac{N\beta J_0}{2\pi}} \int_{-\infty}^{+\infty} e^{-N\beta J m^2/2} (2\cosh(\beta(J_0 m+h)))^N\, dm \end{aligned}$$

from which it follows that

$$\lim_{N\to\infty} \frac{1}{N} \log Z^{CW}_{\beta,h,N} = \max_{m\in\mathbb{R}} \Big[-\beta J_0 m^2/2 + \log\cosh[\beta(J_0 m+h)] + \log 2 \Big].$$

Verify explicitly that this expression is the same as the expression $\psi^{CW}(\beta,h) = \max_{-1\le m\le 1}\{\beta h m - \beta f^{CW}(\beta,m)\}$ computed in class.

6. Let $\psi(\beta,h)$ be the pressure of the d-dimensional Ising model with ferromagnetic interaction $J(x-y)$, and $\psi^{CW}(\beta,h)$ the pressure of the Curie-Weiss model with coupling $J_0 := \sum_{x\neq 0} J(x)$. Prove that, if $h \ge 0$, then $\psi(\beta,h) \ge \psi^{CW}(\beta,h)$, via the following steps.

   (a) Prove that, for any $m \in \mathbb{R}$, $H^{per}_{h,\Lambda}(\sigma)$ can be re-written as $H^{per}_{h,\Lambda}(\sigma) = H^{per,0}_{h,\Lambda}(\sigma) + H^{per,1}_{h,\Lambda}(\sigma)$, where

$$\begin{aligned} H^{per,0}_{h,\Lambda}(\sigma) &= \frac{J_0}{2} m^2 |\Lambda| - (J_0 m + h) \sum_{x\in\Lambda} \sigma_x, \\ H^{per,1}_{h,\Lambda}(\sigma) &= -\frac{1}{2} \sum_{\substack{x,y\in\Lambda: \\ x\neq y}} J(x-y)(\sigma_x - m)(\sigma_y - m). \end{aligned}$$

   (b) Using the previous rewriting, recognize that the grand-canonical partition function of $H^{per}_{h,\Lambda}(\sigma)$ can be rewritten as

$$Z^{per}_{\beta,h,\Lambda} = Z^{per,0}_{\beta,h,\Lambda} \langle e^{-\beta H^{per,1}_{h,\Lambda}} \rangle^{per,0}_{\beta,h,\Lambda},$$

   where $Z^{per,0}_{\beta,h,\Lambda}$ is the grand-canonical partition function of $H^{per,0}_{h,\Lambda}(\sigma)$ and $\langle(\cdot)\rangle^{per,0}_{\beta,h,\Lambda}$ is the average with respect to the grand-canonical distribution associated with $H^{per,0}_{h,\Lambda}(\sigma)$ at inverse temperature β.

(c) Use Jensen's inequality (stating that $\int \mu(dx) f(x) \geq f(\int \mu(dx) x)$ for any probability measure μ and any **convex** function f), to conclude that

$$Z^{per}_{\beta,h,\Lambda} \geq Z^{per,0}_{\beta,h,\Lambda} e^{-\beta \langle H^{per,1}_{h,\Lambda} \rangle^{per,0}_{\beta,h,\Lambda}}.$$

Compute the right side explicitly as a function of m. Show that, by fixing m to be the largest solution of $m = \tanh[\beta(J_0 m + h)]$, one obtains, after having taken the thermodynamic limit,

$$\psi(\beta, h) \geq \psi^{CW}(\beta, h), \quad \forall h \geq 0,$$

as desired.

7. Consider the Curie-Weiss model $H^{CW}_{h,N}(\sigma)$ with coupling $J_0 > 0$ and magnetic field $h \geq 0$. Let $m^*(\beta, h)$ be the largest solution of $m = \tanh \beta(J_0 m + h)$.

(a) Prove that, if $h > 0$, or if $h = 0$ and $\beta < \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\beta,h,N}(|m - m^*(\beta, h)| > N^{-1/2+\epsilon}) = 0,$$

where $\mathbb{P}_{\beta,h,N}$ is the probability with respect to the grand-canonical distribution associated with $H^{CW}_{h,N}(\sigma)$ at inverse temperature β.

(b) Similarly, prove that, if $h = 0$ and $\beta > \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\beta,0,N}\big(\big||m| - m^*(\beta, 0)\big| > N^{-1/2+\epsilon}\big) = 0.$$

(c) Finally, prove that, if $h = 0$ and $\beta = \beta_c$, then, for all $\epsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\beta_c,0,N}(|m| > N^{-1/4+\epsilon}) = 0.$$