Physical Implications of the Chiral Anomaly – *from Condensed Matter Physics to Cosmology*

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Introduction

Abstract

Starting with an analysis of chiral edge currents in 2D electron gases exhibiting the quantum Hall effect, I will discuss the role of anomalous chiral edge currents and of anomaly inflow in 2D insulators with explicitly broken parity and time-reversal and in time-reversal invariant 2D topological insulators exhibiting edge spin-currents. I will derive the topological Chern-Simons theories that yield the correct response equations for the 2D bulk of such materials.

Anomalous commutators would appear in an analysis of conductance quantization in quantum wires and of the ED of left-right asymmetric plasmas in the early Universe that exhibit a magnetic instability. These matters were discussed on previous occasions, so I won't discuss them.

After an excursion into the theory of 3D topological insulators, including "axionic insulators", I discuss a model of Dark Matter and Dark Energy involving an axion coupled to the instanton density of a gauge field.

Credits and Contents

Credits

R. Morf (mentor in matters of the QHE) - Various collaborations with, among others: Alekseev, Alexander, Bieri, Boyarsky, Brandenberger, Cheianov, Graf, Kerler, Levkivskyi, Pedrini, Ruchayskiy, Schweigert, Studer, Sukhorukov, Thiran, Walcher, Werner, Zee.

Contents

1. Anomalous Chiral Edge Currents in Incompressible Hall Fluids
2. Chiral Spin Currents in Planar Topological Insulators
3. 3D Topological Insulators, Axions
4. Implications of the Chiral Anomaly in Cosmology

General goals of analysis

- Classify bulk- and surface states of (condensed) matter/of matter and energy in the Universe, etc., using concepts and results from gauge theory, current algebra & GR: Effective actions (= generating functionals of connected current Green fcts.$\rightarrow$ transport coefficients!), gauge-invariance, anomalies & their cancellation, "holography", etc.
- Extend Landau Theory of Phases and Phase Transitions to a Gauge Theory of Phases of Matter.

Applications

- Fractional Quantum Hall Effect (1989 - 2012)
- Topological Insulators and -Superconductors (1994 - 2014)
- Higher-dimensional cousins of QHE $\Rightarrow$ Cosmology: Primordial magnetic fields in the universe, matter-antimatter asymmetry, dark energy, etc. (2000 - $\cdots$)

The chiral anomaly

Anomalous axial currents (for massless fermions):

In 2D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{2\pi} E, \quad \alpha := \frac{e^2}{\hbar}, \quad [j_5^0(\vec{y},t), j^0(\vec{x},t)] \overset{(ACC)}{=} i\alpha\delta'(\vec{x}-\vec{y})$$

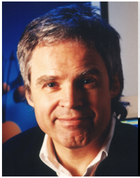
In 4D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{\pi} \vec{E} \cdot \vec{B},$$

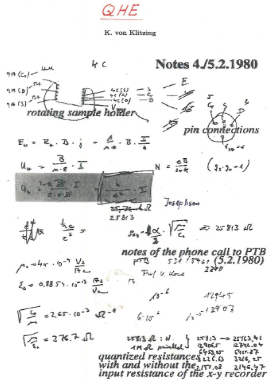
and

$$[j_5^0(\vec{y},t), j^0(\vec{x},t)] \overset{(ACC)}{=} i\frac{\alpha}{\pi} \vec{B}(\vec{y},t) \cdot \nabla_{\vec{y}} \delta(\vec{x}-\vec{y})$$

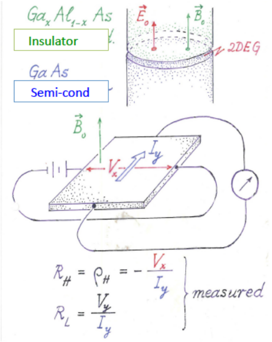
1. Anomalous Chiral Edge Currents in Incomp. Hall Fluids

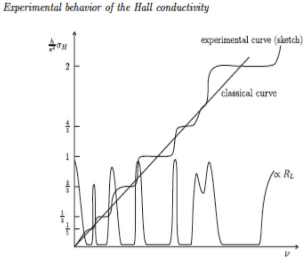



From von Klitzing's lab journal ($\Rightarrow$ 1985 Nobel Prize in Physics):



Setup & basic quantities





2D EG confined to $\Omega \subset xy$ - plane , in mag. field $\vec{B}_0 \perp \Omega$; ν such that $R_L = 0$. Response of 2D EG to small perturb. em field, $\vec{E} \| \Omega$, $\vec{B} \perp \Omega$, with $\vec{B}^{tot} = \vec{B}_0 + \vec{B}$, $\quad B := |\vec{B}|$, $\quad \underline{E} := (E_1, E_2)$.

Field tensor: $$F := \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix} = dA, \quad \text{(A: vector pot.)}$$

Electrodynamics of 2D incompressible e^--gases

Def.:

$$j^\mu(x) = \langle J^\mu(x) \rangle_A, \quad \mu = 0, 1, 2.$$

(1) Hall's Law

$$\underline{j}(x) = \sigma_H \big(\underline{E}(x)\big)^*, \quad (R_L = 0!) \;\rightarrow\; \text{broken } P, T \tag{1}$$

(2) Charge conservation

$$\frac{\partial}{\partial t}\rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0 \tag{2}$$

(3) Faraday's induction law

$$\frac{\partial}{\partial t} B_3^{tot} + \underline{\nabla} \wedge \underline{E}(x) = 0 \tag{3}$$

Then

$$\frac{\partial \rho}{\partial t} \overset{(2)}{=} -\underline{\nabla} \cdot \underline{j} \overset{(1)}{=} -\sigma_H \underline{\nabla} \wedge \underline{E} \overset{(3)}{=} \sigma_H \frac{\partial B}{\partial t} \tag{4}$$

ED of 2D e^--gases, ctd.

Integrate (4) in t, with integration constants chosen as follows:

$$j^0(x) := \rho(x) + e \cdot n, \;\; B(x) = B_3^{tot}(x) - B_0 \qquad \Rightarrow$$

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x) \tag{5}$$

$$\text{Eqs. (1) and (5)} \quad \Rightarrow \boxed{j^\mu(x) = \sigma_H \, \varepsilon^{\mu\nu\lambda} \, F_{\nu\lambda}(x)} \tag{6}$$

Now

$$0 \overset{(2)}{=} \partial_\mu j^\mu \overset{(3),(6)}{=} \varepsilon^{\mu\nu\lambda}(\partial_\mu \sigma_H) F_{\nu\lambda} \;\; \neq 0, \tag{7}$$

wherever $\sigma_H \neq$ *const*., e.g., at $\partial\Omega$. – Actually, j^μ is *bulk* current density, (j^μ_{bulk}), $\neq$ conserved *total* electric current density:

$$j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{edge}, \quad \partial_\mu j^\mu_{tot} = 0, \;\text{but}\;\; \partial_\mu j^\mu_{bulk} \overset{(7)}{\neq} 0 \tag{8}$$

Anomalous chiral edge currents

We have that

$$\text{supp } j^\mu_{edge} = \text{supp}(\underline{\nabla}\sigma_H) \supseteq \partial\Omega, \qquad \underline{j}_{edge} \perp \underline{\nabla}\sigma_H.$$

"Holography": On $\text{supp}(\underline{\nabla}\sigma_H)$,

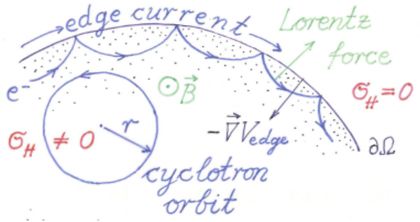
$$\boxed{\partial_\mu j^\mu_{edge} \overset{(8)}{=} -\partial_\mu j^\mu_{bulk}|_{\text{supp}(\underline{\nabla}\sigma_H)} \overset{(6)}{=} -\sigma_H E_{\|}|_{\text{supp}(\underline{\nabla}\sigma_H)}} \tag{9}$$

Chiral anomaly in 1+1 dimensions!

Edge current, $j^\mu_{edge} \equiv j^\mu_5$, is anomalous chiral current in $1+1$ D: At edge,

$$\frac{e}{c} B^{tot} v_{\|} = (\underline{\nabla} V_{edge})^*, \quad V_{edge}: \text{ confining edge pot.}$$

Skipping orbits, hurricanes and fractional charges



Analogous phenomenon in classical physics: *Hurricanes!*

$$\vec{B} \to \vec{\omega}_{earth}, \text{ Lorentz force } \to \text{ Coriolis force }, \underline{\nabla} V_{edge} \to \underline{\nabla} \text{ pressure }.$$

Chiral anomaly in $(1+1)$D:

$$\partial_\mu j_5^\mu = -\frac{e^2}{h}\Big(\sum_{species\ \alpha} Q_\alpha^2\Big)\ E_\| \quad \overset{\text{with (9)}}{\Rightarrow} \quad \boxed{\sigma_H = \frac{e^2}{h}\sum_\alpha Q_\alpha^2,} \qquad (10)$$

where $Q_\alpha \cdot e$ is fractional electric charge of quasi-particle species α.

Edge- and bulk effective actions

Apparently, if $\sigma_H \notin \frac{e^2}{h}\mathbb{Z}$ then there exist fractionally charged quasi-particles propagating along supp$(\underline{\nabla}\sigma_H)$!
Chiral edge current d. J^{μ}_{edge} = generator of $U(1)$- current algebra (free massless fields!) Green functions of J^{μ}_{edge} obtained from 2D anomalous effective action $\Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|}) = \cdots$, where $A_{\|}$ is restriction of vector potential, A, to boundary $\partial\Omega \times \mathbb{R}$.
Anomaly of $\sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|})$ – consequence of fact that J^{μ}_{edge} is *not* cons. – is cancelled by the one of bulk effective action, $S_{\Omega\times\mathbb{R}}(A)$:

$$\begin{aligned} j^{\mu}_{bulk}(x) &= \langle J^{\mu}(x)\rangle_A \equiv \frac{\delta S_{\Omega\times\mathbb{R}}(A)}{\delta A_{\mu}(x)} \\ &\overset{(6)!}{=} \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad x \notin \ \partial\Omega \times \mathbb{R} \end{aligned}$$

$$\Rightarrow \quad \boxed{S_{\Omega\times\mathbb{R}}(A) = \frac{\sigma_H}{2}\int_{\Omega\times\mathbb{R}} A \wedge \mathrm{d}A + \sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_{\|})} \qquad (11)$$

Chern-Simons action on manifold with boundary!

Classification of abelian 2D EG's

(10) $\Rightarrow$ N species of gapless quasi-particles, charges $eQ_1, \dots, eQ_N$, propagating along edge $\leftrightarrow$ N chiral scalar Bose fields $(\varphi_\alpha)_{\alpha=1}^N$, $J_\alpha^\mu = \partial^\mu \varphi_\alpha$, with propagation speeds $(v_\alpha)_{\alpha=1}^N$, such that:

1. Anomalous edge current:

$$J^\mu_{edge} = e \sum_{\alpha=1}^{N} Q_i \partial^\mu \varphi_\alpha, \quad Q = (Q_1, \dots, Q_N), \quad \sigma_H = \frac{e^2}{h} Q \cdot Q$$

2. Vertex operators creating physical states (charge $\leftrightarrow$ statistics)

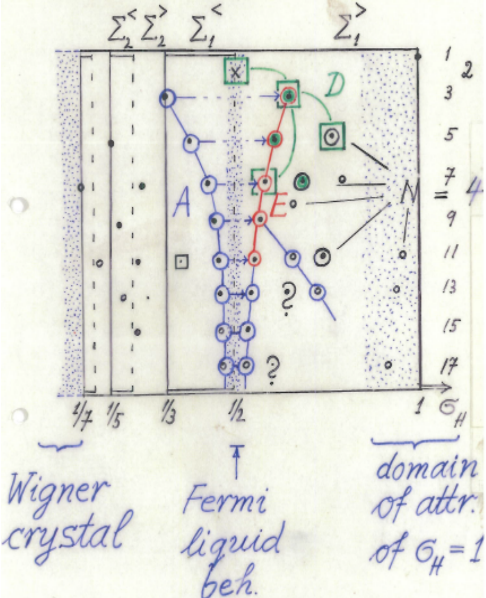
$$: \exp i \Big(\sum_{\alpha=1}^{N} q_j^\alpha \; \varphi_\alpha \Big) : , \; q_j = \begin{pmatrix} q_j^1 \\ \vdots \\ q_j^N \end{pmatrix} \in \Gamma, \quad j = 1, \dots, N \tag{12}$$

3. $\left(q_j^\alpha\right)$ analogous to CKM matrix, $Q \in \Gamma^*$. Classifying data are:

$$\{\Gamma, \, Q \in \Gamma^*, \, (q_j)_{j=1}^N, \, v = (v_1, \dots, v_N)\},$$

Success of classification – comparison with data

Γ = odd-integral lattice, $Q \in \Gamma^*$ $\Rightarrow$ $(\frac{e^2}{h})^{-1}\sigma_H \in \mathbb{Q}\,(!)\,,\ldots$



2. Chiral Spin Currents in Planar Topological Insulators

So far, we have not paid attention to electron spin, although there are 2D EG exhibiting the fractional quantum Hall effect where spin plays an important role. Won't study these systems, today. Instead, we consider time-reversal-invariant 2d topological insulators (2D TI) exhibiting chiral spin currents.

Pauli Eq. for a spinning electron:

$$i\hbar D_o \Psi_t = -\frac{\hbar^2}{2m} g^{-1/2} D_k \left(g^{1/2} g^{kl}\right) D_l \Psi_t, \tag{13}$$

where m is the mass of an electron, (g_{kl}) = metric of sample,

$$\Psi_t(x) = \begin{pmatrix} \psi_t^{\uparrow}(x) \\ \psi_t^{\downarrow}(x) \end{pmatrix} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 : \qquad \text{2-component Pauli spinor}$$

$$i\hbar D_0 = i\hbar \partial_t + e\varphi - \underbrace{\vec{W}_0 \cdot \vec{\sigma}}_{\text{Zeeman coupling}} , \quad \vec{W}_0 = \mu c^2 \vec{B} + \frac{\hbar}{4} \vec{\nabla} \wedge \vec{V} \tag{14}$$

$U(1)_{em} \times SU(2)_{spin}$-gauge invariance

$$\frac{\hbar}{i} D_k = \frac{\hbar}{i}\partial_k + eA_k - m_0 V_k - \vec{W}_k \cdot \vec{\sigma}, \tag{15}$$

where $\vec{A}$ is em vector potential, $\vec{V}$ is velocity field describing mean motion (flow) of sample, $(\vec{\nabla} \cdot \vec{V} = 0)$,

$$\vec{W}_k \cdot \vec{\sigma} := \underbrace{[(-\tilde{\mu}\vec{E} + \frac{\hbar}{c^2}\dot{\vec{V}}) \wedge \vec{\sigma}]_k}_{\text{spin-orbit interactions}},$$

and $\tilde{\mu} = \mu + \frac{e\hbar}{4mc^2}$ ($\leftarrow$ Thomas precession).
Note that the Pauli equation (13) respects $U(1)_{em} \times SU(2)_{spin}$ - gauge invariance.
We now consider an interacting 2D gas of electrons confined to a region Ω of the xy- plane, with $\vec{B} \perp \Omega$ and $\vec{E}, \vec{V} \| \Omega$. Then the $SU(2)$ - conn., $\vec{W}_\mu$, is given by $W^3_\mu \cdot \sigma_3$, $(W^M = 0,$ for $M = 1, 2)$.

Effective action of a 2D TI

Thus the connection for parallel transport of the component $\psi^\uparrow$ of Ψ is given by $a + w$, while parallel transport of $\psi^\downarrow$ is determined by $a - w$, where $a_\mu = -eA_\mu + mV_\mu, w_\mu = W^3_\mu$. These connections are abelian, (phase transformations). Under time reversal,

$$a_0 \to a_0,\ \ a_k \to -a_k,\ \ \text{but } w_0 \to -w_0,\ \ w_k \to w_k. \tag{16}$$

The dominant term in the effective action of a 2D insulator is a Chern-Simons term. If there were only the gauge field a, with $w \equiv 0$, or only the gauge field w, with $a \equiv 0$, a Chern-Simons term would *not* be invariant under time reversal, and the dominant term would be given by

$$S(a) = \int \mathrm{d}t\mathrm{d}^2x\ \{\varepsilon \underline{E}^2 - \mu^{-1}B^2\} \tag{17}$$

But, in the presence of two gauge fields, a and w, satisfying (16):

Effective action of a 2D TI, ctd.

Combination of two Chern-Simons terms is time-reversal invariant:

$$\begin{aligned} S(a,w) &= \frac{\sigma}{2}\int\{(a+w)\wedge\ \mathrm{d}(a+w)-(a-w)\wedge\ \mathrm{d}(a-w)\} \\ &= \sigma\int\{a\wedge\ \mathrm{d}w+w\wedge\ \mathrm{d}a\} \end{aligned}$$

This reproduces (17) for phys. choice of w! ($\nearrow$ J.F., Les Houches '94!) – The gauge fields a and w transform independently under gauge transformations, and the Chern-Simons action is anomalous under these gauge trsfs. on a 2D sample space-time $\Lambda = \Omega \times \mathbb{R}$ with a non-empty boundary, $\partial\Lambda$. The anomalous chiral boundary actions,

$$\pm\sigma\Gamma\big((a\pm w)|_{\|}\big),$$

cancel anomaly of bulk action! Are generating functionals of conn. Green functions of two counter-propagating chiral edge currents:

Edge degrees of freedom: Spin currents

One of the two counter propagating edge currents has "spin-up" (in +3-direction, $\perp \Omega$), the other one has "spin down". Thus, a net chiral spin current, s^3_{edge}, can be excited to propagate along the edge; but there is no net electric edge current!

Response Equations, (2 oppositely (spin-)polarized bands):

$$\underline{j}(x) = 2\sigma(\underline{\nabla} B)^*, \quad \text{and}$$

$$s_3^\mu(x) = \frac{\delta S(a,w)}{\delta w_\mu(x)} = 2\sigma \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x) \tag{18}$$

$\Rightarrow$ edge spin current – as in (7)!

We should ask what kinds of quasi-particles may produce the (bulk) Chern-Simons terms

$$S_\pm(a \pm w) = \pm\frac{\sigma}{2}\int \{(a \pm w) \wedge \, d(a \pm w),$$

where, apparently $+$ stands for "spin-up" and $-$ stands for "spin-down". Well, it has been known ever since the seventies [1] that a two-component relativistic Dirac fermion with mass $M > 0$ ($M < 0$), coupled to an abelian gauge field A, breaks parity and time-reversal invariance and induces a Chern-Simons term

$$\underset{(-)}{+}\frac{1}{2\pi}\int A \wedge \, \mathrm{d}\, A$$

We thus argue that a 2D time-reversal invariant topological insulator with chiral edge spin-current exhibits two species of charged quasi-particles in the bulk, with one species (spin-up) related to the other one (spin-down) by time reversal, and each species has two degenerate states per wave vector mimicking a 2-component Dirac fermion (at small wave vectors).

[1] the first published account of this observation – originally due to Magnen, Sénéor and myself – appears in a paper by Deser, Jackiw and Templeton of 1982

3. 3D Topological Insulators, Axions

Encouraged by the findings of the last section, we propose to consider insulators in 3D with two filled bands "communicating" with each other, confined to a region

$$\Lambda := \Omega \times \mathbb{R}, \quad \partial\Lambda \neq \emptyset$$

of space-time. Again, we are interested in the general form of the effective action describing the response of such materials to turning on an external em field. Until the mid nineties:

$$S_\Lambda(A) = \frac{1}{2} \int_\Lambda \mathrm{d}t \; \mathrm{d}^3x \; \{\vec{E} \cdot \varepsilon \vec{E} - \vec{B} \cdot \mu^{-1} \vec{B}\}, \tag{19}$$

where ε is the tensor of dielectric constants of the material and μ is the magnetic permeability tensor. The action in (19) is dimensionless. Particle theorists taught us in the seventies that one could add another dimensionless term:

An effective action with a topological term

$$S_\Lambda(A) \to S_\Lambda^{(\theta)}(A) := S_\Lambda(A) + \theta\; I_\Lambda(A), \tag{20}$$

where I_Λ is a topological term ("instanton number") given by

$$\boxed{(2\pi^2)\, I_\Lambda(A) = \frac{1}{2}\int_\Lambda \mathrm{d}t\; \mathrm{d}^3x\; \vec{E}\cdot\vec{B} = \int_\Lambda F_A \wedge F_A \underset{Stokes}{=} \int_{\partial\Lambda} A \wedge \mathrm{d}A} \tag{21}$$

The partition function of an insulator is given by

$$\mathcal{Z}_\Lambda^{(\theta)}(A) = \exp\big(iS_\Lambda^{(\theta)}(A)\big),$$

with $S_\Lambda^{(\theta)}$ as in (20). In the thermodynamic limit, $\Omega \nearrow \mathbb{R}^3$, it is *periodic* in θ with period 2π and *invariant under time reversal* iff

$$\boxed{\theta = 0, \pi}$$

Surface degrees of freedom

Conventional insulator $\leftrightarrow \theta = 0$. For $\theta = \pi$, $\mathcal{Z}_{\Lambda}^{(\theta)}(A)$ displays a boundary term given by

$$\exp\Big(\frac{i}{2\pi}\int_{\partial\Lambda} A \wedge \mathrm{d}A\Big), \tag{22}$$

see (21). This is the partition function of $(2+1)$D two-component charged, "relativistic" Dirac fermions on $\partial\Lambda$, coupled to external em field; ('76, '82, '84,...). Two species of charged quasi-particles ("spin-up" and "spin-down") with a conical Fermi surface that propagate along the surface of an insulator may appear in certain insulators with two bands communicating with each other. (Further possible surface degrees of freedom will be considered elsewhere.)
One may wonder whether it might make sense to view θ as the (ground-state) expectation value of a dynamical field, φ, and replace the term $\theta\ I_{\Lambda}(A)$ by

$$I_{\Lambda}(\varphi, A) := \frac{1}{2\pi^2}\int_{\Lambda} \varphi\ F_A \wedge F_A + S_0(\varphi), \tag{23}$$

Axionic topological insulators

where $S_0(\varphi)$ is invariant under shifts

$$\varphi \mapsto \varphi + n\pi, \quad n \in \mathbb{Z} \tag{24}$$

with minima at $\varphi = n\pi, n \in \mathbb{Z}$. The field φ is a pseudo-scalar field, called "axion field". As a speculation, one may argue that axions may emerge in certain topological insulators with anti-ferromagn. short-range order and with two bands with conical Fermi surfaces and communicating with each other $\rightarrow$ anomalous axial vector current, j_5^μ! The time derivative of the axion then would have the interpretation of a chemical potential conjugate to axial-vector charge, ($\nearrow$ F-Pedrini, 2000(!), ... , Hehl et al., 2008, S.-C. Zhang et al., 2010):

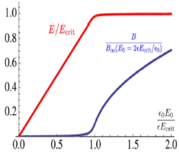
$$\int \varphi \; F_A \wedge F_A = -\text{const.} \int \mathrm{d}\varphi \wedge j_5^* \quad (\textit{chiral anomaly}!) \tag{25}$$

Instabilities of axionic topological insulators

Assuming that (23) might be term in the effective action of a new kind of 3D topological insulator, one may want to study its properties:

By (24), the bulk of such a material will exhibit domain walls across which φ changes by π. Applying insight described after (22), we predict that domain walls carry gapless two-component charged Dirac fermions, which give rise to a non-vanishing conductivity, (i.e., to a break-down of insulating nature of the material, ↗ F-Werner, 2014).

Axion electrodynamics exhibits interesting instabilities: Time Reversal Invariance and Parity can be spontaneously broken inside bulk of axionic TI, (F-Pedrini, 2000). A related instability has been pointed out by Ooguri and Oshikawa (2012) on the basis of simple calculations: Above a certain critical field strength, E_c, an external electric field applied to an axionic topological insulator is screened, with excess field converted into a magnetic field!



Courtesy Ooguri & Oshikawa

4. Implications of Chiral Anomaly in Cosmology?

$\geq$ 6 basic puzzles in cosmology:

1. Matter–Antimatter asymmetry in the Universe
2. The Universe is expanding (for ever?)
3. The expansion of the Universe is accelerated $\leftarrow$ DE?
4. There appear to be comparable amounts of Visible Matter, Dark Matter and Dark Energy in the Universe [2]; *but what are* Dark Matter *and* Dark Energy *made of*?
5. There exist tiny, but very homogeneous magnetic fields extending over intergalactic distances in the Universe
6. Neutrini have a tiny rest mass – why & what is their role in cosmology?

There are fairly precise empirical data supporting these claims; but we don't know the nature of Dark Matter and Dark Energy!

[2] was this always the case?

The ABF-proposal in a nutshell

Party line – until proven wrong: There must be *one mechanism* at the root of all these facts and solving the puzzles.
Our proposal: A new axion-type field, φ, is introduced in the standard model; φ may actually describe new gravitational degrees of freedom.
A toy model: φ is a scalar field with a self-interaction potential

$$V(\varphi) = \lambda e^{\varphi/f}$$

and is coupled to an invisible gauge field through an anomalous current as in (23), (25): A term

$$\alpha \ (\varphi/f) \ \mathrm{tr}(\vec{E} \cdot \vec{B})$$

then appears in the Lagrangian density, leading to an equation of motion

$$\lambda e^{\varphi/f} = \alpha \ \mathrm{tr}(\vec{E} \cdot \vec{B}), \qquad (**)$$

assuming that $\ddot{\varphi}, \ H\dot{\varphi} \approx 0, \ (H = \dot{a}/a = \text{Hubble “constant”})$

An instability leading to Dark Energy

The equations of motion for $\vec{E}, \vec{B}$ exhibit an instability leading to growth of the "instanton density" $\mathrm{tr}(\vec{E} \cdot \vec{B})$, which may approach a non-zero constant, as time increases: "Instanton condensation"; (same mechanism as in Ooguri-Oshikawa!) Then the potential energy density of φ,

$$V(\varphi) \overset{(**)}{=} \alpha \, \mathrm{tr}(\vec{E} \cdot \vec{B}),$$

approaches a constant; but its kinetic energy density tends to 0. $\Rightarrow \varphi$ leads to Dark Energy, while $\vec{E}$ and $\vec{B}$ describe Dark Matter:

$$0 < |\mathrm{tr}(\vec{E} \cdot \vec{B})| \leq \frac{1}{2} \, \mathrm{tr}(\vec{E}^2 + \vec{B}^2) \Rightarrow \rho_{DM} \neq 0!$$

If α is fairly large (strong coupling), Dark Energy may dominate over Dark Matter. In any event, they appear to have comparable values: Tracking!

Conclusions

Well, this is a little sketchy; but I hope the main ideas are plausible.

To conclude, I hope that I may have convinced you that the Chiral Anomaly and axions have diverse applications to concrete problems in Physics. This talk is a report on what I consider to be some of my main contributions to Physics... I hope it has been entertaining!

Sincerely,

Appendix

This appendix contains extracts of my 1994 Les Houches Lectures and of a contribution to the book "Mathematical Physics 2000" (A. Fokas (Imperial College, London), A. Grigoryan (Imperial College, London), T. Kibble (Imperial College, London), and B. Zegarlinski (Imperial College, London). I especially recommend Eqs. (6.26), (6.52) and (6.54) of the first reference to the reader's attention: Here time-reversal invariant 2D topological insulators with edge spin-currents are described (implicitly) for the first time – among many other things that look somewhat interesting.

QUANTUM THEORY OF LARGE SYSTEMS OF NON-RELATIVISTIC MATTER

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† Work supported in part by the Swiss National Foundation

Contents

1 Introduction

These are some notes to the course taught by J. Fröhlich at the 1994 summer school at Les Houches. While teaching and being taught at that school – and the hiking, early in the morning and, sometimes, during week ends – were extremely pleasant and rewarding, the preparation of publishable notes superceeding those more than three hundred pages of handwritten lecture notes prepared and distributed during the school turned out to be a very painful process. It constantly collided with more urgent duties (which ETH pays us for).

The reasons why we finally managed to produce some notes are, first of all, the infinite patience of François David and Paul Ginsparg, and, second, the circumstance that, in these times of computers and of TeX, we could adapt and modify TeXfiles of papers written for other purposes.

1.1 Sources, and acknowledgements

Here are the main sources for this first part of our notes:

J. Fröhlich, R. Götschmann, and P.-A. Marchetti, J. Phys. A **28**, 1169 (1995): for Chapter 5.

J. Fröhlich, R. Götschmann, and P.-A. Marchetti, "The Effective Gauge Field Action of a System of Non-Relativistic Electrons", to appear in Commun. Math. Phys. (1995): for Chapter 5.

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$\mathbf{J}_c$ does not enter into the bulk of the superconductor, i.e., $\mathbf{J}_c = 0$ inside the superconducting region, the London equation (4.4) immediately implies the equation

$$\Delta \mathbf{A}^T = \frac{q^2 n_s}{Mc^2} \mathbf{A}^T , \tag{4.5}$$

which shows that, in a stationary state, currents and magnetic fields in superconductors can exist only within a surface layer of thickness $\Lambda := \left(\frac{Mc^2}{q^2 n_s}\right)^{1/2}$, the so-called London penetration depth (see, e.g., de Gennes, 1966). This is the *Meissner-Ochsenfeld effect*. Note that by Eq. (4.4) a *supercurrent* $\mathbf{J}_s$ is really a sign for the presence of a *vector potential*, $\mathbf{A}^T$, and thus can be used for experimental tests of the Aharonov-Bohm effect. However, if $\oint_\Gamma \mathbf{A}^T \cdot d\boldsymbol{l} = n \frac{hc}{q}$, $n \in \mathbb{Z}$, for any closed curve Γ contained in the superconducting phase, then the Aharonov-Bohm phase factors of the charged bosons are trivial (see Subsect. 2.2), and no supercurrent results. This is the phenomenon of flux quantization.

The expulsion of a magnetic field from the interior of a superconductor is related to the fact that, inside a superconductor, "photons are massive", i.e., one observes the phenomenon of the *Anderson-Higgs mechanism*. We now show that, in a superconductor, the presence of a "*mass term for the photons*" can also be inferred directly from the London equation (4.4): Since the electric current density, $\mathbf{J}$, and the *free energy*, $F(\mathbf{A})$, of a system in the presence of a (static) electromagnetic field with vector potential $\mathbf{A} := (A_1, A_2, A_3)$, are related by

$$\mathbf{J}(\mathbf{x}) = -c \frac{\delta F(\mathbf{A})}{\delta \mathbf{A}(\mathbf{x})} , \tag{4.6}$$

it follows from Eq. (4.4) that, in the bulk of a superconductor,

$$F(\mathbf{A}) = \frac{1}{2\Lambda^2} \int d^3\mathbf{x}\, \mathbf{A}^T(\mathbf{x}) \cdot \mathbf{A}^T(\mathbf{x}) + \cdots , \tag{4.7}$$

where $A_i^T(\mathbf{x}) := \left[\delta_i^j - \partial_i \Delta^{-1} \partial^j\right] A_j(\mathbf{x})$, with Δ the three-dimensional Laplacian, and the dots stand for higher derivative terms. Notice that (4.7) is a *non-local* functional of $\mathbf{A}$ which is manifestly *invariant* under $U(1)$ gauge transformations, $\mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla}\chi$. It provides a "mass term for the photons" in the bulk of a superconductor.

4.5 Quantum Hall Effect

Just as the Aharonov-Bohm effect reflects the $U(1)_{em}$ *gauge invariance* of quantum theory, so does the *quantum Hall (QH) effect for the electric current*, as emphasized by Laughlin (1981; see also Halperin, 1982). In the same vein, the Aharonov-Casher effect and the *quantum Hall effect for the spin current* reflect the $SU(2)_{spin}$ *gauge invariance* of non-relativistic quantum theory, as emphasized by Fröhlich and Studer (1992a, 1992c). In this subsection, we review some basic facts concerning the integer (von Klitzing, Dorda, and Pepper, 1980) and fractional (Tsui, Stormer, and Gossard, 1982) quantum Hall effect; for comprehensive reviews see Chakraborty and Pietiläinen (1988), Morandi (1988), Prange and Gervin (1990), Wilczek (1990), and Stone (1992). The purpose of this review is to set the stage for Sects. 6–8, where we attempt to unravel the *universal* aspects of the quantum Hall effect in two-dimensional, *incompressible* quantum fluids.

Experimentally, the *QH effect* is observed in two-dimensional systems of electrons confined to a planar region Ω and subject to a strong, uniform magnetic field $\mathbf{B}_c$ transversal to Ω. For definiteness, we choose the region Ω to be a rectangle in the (x, y)-plane Ω with dimensions l_x and l_y in the x- and y-directions, respectively. By tuning the y-component, I_y, of the total electric current to some value and then measuring the voltage drop, V_x, in the x-direction – i.e., the difference in the chemical potentials of the electrons at the two edges parallel to the y-axis – one can calculate the *Hall resistance*

$$R_H := \frac{V_x}{I_y} , \tag{4.8}$$

and finds that, for a fixed density, n, of electrons and at temperatures close to $0\,K$, the value of R_H is independent of the current I_y. It depends only on the external magnetic field $\mathbf{B}_c$. If the electrons are treated classically one finds, by equating the electrostatic - to the Lorentz force, that

$$R_H = \frac{B_c}{nec} , \tag{4.9}$$

where B_c is the z-component of $\mathbf{B}_c$ perpendicular to the plane of the system, e is the elementary electric charge, and c is the velocity of light.

By also measuring the voltage drop, V_y, in the y-direction, one can determine the longitudinal resistance, R_L, from the equation

$$R_L := \frac{V_y}{I_y} . \tag{4.10}$$

Neither classical, nor quantum theory make simple predictions about the behaviour of R_L, but $R_L > 0$ means that there are *dissipative processes* in the system.

Two-dimensional systems of electrons (and, similarly, of holes) are realized, in the laboratory, as *inversion layers* which form at the interface between an insulator and a semiconductor when an electric field (gate voltage) perpendicular to the interface, the plane of the system, is applied. An example of such a material is a heterojunction (a "sandwich") made from $GaAs$ and $Al_xGa_{1-x}As$. The quantum-mechanical motion of the electrons in the z-direction perpendicular to the interface is then constrained by a deep potential well with a minimum on the interface. Quantum theory predicts that electrons of sufficiently low energy, i.e., at low enough temperatures, remain bound to the interface and form a very nearly two-dimensional system.

In classical physics, the connection between the electric field $\vec{E} = (E_x, E_y)$ in the plane of the system and the electric current density $\vec{J} = (J_x, J_y)$ is given by the *Ohm-Hall law*

$$\vec{E} = \boldsymbol{\rho} \vec{J}, \quad \text{with} \quad \boldsymbol{\rho} := \begin{pmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix}, \tag{4.11}$$

where the components of the resistivity tensor $\boldsymbol{\rho}$ are given as follows: $\rho_{xx} = R_L\, l_y/l_x$, $\rho_{yy} = R_L\, l_x/l_y$, and $\rho_H = R_H$. This is a phenomenological law valid on macroscopic distance scales and at low frequencies.

It is convenient to introduce a dimensionless quantity, the so-called *filling factor* ν, by setting

$$\nu := \frac{n\,\frac{hc}{e}}{B_c} \tag{4.12}$$

where $\frac{hc}{e}$ is the flux quantum. Then the *classical* Hall law (4.9) says that R_H^{-1} rises *linearly* in ν, $R_H^{-1} = \frac{e^2}{h}\,\nu$, the constant of proportionality being a constant of nature, $\frac{e^2}{h}$. Since, experimentally, B_c can be varied and n can be varied (by varying the gate voltage), this prediction of classical theory can be put to experimental tests. Experiments at very low temperatures and for rather pure inversion layers yield the following very surprizing data some of which are summarized in Fig. 4.1 below:

(D1) $\sigma_H = \frac{h}{e^2}\,R_H^{-1}$ has plateaux at *rational heights*, i.e., $\sigma_H = n_H/d_H$, with n_H, d_H relatively prime integers (see, e.g., Tsui, 1990; Stormer, 1992; Du *et al.*, 1993). Typically d_H is *odd*, but lately plateaux at $\sigma_H = \frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein *et al.*, 1988; Eisenstein *et al.*, 1990) and $\sigma_H = \frac{1}{2}$ (Suen *et al.*, 1992; Eisenstein *et al.*, 1992) have been observed. The plateaux at integer height occur with an astronomical accuracy (measurements are precise to one part in 10^8!).

(D2) When (ν, σ_H) belongs to a plateau the longitudinal resistance, R_L, very nearly *vanishes*, i.e., in plateau regions the system is *dissipationless*. Inverting the resistivity tensor $\boldsymbol{\rho}$ (see (4.11)) to obtain the conductivity tensor, $\boldsymbol{\sigma} = \boldsymbol{\rho}^{-1}$, yields the result that the diagonal part of $\boldsymbol{\sigma}$ *vanishes* on a plateau.

(D3) The precision of the plateau quantization is insensitive to details of sample preparation and geometry, hence this quantization is a "*universal*" phenomenon.

(D4) One has found (Clark *et al.*, 1988; Chang and Cunningham, 1989; Simmons *et al.*, 1989; Clark *et al.*, 1990; Hwang *et al.*, 1992) that, when (ν, σ_H) belongs to a plateau at non-integer height, then the system exhibits *fractionally charged* excitations, (the fractions of e being related to the value of d_H).

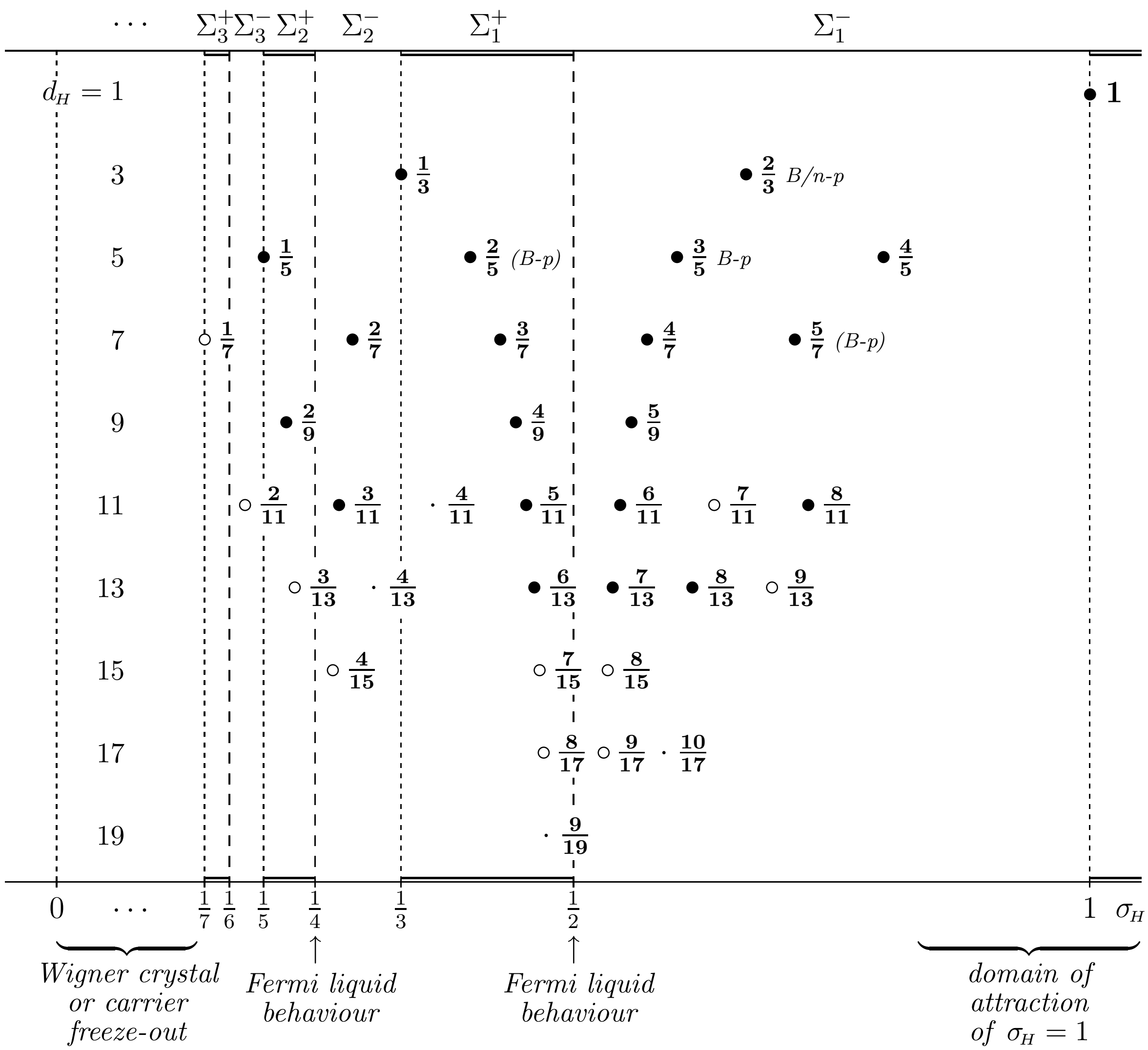

Figure 4.1. *Observed Hall fractions $\sigma_H = n_H/d_H$ in the interval $0 < \sigma_H \leq 1$, and their experimental status in single-layer quantum Hall systems.*

Well established Hall fractions are indicated by "$\bullet$". These are fractions for which an R_{xx}-minimum and a plateau in R_H have been clearly observed, and the quantization accuracy of $\sigma_H = 1/R_H$ is typically better than 0.5%. Fractions for which a minimum in R_{xx} and typically an inflection in R_H (i.e., a minimum in $dR_H/dB_c^{\perp}$, but no well developed plateau in R_H) have been observed are indicated by "$\circ$". If there are only very weak experimental indications or controversial data for a given Hall fraction, the symbol "$\cdot$" is used. Finally, "*B/n-p*" is appended to fractions at which a magnetic field *(B)* and/or density *(n)* driven phase transition has been observed.

(D5) Studies in "*tilted magnetic fields*" provide evidence that, when (ν, σ_H) belongs to a plateau at height $\sigma_H = \frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein, Willett *et al.*, 1988, 1990), $\frac{4}{3}$ (Clark *et al.*, 1989; Clark *et al.*, 1990), $\frac{8}{5}$ (Eisenstein, Stormer *et al.*, 1989, 1990a), or $\frac{2}{3}$ (Clark *et al.*, 1990; Eisenstein, Stormer *et al.*, 1990b), then the ground state of the system can be *spin-unpolarized*. For certain plateaux it might be a *spin-singlet* state.

(D6) Magnetic field and density driven *phase transitions* have been reported at $\sigma_H = 2/3$ (Clark *et al.*, 1990; Eisenstein, Stormer *et al.*, 1990b; Engel *et al.*, 1992). A magnetic field driven phase transition has been established at $\sigma_H = 3/5$ (Engel *et al.*, 1992), and a possible observation of such a phase transition at $\sigma_H = 5/7$ has been discussed in Sajoto *et al.* (1990).

Next, we propose to study what the Ohm-Hall law (4.11) tells us about a two-dimensional system of electrons in an external magnetic field when (ν, σ_H) belongs to a *plateau*. As noted in **(D2)**, experimentally one finds that, in this situation, the longitudinal resistance R_L vanishes. This signals the *absence of dissipative processes*. The absence of dissipative processes could be explained if one succeeded in showing that the spectrum of the many-electron Hamiltonian of the system exhibits an *energy gap*, $\delta > 0$, above the ground-state energy, (or, at least, that states of very small energy above the ground-state energy are *localized*). To exhibit a positive energy gap for certain values of the filling factor ν, physically interpreted as *incompressibility* of the system, poses difficult analytical problems. Some recent ideas about how to establish incompressibility at particular filling factors can be found in the following papers: studies of Laughlin states are given in Haldane (1983, 1990), Halperin (1983, 1984), Laughlin (1983a, 1983b, 1984, 1990), Arovas, Schrieffer, and Wilczek (1984), and Trugman and Kivelson (1985); off-diagonal long-range order and Chern-Simons-Landau-Ginzburg theory in fractional QH fluids have been studied in Girvin and MacDonald (1987), Read (1989), Zhang, Hansson, and Kivelson (1989), Lee and Zhang (1991), Fröhlich (1992), Fröhlich, Kerler, and Marchetti (1992), and Zhang (1992); finally, for some numerical studies concerning the question of incompressibility see, e.g., Fano, Ortolani, and Colombo (1986), Yoshioka (1986), Chakraborty and Pietiläinen (1987, 1988), Rezayi (1987), and d'Ambrumenil and Morf (1989). What is easier to show is for *which values* of the parameter $\sigma_H = \frac{h}{e^2} R_H^{-1}$ a positive energy gap δ *cannot* occur; more precisely, to prove a "*gap labelling theorem*". Such a theorem is stated at the end of Subsect. 7.2, and will form the main theme of Sect. 8.

Thus, if the system is incompressible, in the sense that $R_L = 0$, then we have the following form for the *Hall law* (we use units such that $e^2/h = 1$):

$$\vec{J} = \boldsymbol{\sigma}\,\vec{E}\,, \quad \text{with} \quad \boldsymbol{\sigma} := \boldsymbol{\rho}^{-1} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix}\,, \tag{4.13}$$

where $\sigma_H = R_H^{-1}$. This is a *phenomenological* law valid on macroscopic distance scales and at low frequencies, as mentioned after (4.11). More fundamental are the following two laws: *Charge conservation*

$$\frac{1}{c}\frac{\partial}{\partial t} J^0 + \vec{\nabla} \cdot \vec{J} \;=\; 0 \;, \tag{4.14}$$

(continuity equation), where J^0 is (c times) the electric charge density, and *Faraday's induction law*,

$$\frac{1}{c}\frac{\partial}{\partial t} B + \vec{\nabla} \times \vec{E} \;=\; 0 \;, \tag{4.15}$$

where B denotes the component of the magnetic field perpendicular to the plane of the system, and $\vec{E}$ is the electric field in the plane of the system. We note that the dynamics of charged, spinless particles confined to a plane only depends on the component of the magnetic field perpendicular to the plane of the system and the components of the electric field in that plane. Combining Eqs. (4.13) through (4.15), we find that

$$\frac{\partial}{\partial t} J^0 \;=\; \sigma_H \frac{\partial}{\partial t} B \;. \tag{4.16}$$

Eq. (4.16) can be integrated with respect to time t. By $J^0 = J^0_{tot} - nec$ we denote the difference between the total electric charge density, J^0_{tot}, and the uniform background density, nec, of a system in a uniform background magnetic field B_c. Likewise, B denotes the difference between the total magnetic field, B_{tot}, and the uniform background field B_c. Then Eq. (4.16) implies the *charge-flux relation*

$$J^0 \;=\; \sigma_H \, B \;. \tag{4.17}$$

It is convenient to introduce the electromagnetic field tensor which is a 2-form, F, given by

$$F \;=\; \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \;, \quad \text{with} \quad (F_{\mu\nu}) \;=\; \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{pmatrix} \;, \tag{4.18}$$

and the 2-form $\mathcal{J}$ dual to the current density $J^\mu = (J^0, \vec{J})$, i.e.,

$$\mathcal{J} = \frac{1}{2}\sum_{\mu,\nu} \mathcal{J}_{\mu\nu}\, dx^\mu \wedge dx^\nu \;, \quad \text{with} \quad \mathcal{J}_{\mu\nu} = \varepsilon_{\mu\nu\rho}\, J^\rho \;. \tag{4.19}$$

Then Eqs. (4.13) and (4.17) can be combined into one equation,

$$\mathcal{J} = -\sigma_H F , \tag{4.20}$$

while current conservation (4.14) is expressed as

$$\mathrm{d}\mathcal{J} = \frac{1}{2} \sum_{\alpha,\mu,\nu} \partial_\alpha \mathcal{J}_{\mu\nu} \, dx^\alpha \wedge dx^\mu \wedge dx^\nu = 0 , \tag{4.21}$$

and Faraday's induction law (4.15) becomes

$$\mathrm{d}F = 0 . \tag{4.22}$$

Eqs. (4.20) through (4.22) are compatible with each other if, and only if, σ_H is *constant*. If the values of σ_H along the two sides of a curve Γ differ from each other – which happens, for example, at the boundary of the system – then an *additional* current, $\vec{\mathcal{I}}$, *not* described by Eq. (4.13), is observed in the vicinity of Γ, in order to reconcile charge conservation with the induction law. [For time-independent fields one finds that $\vec{\nabla} \cdot \vec{\mathcal{I}} = (\vec{\nabla}\sigma_H) \times \vec{E}$; see also Halperin (1982) and Sect. 7 below.]

Note that Eqs. (4.20) through (4.22) are *generally covariant* and independent of metric properties of the system. Eqs. (4.21) and (4.22) can be integrated by introducing the 1-forms (or "vector potentials") A and b, with

$$\mathcal{J} = \mathrm{d}b , \quad \text{i.e.,} \quad \mathcal{J}_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu , \quad \text{and} \quad F = \mathrm{d}A . \tag{4.23}$$

Eq. (4.20) then reads

$$\mathrm{d}b = -\sigma_H \, \mathrm{d}A . \tag{4.24}$$

Eq. (4.24) is the Euler-Lagrange equation derived form an *action principle*! The corresponding action, $S(b; A)$, is given by

$$\begin{aligned} S(b;A) &:= \frac{1}{2c^2\sigma_H} \int_\Lambda b \wedge \mathrm{d}b + \frac{1}{c^2} \int_\Lambda A \wedge \mathrm{d}b + \mathrm{B.T.}(A|_{\partial\Lambda}, b|_{\partial\Lambda}) \\ &= \frac{1}{2c^2\sigma_H} \int_\Lambda (\mathrm{d}^{-1}\mathcal{J}) \wedge \mathcal{J} + \frac{1}{c^2} \int_\Lambda A \wedge \mathcal{J} + \mathrm{B.T.}(A|_{\partial\Lambda}, \mathcal{J}|_{\partial\Lambda}) , \end{aligned} \tag{4.25}$$

where $\Lambda := \mathbb{R} \times \Omega$ is the space-time domain to which the system is confined, and "B.T." stands for *boundary terms* which only depend on the restrictions $A|_{\partial\Lambda}$ and $b|_{\partial\Lambda}$ of A and b to the space-time boundary $\partial\Lambda$, (see the remark after (4.27) below). Moreover, $S(b; A)$

shall be varied with respect to the *dynamical* variable, that is, with respect to b; (the vector potential A of the electromagnetic field is a tunable, external field).

Why is the result (4.25) interesting? It is interesting, because an equation of motion, like (4.24), that can be derived form an action principle can be quantized easily; for example by using *Feynman path integrals*. Clearly, the current density $\mathcal{J}$ of a system of electrons must be interpreted as a quantum-mechanical operator-valued distribution. Hence (4.24) must be *quantized.* We note that, in the present example, Feynman path integral quantization has a mathematically rigorous interpretation. In formal, "physical" notation, Feynman's path space measure is given by

$$dP_A(b) \;=\; Z(A)^{-1}\,\exp\Big[\frac{i}{\hbar}S(b;A)\Big]\,\mathcal{D}b\;, \tag{4.26}$$

where the *partition* - or *generating function* $Z(A)$ is chosen such that (formally) $\int dP_A(b) = 1$. This implies that

$$Z(A) \;=\; Z_o\,\exp\Big[-\frac{i\,\sigma_H}{2\hbar c^2}\int_\Lambda A\wedge \mathrm{d}A \;+\; \mathrm{B.T.}(A|_{\partial\Lambda})\Big]\;, \tag{4.27}$$

where Z_o is a constant *independent* of A. [In (4.26), we have omitted a gauge fixing term for the integration over the field b.] At this point, we emphasize that a non-trivial boundary term, $\mathrm{B.T.}(A|_{\partial\Lambda})$, in Eq. (4.27) is a necessity forced upon us by the $U(1)_{em}$ *gauge invariance of quantum mechanics* (see Sect. 3): Under a $U(1)$ gauge transformation, $A\mapsto A+\mathrm{d}\chi$, the Chern-Simons term in (4.27) transforms according to

$$\int_\Lambda A\wedge\mathrm{d}A \;\;\mapsto\;\; \int_\Lambda A\wedge\mathrm{d}A \;-\; \int_{\partial\Lambda}\mathrm{d}\chi\wedge A\;, \tag{4.28}$$

i.e., there is a *gauge anomaly* localized at the boundary $\partial\Lambda$. Hence, in order for the partition function $Z(A)$ to be $U(1)$ gauge invariant, the presence of a boundary term, $\mathrm{B.T.}(A|_{\partial\Lambda})$, exhibiting a gauge anomaly cancelling the one in (4.28) is indispensable! We shall see that the anomalous part of $\mathrm{B.T.}(A|_{\partial\Lambda})$ turns out to be the generating functional of the connected Green functions of chiral current operators generating a $\hat{u}(1)$ current (Kac-Moody) algebra which, physically, describes charged, chiral waves circulating at the edge of the QH sample. Sect. 7 will be devoted to an investigation of this current algebra. Our analysis will lead to a list of the possible (quantized) values of the response coefficient of the Hall conductivity σ_H; see also remark **(iii)** at the end of this subsection.

Given the expression (4.27) for the partition function $Z(A)$, we may ask whether there is a simple way of recovering the action $S(b;A)$ as a functional of the vector potential b of the conserved current density $\mathcal{J}$, as given in (4.25). The answer is yes. It is provided by the following *functional Fourier transform* identity:

$$\exp\Big[\frac{i}{\hbar}S(b;A)\Big] \;=\; \text{const.} \int \mathcal{D}\alpha\ \exp\Big[-\frac{i}{\hbar c^2}\int \alpha\wedge \mathrm{d}b\Big]\ Z(A+\alpha)\ , \tag{4.29}$$

where we again omit a suitable gauge fixing term for the integral over α; see Fröhlich and Kerler (1991), and the general discussion in Sect. 5 below.

Remarks. **(i)** Defining the *effective action*, $S^{\text{eff}}(A)$, of a two-dimensional electronic system by

$$S^{\text{eff}}(A) \;:=\; \frac{\hbar}{i}\ \ln Z(A)\ , \tag{4.30}$$

our circle of arguments can be closed from Eq. (4.27) back to the starting point of the Hall law (4.13) (and (4.17)) by noting that

$$J^i \;=\; c^2\,\frac{\delta S^{\text{eff}}(A)}{\delta A_i} \;=\; \sigma_H\,\varepsilon^{ij}\,E_j\ , \tag{4.31}$$

where $E_j = \partial_j A_0 - \partial_0 A_j$, for $j = 1, 2$.

Eqs. (4.31), (4.30) and (4.29) make it clear that one approach leading to an understanding of the QH effect is to derive, for a QH fluid at particular values of the filling factor ν, the effective action $S^{\text{eff}}(A)$ corresponding to Eq. (4.27) from “first principles”. In Subsect. 6.1, we show that *gauge invariance* of non-relativistic quantum mechanics and the single assumption of *incompressibility* of QH fluids are sufficient to uniquely determine their effective action $S^{\text{eff}}(A)$ in the “scaling limit” and thereby to derive (4.27). Hence the phenomenology of QH systems at low frequencies and on large distance scales, including the quantization of the Hall conductivity σ_H (see remark **(iii)** below), can be derived from gauge invariance and incompressibility. This shows that a proof of incompressibility of electronic systems at particular values of the filling factor ν is really the essential problem in the theory of the QH effect in need of further investigation.

(ii) It can be seen directly from the Ohm-Hall law (4.11) that the incompressibility of QH fluids is a crucial property which allows for a description of these systems in terms of an effective action formalism: Only for *dissipationless* systems, i.e., for systems with an *antisymmetric* conductivity tensor $\boldsymbol{\sigma}$, it is possible to functionally integrate relation (4.31) in order to obtain an effective action. In other words, for electronic systems with *dissipation*, one *cannot* formulate the Ohm-Hall law (coming from transport theory) within an effective action formalism.

(iii) Clearly, we must require that the quantum theory with action $S(b;A)$ given by (4.25), defined by the Feynman path integral (4.26), describe *localized, particle-like excitations* with the quantum numbers of the *electron* or *hole*, i.e., with electric charge $\pm e$ and Fermi statistics. We shall see in Sects. 7.2 and 8.2 that this requirement implies that, for *consistency* of the theory, the dimensionless Hall conductivity (*Hall fraction*) $\sigma = \frac{h}{e^2}\,\sigma_H$ must be a *rational number*.

5 Scaling Limit of the Effective Action of Fermi Systems, and Classification of States of Non-Relativistic Matter

In this section, we generalize the observations made at the end of the last subsection. We review a conceptual framework (Fröhlich, Götschmann, and Marchetti, 1995a, 1995b), based on the bosonization of quantum systems with infinitely many degrees of freedom, which has proven to be useful in attempting to classify states of non-relativistic matter at very low temperatures. In this section, we focus our attention on the analysis of electric charge transport properties. Magnetic properties can be studied in a similarly general manner; for an example see Sect. 6 where two-dimensional, incompressible quantum fluids are treated, *including* their magnetic properties.

The basic ideas underlying our approach are very simple: The starting point is to study the response of a quantum system of charged particles to perturbations by external electromagnetic fields. Thus we couple the electric current density, J^μ, to an arbitrary, smooth, external electromagnetic vector potential (1-form), A, and then attempt to calculate the partition function, $Z(A)$, of the system as a functional of A.

Of course, this is a very complicated task. However, in order to *classify electric properties of non-relativistic matter*, we are really only interested in understanding the behaviour of the *effective action* (see also (4.30))

$$S^{\mathrm{eff}}(A) := \frac{\hbar}{i} \ln Z(A) , \tag{5.1}$$

on very large distance scales and at very low frequencies. We thus study families of systems confined to ever larger cubes, $\Omega^{(\theta)} := \{\vec{x} \,|\, \vec{x}/\theta \in \Omega\}$, in physical space $\mathbb{E}^d$, $d = 1, 2, 3$, where Ω is a fixed compact subset of $\mathbb{E}^d$, and $1 \leq \theta < \infty$ is a scale parameter. We keep the particle density, n, and the temperature T ($\approx 0\,K$) constant. We then couple the electric current density of the system confined to $\Omega^{(\theta)}$ to a vector potential $A^{(\theta)}$ given by

$$A^{(\theta)}(t, \vec{x}\,) := \theta^{-1} A(\frac{t}{\theta}, \frac{\vec{x}}{\theta}) , \qquad \frac{\vec{x}}{\theta} \in \Omega , \tag{5.2}$$

where A is an arbitrary, but θ-independent vector potential on $\mathbb{R} \times \Omega$. We then study the behaviour of $S^{\mathrm{eff}}_{\Omega^{(\theta)}}(A^{(\theta)})$ when θ becomes large.

More precisely, we attempt to expand $S^{\mathrm{eff}}_{\Omega^{(\theta)}}(A^{(\theta)})$ in powers of θ^{-1} around $\theta = \infty$ (to a finite order) and define the *scaling limit*, $S^*(A)$, *of the effective acction* to be the coefficient of the leading power of θ in that expansion; more details are given in Subsect. 6.1.

Remarkably, all many-body systems of non-relativistic electrons that can be controled analytically have the property that $S^*(A)$ is *quadratic* in A; this holds, in particular, for insulators, Landau-Fermi (non-interacting electron) liquids, metals, incompressible quantum

6 Scaling Limit of the Effective Action of a Two - Dimensional, Incompressible Quantum Fluid

In this section, we study the *partition* - or *generating function* (at $T = 0$ and for *real* time) of a two-dimensional, non-relativistic quantum system confined to a space-time region $\Lambda = \mathbb{R} \times \Omega$ and coupled to external electromagnetic, "tidal", and geometric fields:

$$Z_\Lambda(a, w) \;:=\; \int \mathcal{D}\psi^* \mathcal{D}\psi \, \exp\Big[\frac{i}{\hbar} S_\Lambda(\psi^*, \psi \,;\, a, w)\Big] \,, \tag{6.1}$$

where the gauge potentials a and w have been introduced in (3.26)–(3.31), and $S_\Lambda(\psi^*, \psi\,; a, w)$ is the action of the system given in (3.34) and (3.35); see also (3.55)–(3.58). The integration variables ψ^* and ψ are Grassmann variables (i.e., anticommuting c-numbers) for Fermi statistics, and complex c-number fields for Bose statistics.

We have not displayed the metric, g_{ij}, on the background space, M, explicitly, since it will be kept fixed, and usually $M = \mathbb{E}^2$ with $g_{ij} = \delta_{ij}$, for simplicity. We note, however, that for the study of the *stress tensor*, pressure - and density fluctuations, and curvature - and torsion effects, we would have to choose a variable external metric (or, at least, a variable conformal factor in g_{ij}). This is important in the study of density waves, in particular of surface density waves (which are interesting in two-dimensional quantum fluids), and of *critical phenomena* in the theory of phase transitions. We note, however, that curvature - and torsion effects can be studied by analyzing the dependence of $Z_\Lambda(a, w)$ on w which contains the affine spin connection, ${\omega^A}_B$; see (3.28) and (3.29), and the remarks at the end of Subsect. 3.1 and after (3.39).

Calculating the partition function (6.1) for an arbitrary (two-dimensional) non-relativistic quantum system is surely a major task. In the first part of this section, we show how the calculation can be carried out for *incompressible* systems, provided one passes to the *scaling limit*. Once we have an explicit expression for the partition function of a system, many of its physical properties can be derived. For incompressible systems, this is the topic of the rest of this section and of Sects. 7 and 8 where we review, in partucular, a classification of incompressible quantum Hall fluids. Since we are working in the scaling limit, we can only analyze *universal* properties of such systems, (i.e., properties independent of the small-scale structure of the system).

6.1 Scaling Limit of the Effective Action

In this subsection, we sketch how one calculates, for *incompressible* systems, the scaling limit of the effective action (see (6.5) below) associated with the partition function (6.1).

One of our main motivations for studying two-dimensional, incompressible quantum fluids comes from the phenomenology of the quantum Hall effect; see Subsect. 4.5. In discussing quantum Hall fluids, it is often assumed that the magnetic field transversal to the samples is so strong that the Zeeman energies are large enough for the systems to be totally spin-polarized. Moreover, spin-orbit interaction terms are expected to be negligible in quantum Hall fluids. One might therefore ask why, when studying quantum Hall fluids, one should worry about the dependence of the partition function $Z_\Lambda(a,w)$ on the $SU(2)$ connection w, thereby taking into account Zeeman and spin-orbit interaction effects?

To answer this question, we first argue that it is of principal, theoretical interest to know how to incorporate the *spin degrees of freedom* in a consistent way into the description of two-dimensional electronic systems.

Second, as first pointed out by Halperin (1983), in *GaAs* (for example) the g-factor of the electron is $\frac{1}{4}$ of the value in empty space, and the effective mass, m, of the electron is about $\frac{7}{100}$ of the mass, m_o, in the vacuum. Thus, in *GaAs*, the Zeeman energies are only approximately $\frac{1}{60}$ of the cyclotron energy (i.e., the splitting between Landau levels). Furthermore, they are of the same magnitude as the quasi-particle energies of the fractional quantum Hall states in magnetic fields of the order of $10\,\mathrm{T}$. One expects therefore that, at some values of the filling factor ν, the ground state of the system will contain electrons with reversed spins. Experimental evidence that *spin-unpolarized* quantum Hall fluids exist has been given in the works cited in **(D5)** in Subsect. 4.5. We emphasize that unpolarized (or partially polarized) quantum Hall fluids can arise in two fundamentally different ways: either through the presence of two (or more) *independent*, but oppositely polarized bands, or through the formation of *spin-singlet* bands; see Subsect. VI.C in Fröhlich and Studer (1993b) and Fröhlich and Thiran (1994).

Third, we will show in Sects. 7 and 8 how one can infer, from the form of $Z_\Lambda(a,w)$, *universal* properties of quantum Hall fluids that will lead to a classification of such systems in terms of "universality classes".

Fourth, in Subsect. 6.2, we sketch how one can derive the linear response theory of quantum Hall fluids from $Z_\Lambda(a,w)$ describing, among other effects, a *quantum Hall effect for spin currents*. For a discussion of possible Hall systems where this effect might be tested experimentally, see Subsect. VII.A in Fröhlich and Studer (1993b).

We define the *electric charge* - and *current densities*, $j^0(x)$ and $\vec{j}(x)$, by

$$\begin{aligned} j^0(x) &:= \psi^*(x)\psi(x)\ , \\ j^k(x) &:= -\frac{i\hbar}{2mc}\, g^{kl}(x)\,[(\mathcal{D}_l\psi)^*(x)\psi(x) - \psi^*(x)(\mathcal{D}_l\psi)(x)]\ , \end{aligned} \tag{6.2}$$

and the *spin* - and *spin current densities*, $s_A^0(x)$ and $\vec{s}_A(x)$, by

$$\begin{aligned} s_A^0(x) &:= \psi^*(x)\, L_A^{(s)}\, \psi(x)\ , \\ s_A^k(x) &:= -\frac{i\hbar}{2mc}\, g^{kl}(x)\left[(\mathcal{D}_l\psi)^*(x)\, L_A^{(s)}\, \psi(x) - \psi^*(x)\, L_A^{(s)}\, (\mathcal{D}_l\psi)(x)\right]\ , \end{aligned} \tag{6.3}$$

it is satisfied. [For some recent ideas about how to establish it for quantum Hall fluids at certain filling factors, see the references given after **(D5)** in Subsect. 4.5.] What we propose to do here is to use it to calculate the general form of $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, the scaling limit of the effective action of the system, thereby elucidating the *universal* properties of two-dimensional, incompressible quantum fluids. We only sketch some ideas; for details, see Fröhlich and Studer (1992b), and Appendix A in Fröhlich and Studer (1993b).

The calculation is based on the following four principles:

(P1) *Incompressibility*: For all n and m, with $2 \le n+m \le 4$, the distributions $\varphi^{(\theta)\,\underline{\mu},\underline{\nu}}_{\quad\underline{A}}(a_c, w_c; \underline{\xi}, \underline{\eta})$ "converge" (in the bulk), as $\theta \to \infty$, to local distributions, as specified in Eq. (6.21).

(P2) $U(1) \times SU(2)$ *gauge invariance*: Ward identities (6.9) to (6.12).

(P3) *Only relevant and marginal terms are kept in* $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$.

(P4) *Extra symmetries of the system*, e.g., for $w_{c,\mu A}(x) = \delta_{A3}\, w_{c,\mu 3}(x)$ (and hence, by (3.38), for $a_{c,\mu}(x)$ such that $E_{c,3}(x) = 0$), global rotations around the 3-axis in spin space are a continuous, global symmetry of the system with an associated *conserved* Noether current, $s_3^\mu(x)$; or translation invariance in the scaling limit ($\theta \to \infty$), ... , *are exploited to reduce the number of terms.*

From **(P1)** and Eqs. (6.16) and (6.17) it immediately follows that all terms contributing to $S^{\text{eff}}_{\theta\Lambda_o}(a^{(\theta)}, w^{(\theta)})$ of order 4 or higher in $\tilde{a}(\xi)$ and $\tilde{w}(\xi)$ are *irrelevant*, i.e., they scale like θ^{-D}, with $D > 0$. In particular, a fourth-order remainder term does *not* contribute to $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, (principle **(P3)**). We present the final result of our analysis in the special case of a system which is incompressible for a choice of w_c satisfying

$$w_{c,\mu A}(x) \;=\; \delta_{A3}\, w_{c,\mu 3}(x) \;, \tag{6.23}$$

or, in view of Eqs. (3.28)–(3.30), (3.37) and (3.38), for a background electromagnetic field $(\mathbf{E}_c(x), \mathbf{B}_c(x))$ with

$$\mathbf{E}_c(x) \;=\; (E_{c,1}(x), E_{c,2}(x), 0) \;, \quad \mathbf{B}_c(x) \;=\; (0, 0, B_c(x)) \;, \tag{6.24}$$

and possibly for some affine spin connection, ω, of the following form (see (3.29), (3.11), (3.12), and (3.31), as well as the remarks about the physical relevance of ω at the end of Subsect. 3.1 and after (3.39)):

$$
\left(\omega^{A}_{\ B\mu}(x)\right) = \begin{pmatrix} 0 & \omega_\mu(x) & 0 \\ -\,\omega_\mu(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{6.25}
$$

relative to some orthonormal frames $(e^1(x),\, e^2(x),\, e^3(x))$. [It is natural to work in an $SU(2)$ gauge which respects our convention of choosing $e^3(x)$ perpendicular to the cotangent plane $T^*_{\vec{x}}(M)$ at $\vec{x} \in M$, for all times t; see the discussion preceding (3.1). E.g., if M were the (x, y)-plane in $\mathbb{E}^3$, we would choose $e^3(x)$ to coincide with dz. Hence the choice of the electromagnetic *background* field specified by (6.24) corresponds to an electric field, $\mathbf{E}_c$, which is tangential to the sample and to a magnetic field, $\mathbf{B}_c$, which is perpendicular to it.] In this situation, the *scaling limit of the effective action* is given by

$$
\begin{aligned}
-\frac{1}{\hbar}\, S^*_{\Lambda_o}(\tilde{a}, \tilde{w}) \;=\; & \int_{\Lambda_o} j^\mu_c\, \tilde{a}_\mu\, dv \;+\; \int_{\Lambda_o} m^\mu_3\, \tilde{w}_{\mu 3}\, dv \\
& +\; \sum_{A=1}^{2} \int_{\Lambda_o} \tau^{\mu\nu}_1\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu A}\, dv \;+\; \sum_{A,B=1}^{2} \int_{\Lambda_o} \tau^{\mu\nu}_2\, \varepsilon_{AB}\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu B}\, dv \\
& +\; \frac{k}{4\pi} \int_{\Lambda_o} \mathrm{tr}\Big(w \wedge \mathrm{d}w + \frac{2}{3}\, w \wedge w \wedge w\Big) \\
& +\; \frac{\sigma}{4\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{a} \;+\; \frac{\chi_s}{2\pi} \int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{w}_3 \;+\; \frac{\sigma_s}{4\pi} \int_{\Lambda_o} \tilde{w}_3 \wedge \mathrm{d}\tilde{w}_3 \\
& +\; \sum_{A,B,C=1}^{3} \int_{\Lambda_o} \eta^{\mu\nu\rho}_{ABC}\, \tilde{w}_{\mu A}\, \tilde{w}_{\nu B}\, \tilde{w}_{\rho C}\, dv \;+\; \mathrm{B.T.}(a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o}) ,
\end{aligned} \tag{6.26}
$$

where $j^\mu_c(\xi)$ is an electric - and $m^\mu_3(\xi)$ a magnetic current circulating in the system when $\tilde{a}(\xi) = 0 = \tilde{w}(\xi)$; $\tau^{\mu\nu}_1(\xi)$ is a function symmetric in μ and ν, while $\tau^{\mu\nu}_2(\xi)$ is antisymmetric in μ and ν; the function $\eta^{\mu\nu\rho}_{ABC}(\xi)$ is symmetric under interchanges of (μA), (νB) and (ρC) and vanishes if two or more of the indices A, B, C are equal to 3; $dv := \sqrt{\gamma(\xi)}\, d^3\xi$, where $\gamma(\xi) := \det(\gamma_{ij}(\xi))$ is the volume element on the space-time cylinder Λ_o, see (6.13); σ, χ_s, σ_s, and k are real constants; $w(\xi)$ is the total $SU(2)$ connection given by

$$
w(\xi) \;:=\; w^{(\theta)}_c(\xi) + \tilde{w}(\xi) , \qquad \text{with} \qquad w^{(\theta)}_c(\xi) \;:=\; \theta\, w_c(\theta\xi) , \tag{6.27}
$$

see (6.17); and $\mathrm{B.T.}(a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o})$ denotes boundary terms only depending on the gauge potentials $a|_{\partial\Lambda_o}, w|_{\partial\Lambda_o}$ restricted to the boundary, $\partial\Lambda_o$, of the space-time cylinder Λ_o. They will be discussed in Sect. 7. Moreover, on the r.h.s. of (6.26), we are using the notation:

$$
\begin{aligned}
\tilde{a} \;&=\; \sum_{\mu=0}^{2} \tilde{a}_\mu(\xi)\, d\xi^\mu \;, \qquad & \mathrm{d}\tilde{a} \;&=\; \sum_{\mu,\nu=0}^{2} (\partial_\mu \tilde{a}_\nu)(\xi)\, d\xi^\mu \wedge d\xi^\nu \;, \\
\tilde{w}_3 \;&=\; \sum_{\mu=0}^{2} \tilde{w}_{\mu 3}(\xi)\, d\xi^\mu \;, \qquad & \tilde{w} \;&=\; i \sum_{\mu=0}^{2} \sum_{A=1}^{3} \tilde{w}_{\mu A}(\xi)\, L_A^{(s)}\, d\xi^\mu \;,
\end{aligned} \tag{6.28}
$$

where $\partial_\mu = \frac{\partial}{\partial \xi^\mu}$ whenever we are working in rescaled ξ-coordinates. Finally, we note that the terms in (6.26) are ordered according to their scaling dimensions.

In Fröhlich and Studer (1993b), we have used results on chiral $\hat{u}(1)$ - and $\hat{su}(2)$ current algebras to determine the possible values of the constants σ, χ_s, σ_s, and k; see Sect. 7 below.

Here we wish to point out that the functions j_c^μ, m_3^μ, $\tau_1^{\mu\nu}$, $\tau_2^{\mu\nu}$, and $\eta_{ABC}^{\mu\nu\rho}$ are *not* all independent, but are constrained by the infinitesimal Ward identities (6.10) and (6.12): By (6.4) and (6.5)

$$
\langle j^\mu(\xi) \rangle_{a^{(\theta)},\, w^{(\theta)}} \;=\; -\frac{1}{\hbar}\, \frac{\delta S^*_{\Lambda_o}(\tilde{a}, \tilde{w})}{\delta \tilde{a}_\mu(\xi)} + \cdots \;, \tag{6.29}
$$

and

$$
\langle s_A^\mu(\xi) \rangle_{a^{(\theta)},\, w^{(\theta)}} \;=\; -\frac{1}{\hbar}\, \frac{\delta S^*_{\Lambda_o}(\tilde{a}, \tilde{w})}{\delta \tilde{w}_{\mu A}(\xi)} + \cdots \;. \tag{6.30}
$$

The dots stand for contributions from irrelevant terms in the effective action. We calculate the r.h.s. of these equations by using (6.26) and plug the result into Eqs. (6.10) and (6.12). As a result we obtain the following *constraints* (Fröhlich and Studer, 1992b):

(i)
$$
\frac{1}{\sqrt{\gamma(\xi)}}\, \partial_\mu (\sqrt{\gamma}\, j_c^\mu)(\xi) \;=\; 0 \;.
$$

(ii)
$$
\frac{1}{\sqrt{\gamma(\xi)}}\, \partial_\mu (\sqrt{\gamma}\, m_3^\mu)(\xi) \;=\; 0 \;.
$$

(iii)
$$
\begin{aligned}
&\sum_{B=1}^{2} \varepsilon_{AB} \left[m_3^\mu(\xi) - 2\, \tau_1^{0\mu}(\xi)\, w_{c,03}^{(\theta)}(\xi) \right] \tilde{w}_{\mu B}(\xi) \\
&\qquad + 2\, w_{c,03}^{(\theta)}(\xi) \sum_{j=1}^{2} \tau_2^{0j}(\xi)\, \tilde{w}_{jA}(\xi) \;=\; 0 \;, \qquad A = 1,\, 2 \;.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{4} \Big[\sum_{i=1}^{2} g_{\|}^{(i)} \int_{\Lambda_o} \tilde{f}_{0i}^{\,2}\, dv + g_{\perp} \int_{\Lambda_o} \tilde{f}_{12}^{\,2}\, dv \\
&\quad + \sum_{i=1}^{2} l_{\|}^{(i)} \int_{\Lambda_o} \mathrm{tr}\,[\, h_{0i}^2 \,]\, dv + l_{\perp} \int_{\Lambda_o} \mathrm{tr}\,[\, h_{12}^2 \,]\, dv \Big] , \qquad (6.36)
\end{aligned}
$$

where $\tilde{f}_{\mu\nu} := \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$ is the $U(1)$ curvature (or field strength), and likewise $h_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu + [w_\mu,\, w_\nu]$, where w is given by (6.27), is the $SU(2)$ curvature. Moreover, $g_{\|}^{(i)}$, $g_{\perp}$, $l_{\|}^{(i)}$ and $l_{\perp}$ are constants of dimension of a length. Rotation invariance in the scaling limit would imply that $g_{\|}^{(1)} = g_{\|}^{(2)} =: g_{\|}$ and $l_{\|}^{(1)} = l_{\|}^{(2)} =: l_{\|}$. A discussion of the consequences of the $U(1)$ curvature terms can be found in Fröhlich and Studer (1992b). For an application of the $SU(2)$ curvature terms to a spin-pairing mechanism, see the end of Subsect. 6.3.

6.2 Linear Response Theory and Current Sum Rules

Next, we discuss the *linear response equations* (6.29) and (6.30) that follow from our (universal) expression (6.26) for the scaling limit $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$ of the effective action of systems characterized by the conditions (6.23) and (6.33)–(6.35). It is a simple exercise to verify that

$$
\begin{aligned}
\sqrt{\gamma(\xi)}\, \langle j^\mu(\xi) \rangle_{a,\, w} \;&=\; \sqrt{\gamma(\xi)}\, j_c^\mu(\xi) + \frac{\sigma}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{a}_\rho)(\xi) \\
&\quad + \frac{\chi_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{w}_{\rho 3})(\xi) + \cdots \;, \qquad (6.37)
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{\gamma(\xi)}\, \langle s_A^\mu(\xi) \rangle_{a,\, w} \;&=\; \sqrt{\gamma(\xi)}\, \delta_{A3}\, \delta_0^\mu\, m_3^0(\xi) + \delta_{A3}\, \frac{\chi_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{a}_\rho)(\xi) \\
&\quad + \delta_{A3}\, \frac{\sigma_s}{2\pi}\, \varepsilon^{\mu\nu\rho}\, (\partial_\nu \tilde{w}_{\rho 3})(\xi) \\
&\quad - \frac{k}{\pi}\, \varepsilon^{\mu\nu\rho} \Big[(\partial_\nu w_{\rho A})(\xi) - \varepsilon_{ABC}\, w_{\nu B}(\xi) w_{\rho C}(\xi) \Big] \\
&\quad + \sqrt{\gamma(\xi)}\, 2(1 - \delta_{A3})\, \delta_0^\mu\, \tau_1^{00}(\xi)\, \tilde{w}_{0A}(\xi) + \cdots \;, \qquad (6.38)
\end{aligned}
$$

where the dots stand for terms coming from irrelevant terms in the effective action, or from terms of second order in $\tilde{w}$ (e.g. a term proportional to η^{000}_{ABC}) which are of little interest in linear response theory. Furthermore, we recall that $w_{\mu A} = w^{(\theta)}_{c,\mu A} + \tilde{w}_{\mu A}$; see (6.27).

In order to understand the physical contents of these equtions, we should recall the physical meaning of the connections a and w elucidated in Subsect. 3.2: From Eqs. (3.36), (3.55), and (3.59) we know that

$$a_j(x) \;=\; -\frac{q}{\hbar c} A_j(x) - \frac{m}{\hbar} f_j(x) \;, \tag{6.39}$$

where $\mathbf{A}$ is the electromagnetic vector potential, q is the charge and m the effective mass of the particles in the quantum fluid, (for electrons, we have $q = -e$), and $\mathbf{f}$ is a divergence free velocity field generating some incompressible superfluid flow. Furthermore, by (3.36),

$$a_0(x) \;=\; \frac{q}{\hbar c} \Phi(x) \;, \tag{6.40}$$

where Φ is the electrostatic potential.

In our study of two-dimensional, incompressible quantum fluids on a surface M embedded in $\mathbb{E}^3$, it is natural to choose an $SU(2)$ gauge with the property that $e^3(t, \vec{x})$ is orthogonal to the cotangent space of M at $\vec{x}$, for all times t; see (6.23)–(6.25) and the discussion at the beginning of Subsect. 3.1. Then, by (6.25), a possible affine $SU(2)$ connection, $\omega_\mu^{(s)}$, has the form

$$\omega_\mu^{(s)}(x) \;=\; i\,\omega_\mu(x)\, L_3^{(s)} \;. \tag{6.41}$$

It then follows from (3.28)–(3.30), (3.37), (3.55), and (3.60) that

$$w_{0A}(x) \;=\; -\frac{g\,\mu}{2\hbar c} B_A(x) + \delta_{A3}\left[\,\Omega(x) + \omega_0(x)\,\right] \;, \tag{6.42}$$

where, by (3.44), $\mathbf{\Omega}(x) = (0, 0, \Omega(x)) = \frac{1}{2}\, curl\, \mathbf{f}(x)$ with respect to the orthonormal frame $(e^A(x))_{A=1}^3$ at x, and the magnetic moment of the particles is given by $g\mu\vec{L}^{(s)}$, (for electrons, we have $\mu = -\mu_B$; see (2.4)). Moreover, by (3.28)–(3.30), (3.38), (3.50), and (6.41),

$$w_{jA}(x) \;=\; \left(-\frac{g\,\mu}{2\hbar c} + \frac{q}{4mc^2}\right) \sum_{B=1}^{3} \varepsilon_{kAB}(x)\, E_B(x) + \delta_{A3}\left[\,\omega_j(x) + \cdots\right] \;, \tag{6.43}$$

where the dots correspond to terms proportional to derivatives of $\Omega(t', \vec{x})$, $t' \leq t$, (and are generated by the $SU(2)$ gauge transformation $U^{(s)}(R)$ with R defined in (3.45)).

Finally, we define, *in physical units*, the *charge density (operator)* by

$$\rho(\xi) \;:=\; q\,\sqrt{\gamma(\xi)}\, j^0(\xi) \;, \tag{6.44}$$

the *electric current density* by

$$\mathcal{J}^i(\xi) \;:=\; qc\,\sqrt{\gamma(\xi)}\, j^i(\xi) \;, \tag{6.45}$$

the *spin density* by

$$S_A^{\,o}\,(\xi) \;:=\; \frac{\hbar}{2}\,\sqrt{\gamma(\xi)}\;s_A^0(\xi)\;, \tag{6.46}$$

and the *spin current density* by

$$S_A^{\,i}\,(\xi) \;:=\; \frac{\hbar c}{2}\,\sqrt{\gamma(\xi)}\;s_A^i(\xi)\;. \tag{6.47}$$

Then, for the $(\mu = 0)$-component, Eq. (6.37) reads

$$\begin{aligned}\langle\rho(\xi)\rangle_{a,\,w} \;&=\; \rho_c(\xi) - \frac{\sigma_{\scriptscriptstyle H}}{c}\,\tilde{B}_3(\xi) - \sigma\,\frac{2qm}{h}\,\tilde{\Omega}(\xi)\\ &\qquad - \chi_s\left[\,\frac{qg\mu}{4hc}\,\vec{\nabla}\cdot\vec{\tilde{E}}(\xi) - \frac{q}{2\pi}\,\mathcal{R}(\xi)\,\right] + \cdots\;,\end{aligned} \tag{6.48}$$

where the *Hall conductivity (for the electric current)*, $\sigma_{\scriptscriptstyle H}$, is defined by

$$\sigma_{\scriptscriptstyle H} \;:=\; \frac{q^2}{h}\,\sigma\;, \tag{6.49}$$

$\vec{\tilde{E}}(\xi) := (\tilde{E}_1(\xi), \tilde{E}_2(\xi))\,,\ \vec{\nabla} = (\partial_1, \partial_2) := (\frac{\partial}{\partial\xi^1}, \frac{\partial}{\partial\xi^2}),\ \mathcal{R}(\xi) := curl\;\vec{\omega}(\xi)$ is the scalar curvature of M at ξ, and the dots stand for contributions from irrelevant terms. It will turn out that

$$\chi_\perp \;:=\; -\frac{qg\mu}{2hc}\,\chi_s \tag{6.50}$$

is the *magnetic susceptibility* of the system in the 3-direction normal to the surface. In Eq. (6.48) and the following formulas the tildes $\tilde{}$ indicate contributions from the perturbation potentials $\tilde{a}$ and $\tilde{w}$; (we have absorbed the affine spin connection ω into $\tilde{w}$, but without decorating it with a $\tilde{}$). Next, one verifies that

$$\begin{aligned}\langle\mathcal{J}^i(\xi)\rangle_{a,\,w} \;&=\; \mathcal{J}_c^i(\xi) - \sigma_{\scriptscriptstyle H}\,\varepsilon^{ij}\,\tilde{E}_j(\xi) + \sigma\,\frac{qm}{h}\,\varepsilon^{ij}\,\frac{\partial}{\partial\tau}\tilde{f}_j(\xi)\\ &\qquad - \chi_s\left[\,\frac{qg\mu}{2h}\,\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - \frac{qc}{2\pi}\,\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi)\,\right]\\ &\qquad + \chi_s\left[\,\frac{qg\mu}{4hc}\,\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) - \frac{q}{2\pi}\,\varepsilon^{ij}\,\frac{\partial}{\partial\tau}\lambda_j(\xi)\,\right] + \cdots\;,\end{aligned} \tag{6.51}$$

where $\tau := \xi^0/c$ is the rescaled time variable.

From Eq. (6.38) we find, for example, that for $\mu = 0$ and $A = 3$ (i.e., for the spin density along the 3-direction in the spin - or (co)tangent space)

$$\begin{aligned} \frac{g\mu}{\hbar} \langle S_3^{\,0}\,(\xi)\rangle_{a,\,w} \;&=\; \mathcal{M}_c(\xi) + \sigma_H^{\text{spin}} \left[\frac{g\mu}{2\hbar c} \vec{\nabla} \cdot \vec{\tilde{E}}(\xi) - 2\,\mathcal{R}(\xi) \right] + k\, \frac{g^2\mu^2}{4hc} \vec{\nabla} \cdot \vec{\tilde{E}}_c(\xi) \\ &+ \chi_\perp \left[\tilde{B}_3(\xi) - \frac{hc}{g\mu\,\pi}\,\tilde{\Omega}(\xi) \right] + \cdots \,, \end{aligned} \tag{6.52}$$

where $\mathcal{M}_c$ is the *magnetization* of the system in the background field (a_c, w_c) given by (6.23)–(6.25), $\chi_\perp$ is the magnetic susceptibility at (a_c, w_c) defined in (6.50), and

$$\sigma_H^{\text{spin}} \;:=\; \frac{g\mu}{4\pi}\,k - \frac{g\mu}{8\pi}\,\sigma_s \tag{6.53}$$

is the *Hall conductivity for the spin current*. As Eqs. (6.52) and (6.26) show, σ_H^{spin} is a *pseudoscalar*. Note that (when $\tilde{f}_j = \frac{\partial}{\partial\tau}\,\tilde{E}_j = \frac{\partial}{\partial\tau}\,\lambda_j = 0$) equations (6.51), (6.52) and (6.50) imply the *Hall law* (4.13)! Next, for $\mu = i = 1,\,2$ and $A = 3$ (i.e., for the spin current density in the i-direction in the surface M and polarized along the 3-direction in the spin - or (co)tangent space)

$$\begin{aligned} \langle S_3^{\,i}\,(\xi)\rangle_{a,\,w} \;&=\; \sigma_H^{\text{spin}} \left[\varepsilon^{ij}\partial_j \tilde{B}_3(\xi) - \frac{hc}{g\mu\,\pi}\,\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi) - \frac{1}{2c}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) + \frac{h}{g\mu\,\pi}\frac{\partial}{\partial\tau}\lambda^i(\xi) \right] \\ &+ k\,\frac{g\mu}{4\pi}\,\varepsilon^{ij}\partial_j B_c(\xi) + \chi_\perp \left[\frac{\hbar c}{g\mu}\,\varepsilon^{ij}\tilde{E}_j(\xi) - \frac{m\hbar c}{qg\mu}\,\frac{\partial}{\partial\tau}\tilde{f}^i(\xi) \right] + \cdots \,, \end{aligned} \tag{6.54}$$

where the dots stand for terms proportional to $\omega_0(\xi)$ and further irrelevant and higher-order terms. Similar equations hold for the remaining $su(2)$ components of $\langle S_A^{\,\mu}\rangle_{a,\,w}$, but we refrain from displaying them explicitly and refer the reader to the discussion in Fröhlich and Studer (1992b).

We encourage the reader to notice how neatly our formulas summarize the laws of the *Hall effect*, including effects due to *tidal forces* coming from *(superfluid) flow* and effects due to *curvature*. [We believe that the tidal terms might be relevant in the study of transitions between certain plateaux of σ_H in very pure samples.]

Let us comment on the relation of our definition of the *Hall conductivity* $\sigma_H = \frac{e^2}{h}\sigma$ as the coefficint of a Chern-Simons term, $\frac{\sigma}{4\pi}\int_{\Lambda_o} \tilde{a} \wedge \mathrm{d}\tilde{a}$ in the effective action $S^*_{\Lambda_o}(\tilde{a}, \tilde{w})$, see (6.26), of an incompressible quantum Hall fluid to the more conventional definition via the *Kubo formula* (see, e.g., Fradkin, 1991). It follows easily from Eqs. (6.4), (6.5), and (6.26) that σ appears in the following *current sum rules*: For every choice of a permutation $(\mu\,\nu\,\rho)$ of $(0\,1\,2)$,

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New applications of the chiral anomaly [1]

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Abstract

We describe consequences of the chiral anomaly in the theory of quantum wires, the (quantum) Hall effect, and of a four-dimensional cousin of the Hall effect. We explain which aspects of conductance quantization are related to the chiral anomaly. The four-dimensional analogue of the Hall effect involves the axion field, whose time derivative can be interpreted as a (space-time dependent) difference of chemical potentials of left-handed and right-handed charged fermions. Our four-dimensional analogue of the Hall effect may play a significant rôle in explaining the origin of large magnetic fields in the (early) universe.

1 What is the chiral anomaly?

The chiral abelian anomaly has been discovered, in the past century, by Adler, Bell and Jackiw, after earlier work on π^0-decay starting with Steinberger and Schwinger; see e.g. [1] and references given there. It has been rederived in many different ways of varying degree of mathematical

[1] This review is dedicated to the memory of Louis Michel, the theoretician and the friend.

(depending on the sign of σ_H). It is well known that the action $W_{\ell/r}(a)$ is the generating function for the connected Green functions of the chiral current operators, $\mathcal{J}^{\alpha}_{\ell/r}$, on $\partial\Omega$, which generate a $\hat{u}(1)$-current algebra. Formula (2.18) plays an important rôle in understanding the physics of incompressible quantum Hall fluids.

In the next section, we consider systems of massless chiral modes in four-dimensional space-time, with physical properties some of which are related to the four-dimensional chiral anomaly, and which may play a significant rôle in the *physics of the early universe.*

3 Branes, axions and charged fermions

The very early universe is filled with a hot plasma of charged leptons, quarks, gluons, photons, $\dots$. At a time after the big bang when the temperature T is of the order of 80 TeV chirality flips of light charged leptons, in particular of right-handed electrons, constitute a dynamical process slower than the expansion rate of the universe. Thus, for $T \gtrsim 80\ TeV$, the *chiral charges*, N_ℓ and N_r, defined in eq. (1.50) of Sect. 1, are approximately conserved for electrons. They are related to an approximate chiral symmetry of the electronic sector of the standard model. Among other results, we shall attempt to show that if, in the very early universe, the chemical potentials of left-handed and right-handed electrons are different from each other, this may give rise to the generation of large, cosmic magnetic fields, [15]; (see also [7] for a similar, independent suggestion). This effect is, in a sense explained in Sects. 4 and 6, an effect in *equilibrium statistical mechanics*. However, this is precisely what may make it appear quite unnatural and implausible: The chiral charges, N_ℓ and N_r, are not really conserved; leptons are massive. The very early universe is not really in an equilibrium state, and the chemical potentials of left-handed and right-handed electrons neither have an unambiguous meaning, *nor* would they be *space-* and *time-independent*. It may then be wrong, or, at least, misleading, to invoke results from *equilibrium* statistical mechanics to explore effects in the physics of the very early universe.

A way out from these difficulties can be found by seeking inspiration from an analogy with the quantum Hall effect: Consider a quantum Hall fluid (QHF), confined to a strip of macroscopic width ℓ in the plane. If the QHF is *incompressible* then there are no light (gapless) modes propagating through the bulk of the sample; but, as shown in the last section, there are gapless, chiral modes propagating along the boundaries of the sample. Let Ω denote the space-time of the fluid; it is a slab of width ℓ in three-dimensional Minkowski space. The two components of the boundary, $\partial\Omega$, of Ω are denoted by $\partial_+\Omega$, $\partial_-\Omega$, respectively. As shown in the last section, eq. (2.18), (see also [10] for more details) the effective action of such an incompressible QHF (in the scaling limit) is given by

$$S_{\text{eff}}(A) \ = \ \sigma_H \left[W_\ell \left(a_+\right) + W_r \left(a_-\right) - S_{CS}(A) \right] \ , \tag{3.1}$$

(if the direction of the external magnetic field $\vec{B}^{(0)}$ is chosen appropriately, given an orientation of Ω). In (3.1), A is an external electromagnetic vector potential on Ω, and

$$a_\pm \ := \ A \left.\right|_{\partial_\pm \Omega} \ , \tag{3.2}$$

is the restriction of the 1-form A to a component, $\partial_\pm\Omega$, of the boundary of Ω; $W_{\ell/r}(\cdot)$ is the two-dimensional, anomalous effective action for charged, chiral (left-moving, or right-moving, respectively) surface modes propagating along $\partial_+\Omega$, $\partial_-\Omega$, respectively; and $S_{CS}(\cdot)$ is the three-dimensional topological Chern-Simons action, see (2.17). Many universal features of the quantum Hall effect can be derived directly from eq. (3.1).

Suppose, in analogy to what we have just discussed, that the world, as known to us, is a movie showing the dynamics of light modes propagating along two parallel 3-branes in a five-dimensional space-time, M. More precisely, we imagine that M is a slab of width ℓ in five-dimensional space-time, $\mathbb{R}^5$, the two components, $\partial_+ M$ and $\partial_- M$, of the boundary of M being identified with the two parallel 3-branes. Let us imagine that, through the five-dimensional bulk M of the system, a massive, charged, four-component spinor field ψ propagates. We consider the response of this system to coupling the charged fermions described by ψ to a five-dimensional, external electromagnetic vector potential, $\hat{A}$. By $A_\pm$ we denote the four-dimensional vector potentials on $\partial_\pm M$ obtained by restricting $\hat{A}$ to $\partial_\pm M$. As discussed at the end of Sect. 1, there are chiral, left-handed or right-handed, charged, fermionic surface modes propagating along $\partial_+ M$, $\partial_- M$, which are coupled to A_+, A_-, respectively; see [6]. In eq. (1.74), the effective action of this system has been reported. It is given by

$$\begin{aligned} S^E_{\text{eff}}(\hat{A}) \;&=\; W_\ell(A_+) + W_r(A_-) - S_{CS}(\hat{A}) \\ &\quad + (4\,\ell\, e^2)^{-1} \int_M d^5\xi \; F_{\hat{A}}(\xi)^2 + \dots\;, \end{aligned} \tag{3.3}$$

where the dots stand for terms $\sim O\left(\frac{1}{m}\right)$, and the renormalization conditions have been chosen in such a way that the constant e^2 in front of the five-dimensional Maxwell term is the four-dimensional feinstructure constant. The components, $\hat{A}_K$, of $\hat{A}$ are denoted by

$$\hat{A}_\mu \;=:\; A_\mu\;, \quad \mu \;=\; 0,1,2,3, \quad \hat{A}_4 \;=:\; \varphi\;, \tag{3.4}$$

i.e., $(\hat{A}_K) = (A, \varphi)$, $K = 0,1,2,3,4$.

In order to make contact with the laws of physics in four space-time dimensions, we should insist on the requirement that left-handed and right-handed fermions propagating along $\partial_+ M$ and $\partial_- M$, respectively, couple to the *same* electromagnetic vector potential, i.e., that

$$A_+\left(x, x^4 = \ell\right) \;=\; A_-\left(x, x^4 = 0\right) \;\equiv\; A(x)\;. \tag{3.5}$$

This requirement is met if we assume that

$$\hat{A}\left(x, x^4\right) \quad \text{is } \textit{independent} \text{ of } x^4\;. \tag{3.6}$$

In this case,

$$\begin{aligned} S_{CS}(\hat{A}) \;&=\; \frac{i\,\ell}{32\,\pi^2} \int_N \varphi\,(F_A \wedge F_A) \\ &=\; \frac{i\,\ell}{32\,\pi^2} \int_N d^4x\; \varphi(x)\; \varepsilon^{\mu\nu\lambda\rho}\, F_{\mu\nu}(x)\, F_{\lambda\rho}(x) \end{aligned} \tag{3.7}$$

where $N \cong \mathbb{R}^4$ is a slice through M parallel to $\partial_\pm M$, $\mu, \nu = 0,1,2,3$, and $F_A = (F_{\mu\nu})$ is the four-dimensional field tensor; (the trivial integration over x^4 has produced the factor ℓ). Furthermore, the Maxwell term on the R.S. of (3.3) reduces to

$$\frac{1}{4\,e^2}\;\left\{\int_N d^4x\; F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \;2\int_N d^4x\; (\partial_\mu\varphi)\,(x)\,(\partial^\mu\varphi)\,(x)\right\}\;. \tag{3.8}$$

Finally,

$$W_\ell\,(A_+ = A) + W_r\,(A_- = A) \;=\; S^E_{\text{eff}}(A)\;, \tag{3.9}$$

with $S^E_{\text{eff}}(A) = S^E_{\text{eff}}(A, Z = 0)$ as in eqs. (1.29), (1.30). Thus, the complete effective action of the system is given by

$$\begin{aligned} S^E_{\text{eff}}\,(\varphi; A) \;&=\; S^E_{\text{eff}}(A) \;+\; \frac{i\ell}{32\,\pi^2}\int_N \varphi\,(F_A \wedge F_A) \\ &+\; \frac{1}{4\,e^2}\left\{\int_N d^4x\; F_A^2(x) + 2\int_N d^4x\,(\nabla\varphi)^2\,(x)\right\}\;. \end{aligned} \tag{3.10}$$

Clearly, there is something quite unnatural about this approach: It is conditions (3.5) and (3.6)! If A_+ were different from A_- then the fermionic effective action $S^E_{\text{eff}}(A) = S^E_{\text{eff}}(A, Z = 0)$ would be replaced by $S^E_{\text{eff}}(A, Z)$, where $A = \frac{1}{2}(A_+ + A_-)$ and $Z = \frac{1}{2}(-A_+ + A_-)$. Thus the surface modes would not only couple to the electromagnetic field, but also to a chiral gauge field Z for which there is no experimental evidence, and the gauge fields would sample a five-dimensional space-time.

These unnatural features can be avoided by following *Connes' formulation* of gauge theories with fermions [16]. Then the effective action displayed in eq. (3.10) can be reproduced as follows: One sets $M = N \times \mathbb{Z}_2$, $N \cong \mathbb{R}^4$ and treats the discrete "fifth dimension", $\mathbb{Z}_2$, by using elementary tools from non-commutative geometry [16]. By adding a "non-commutative", five-dimensional Chern-Simons action, as constructed in [17], to Connes' version of the Yang-Mills action (for a U(1)-gauge field) and to the standard fermionic effective action, one can reproduce actions like the one in eq. (3.10); see [17]. There is no room, here, to review the details of these constructions.

In analogy to what we have discussed above, one may argue that string theories arise as effective theories of surface modes propagating along 9-branes in an "eleven-dimensional" space-time, starting from eleven-dimensional M-theory, (with anomalies of the surface theories cancelled by certain eleven-dimensional Chern-Simons actions). One realization of this idea appears in [18]. But we shall not pursue these ideas any further, in this review.

Instead, we ask whether the effective action in (3.10) ought to look familiar to people holding a conventional point of view that physical space-time is four-dimensional. The answer is "yes"! The scalar field φ appearing in the effective action on the R.S. of (3.10) can be interpreted as the *axion*. The axion field was originally introduced by Peccei and Quinn [19] to solve the strong CP problem. There are various reasons, including, primarily, experimental ones, to feel unhappy about introducing an axion into the standard model. But there is also a good reason to do so: String theory predicts the existence of an axion, the "model-independent axion" first described by Witten [20].

The argument in favor of the model-independent axion goes as follows: String theory tells us that there must exist a second-rank antisymmetric tensor field, i.e., a two-form, $B_{\mu\nu}$. The gauge-invariant field strength, H, a three-form, corresponding to B is given by

$$H \;=\; dB - \omega_{3YM} + \omega_{3G}\;, \tag{3.11}$$

where d denotes exterior differentiation, and ω_{3YM} and ω_{3G} are the gauge-field ("Yang-Mills") and gravitational (Lorentz) Chern-Simons three-forms. (The coefficients in front of these Chern-Simons forms are proportional to the number, N_f, of species of fermions coupled to the gauge-and gravitational fields. In the following we shall set $N_f = 1$.) The field strength H is invariant under the gauge transformations $B \to B + d\lambda$, where λ is an arbitrary one-form, and under gauge- and local Lorentz transformations accompanied by shifts of B. The equation of motion of H is

$$\partial^\mu \, H_{\mu\nu\lambda} \;=\; 0 \;, \tag{3.12}$$

or $\delta H = 0$, where δ is the co-differential. We consider the components of $B_{\mu\nu}$ with $\mu, \nu = 0, \ldots, 3$ and assume that B is independent of coordinates of *internal* dimensions (of the string theory target). Then, in four-dimensional (non-compact) space-time, the three-form H is dual to a one-form, Z, and the equation of motion (3.12) becomes

$$\partial_\mu \, Z_\nu - \partial_\nu \, Z_\mu \;=\; 0 \;, \quad \text{or} \quad d\, Z \;=\; 0 \;. \tag{3.13}$$

By Poincaré's lemma,

$$Z_\mu \;=\; \partial_\mu \, \alpha \;, \quad \text{or} \quad Z \;=\; d\, \alpha \;, \tag{3.14}$$

where α is a scalar field. By (3.11), the scaling dimension of α is two. Introducing a constant, ℓ, with the dimension of length, we set

$$\alpha \;=\; \frac{1}{\ell \, e^2} \, \varphi \;, \tag{3.15}$$

where φ has scaling dimension $= 1$; (e^2 is the feinstructure constant).

From $d^2 = 0$ and (3.11) we obtain the equation

$$dH(x) \;=\; *\mathcal{A}(x) + \text{const. tr} \; (R(x) \wedge R(x)) \tag{3.16}$$

where

$$*\, \mathcal{A}(x) \;=\; - \, \frac{i}{32 \, \pi^2} \; (F(x) \wedge F(x)) \tag{3.17}$$

is the index density, see eq. (1.59), ($*$ denotes the Hodge dual), and $R(x)$ is the Riemann curvature tensor. Assuming that space-time is flat, hence $R = 0$, and considering the special case, where the electromagnetic field is the only gauge field in the system, we obtain

$$dH \;=\; - \, \frac{i}{32 \, \pi^2} \; (F_A \wedge F_A) \;\; . \tag{3.18}$$

Recalling that

$$H \;=\; * \, \left(\frac{1}{\ell \, e^2} \, d\, \varphi \right) \;,$$

see (3.13)–(3.15), we find that (3.18) yields the following equation of motion for φ:

$$\Box \, \varphi \;=\; - \, \frac{i \ell \, e^2}{32 \, \pi^2} \; * (F_A \wedge F_A) \;\; . \tag{3.19}$$

This equation is the Euler-Lagrange equation corresponding to the action functional

$$\frac{1}{2 \, e^2} \int d^4 x \; (\nabla \varphi)^2 \, (x) + \frac{i \ell}{32 \, \pi^2} \int \varphi \, (F_A \wedge F_A) \;\; , \tag{3.20}$$

which reproduces the R.S. of (3.10), up to the fermionic effective action and the Maxwell term! The second term in (3.20) can be understood as arising from coupling *fermions* to the axion. The term in the bare action of the fermions describing their coupling to the axion is given by

$$\frac{\ell^2}{2} \int d^4x \, H_{\mu\nu\lambda} \, \bar{\psi}\gamma^\mu\gamma^\nu\gamma^\lambda\psi \;=\; \frac{\ell}{2} \int d^4x \; \partial_\mu\varphi\bar{\psi}\gamma^\mu\gamma\psi \;, \tag{3.21}$$

where $\gamma = \gamma^5$. Carrying out the Berezin integral over the fermionic degrees of freedom — see eq. (1.29) — we find an effective action for the fermions given by

$$\begin{aligned} S^E_{\text{eff}}\left(A, Z \;=\; \frac{\ell}{2}\, d\varphi\right) \;&=\; S^E_{\text{eff}}\,(A, Z = 0) \;-\; i\ell \int d^4x \; \varphi(x) \; \mathcal{A}(x) \\ &=\; S^E_{\text{eff}}(A) \;-\; \frac{i\ell}{32\,\pi^2} \int \varphi \, (F_A \wedge F_A) \;\;, \end{aligned} \tag{3.22}$$

in accordance with (3.20). The first equation in (3.22) is eq. (1.41), the second follows from (1.59).

Thus, coupling charged Dirac fermions to an external electromagnetic vector potential A and an axion φ yields the effective action (3.22). Adding to it the Maxwell term and the kinetic energy term for φ, we again obtain the action (3.10)!

One may argue that, in any case, the presence of an axion in the theory may be an indication that there must exist *extra* (classical or, perhaps more plausibly, discrete or "non-commutative") *dimensions*. But, for our applications in Sect. 6, this point is not important. What *will* matter is that the time derivative of the axion field will play the rôle of a, generally speaking, space-time dependent *"chemical potential"* for right-handed leptons.

But, quite independently of the properties of fermions (which, for example, may acquire masses through a Higgs-Kibble mechanism), the axion, φ, will turn out to be the *driving force* for a possible generation of large cosmic magnetic fields.

As our discussion at the beginning of this section, up to eq. (3.10), has shown it is legitimate to view a four-dimensional system of fermions in an external electromagnetic and an external axion field as the four-dimensional analogue of the edge degrees of freedom of an incompressible quantum Hall fluid. It supports electric currents analogous to the diamagnetic edge currents of a quantum Hall fluid.

4 Transport in thermal equilibrium through gapless modes

In this section we prepare the ground for a theoretical explanation of effects such as the ones described in Sects. 2 (Examples 1 through 3) and 3. We consider a quantum-mechanical system $\mathcal{S}$ whose dynamics is determined by a Hamiltonian H, which is a selfadjoint operator on the Hilbert space $\mathcal{H}$ of pure state vectors of $\mathcal{S}$ with discrete energy spectrum. It is assumed that the system obeys conservation laws described by some conserved "charges" $N_1, \ldots, N_L$ commuting with all observables of the system. Hence

$$[H, N_\ell] \;=\; 0 \;, \quad [N_\ell, N_k] \;=\; 0 \;, \quad \ell, k \;=\; 1, \ldots, L \;, \tag{4.1}$$

see also (2.4) (with $\sigma_L = 0$). Thus

$$I_H^{\text{bulk}} \;=\; \sigma_H \int_\gamma \underline{E}\,(\underline{x}, t)\;\cdot\; d\underline{s}(\underline{x})\;. \tag{5.19}$$

We have shown in eq. (5.9) that

$$I_H^{\text{edge}} \;=\; \sigma_H\;(\mu_\ell - \mu_r)\;\;. \tag{5.20}$$

Thus, combining (5.15), (5.19) and (5.20), we conclude that

$$I_H \;=\; I_H^{\text{edge}} + I_H^{\text{bulk}} \;=\; \sigma_H \left(\mu_\ell - \mu_r + \int_\gamma \underline{E}\,(\underline{x}, t)\;\cdot\; d\underline{s}\,(\underline{x})\right)\;\;. \tag{5.21}$$

But the expression in the parenthesis on the R.S. of (5.21) is nothing but the *total voltage drop* V between the outer and the inner edge. Hence (5.21) implies that

$$I_H \;=\; \sigma_H\;V\;\;, \tag{5.22}$$

as desired.

Transport phenomena such as *heat conduction* through a quantum wire or a Hall sample (see Example 3 at the beginning of Sect. 2) can be studied along similar lines: In a physical system where modes of *different* chirality do not interact with each other (such as the modes at the inner and at the outer edge of the sample containing an incompressible Quantum Hall fluid) the left-moving and the right moving modes can be coupled to different reservoirs at *different* temperatures β_ℓ^{-1} and β_r^{-1}. This results in a *non-zero* heat current given by an expectation value of the component T^{01} of the energy-momentum tensor of the conformal field theory describing the chiral modes in an equilibrium state where the left-movers are at temperature β_ℓ^{-1} and the right-movers at temperature $\beta_r^{-1} \neq \beta_\ell^{-1}$. (Such expectation values can be calculated from Virasoro characters.) These ideas lead to a conceptually clean understanding of the effects described in Example 3 at the beginning of Sect. 2.

6 A four-dimensional analogue of the Hall effect, and the generation of large, cosmic magnetic fields in the early universe

In this section, we further explore the four-dimensional analogue of the Hall effect described in Sect. 3. We shall apply our findings to exhibit effects that may play an important rôle in early-universe cosmology. Our results represent an elaboration upon those in [15, 7].

We start our analysis by studying a system of massless Dirac fermions coupled to an external electromagnetic field in four-dimensional Minkowski space. Using results derived in Sects. 1 and 4, we derive equations analogous to eqs. (5.3)–(5.6) for the conductance of a quantum wire.

From Sect. 1 we recall the expression for the *anomalous commutators* between vector- and axial-vector — or chiral currents.

$$\left[\widehat{\mathcal{J}}^0_{\ell/r}\,(t, \underline{x})\,,\;\widehat{\mathcal{J}}^0_{\ell/r}\,\bigl(t, \underline{y}\bigr)\right] \;=\; \pm\; i\;\frac{q^2}{4\pi^2}\;\left(\underline{B}\,(\underline{x}, t)\;\cdot\;\underline{\nabla}\right)\;\delta\left(\underline{x} - \underline{y}\right)\;\;, \tag{6.1}$$

where q is the charge of the fermions — see eq. (1.62) — and

$$\left[\widehat{\mathcal{J}}_\ell^0\,(t,\underline{x})\ ,\ \widehat{\mathcal{J}}_r^0\,(t,\underline{y})\right] \;=\; 0\ . \tag{6.2}$$

With (1.45) and (1.48), these equations yield

$$\left[\widehat{\mathcal{J}}_{\ell/r}^0\,(t,\underline{y})\ ,\ \mathcal{J}^0\,(t,\underline{x})\right] \;=\; \pm\, i\, \frac{q^2}{8\pi^2}\, \left(\underline{B}\left(\underline{y},t\right)\,\cdot\,\underline{\nabla}_{\underline{x}}\right)\, \delta\left(\underline{x}-\underline{y}\right)\ , \tag{6.3}$$

where $\mathcal{J}^\mu$ is the μ-component of the conserved vector current. In Sect. 4, we have introduced the vector potential, $\underline{\varphi}$, of $\mathcal{J}^\mu$:

$$\mathcal{J}^0(x) \;=\; \frac{q}{2\pi}\,\mathrm{div}\,\underline{\varphi}(x)\,,\ \underline{\mathcal{J}}(x) \;=\; -\,\frac{q}{2\pi}\,\frac{\partial}{\partial t}\,\underline{\varphi}(x)\ . \tag{6.4}$$

Eqs. (6.3) and (6.4) imply that

$$\begin{aligned}\left[\widehat{\mathcal{J}}_{\ell/r}^0\left(\underline{y},t\right)\ ,\ \underline{\varphi}\left(\underline{x},t\right)\right] \;&=\; \pm\, i\,\frac{q}{4\pi}\,\underline{B}\left(\underline{y},t\right)\ \delta\left(\underline{x}-\underline{y}\right)\\ &\pm\ \ \mathrm{curl}\,\underline{\Pi}\left(\underline{x}-\underline{y},t\right)\end{aligned} \tag{6.5}$$

where $\underline{\Pi}$ is some vector-valued distribution.

Next, we recall that the operators

$$N_{\ell/r}\ :=\ \int d\underline{y}\ \widehat{\mathcal{J}}_{\ell/r}^0\ \left(\underline{y},t\right) \tag{6.6}$$

are *conserved.* They are interpreted as the electric charge operators for left-handed/right-handed fermionic modes. The chemical potentials conjugate to $N_{\ell/r}$ are denoted by $\mu_{\ell/r}$. Let us imagine that, at *very early times* in the evolution of our universe (or others), there was an asymmetry in the population of left-handed and right-handed fermionic modes, (as argued in [15] for the example of electrons before the electroweak phase transition). Then

$$\mu_\ell\ \neq\ \mu_r\ , \tag{6.7}$$

in the state of the universe at those very early times. Let us furthermore imagine that the state of the universe at those early times was, to a good approximation, a thermal equilibrium state at an inverse temperature β ($\lesssim (80\ TeV)^{-1}$, as argued in [15]) and with chemical potentials μ_ℓ and μ_r. (It may well be that this is an unrealistic assumption. — It will subsequently turn out that it is unimportant!)

Under these assumptions, we may apply the *current sum rule* (4.12) derived in Sect. 4. Combining eqs. (6.5), (6.6) and (4.12), and using that $\int_{\mathbb{R}^3} d\underline{y}\ \mathrm{curl}\ \underline{\Pi}\left(\underline{x}-\underline{y},t\right) = 0$, for all $\underline{x},t$, we find that

$$\begin{aligned}\langle\underline{\mathcal{J}}(x)\rangle_{\beta,\underline{\mu}} \;&=\; \frac{iq}{h}\ \left\{\mu_\ell\langle\,\left[N_\ell,\underline{\varphi}(x)\right]\,\rangle_{\beta,\underline{\mu}}\ +\ \mu_r\ \langle\,\left[N_r,\underline{\varphi}(x)\right]\,\rangle_{\beta,\underline{\mu}}\right\}\\ &=\; -\ \frac{q^2}{4\pi\, h}\ \left(\mu_\ell-\mu_r\right)\ \underline{B}(x)\ ,\end{aligned} \tag{6.8}$$

as claimed in [7]. This equation is the analogue of (5.6).

Treating the electromagnetic field as a classical, but *dynamical* field, its dynamics is governed by Maxwell's equations,

$$\underline{\nabla} \cdot \underline{B} \;=\; 0 \;, \quad \underline{\nabla} \wedge \underline{E} \;+\; \partial_t \, \underline{B} \;=\; 0 \;,$$

and

$$\underline{\nabla} \cdot \underline{E} \;=\; \langle \mathcal{J}^0 \rangle_{\beta, \underline{\mu}} \;, \quad \underline{\nabla} \wedge \underline{B} - \partial_t \, \underline{E} \;=\; \langle \underline{\mathcal{J}} \rangle_{\beta, \underline{\mu}} \;. \tag{6.9}$$

There is no reason to imagine that the charge density, $\langle \mathcal{J}^0 \rangle_{\beta, \underline{\mu}}$, in the very early universe is different from zero. In the last equation of (6.9), the current on the R.S. is given by eq. (6.8). Actually, assuming that there are some *dissipative processes* evolving in the early universe, an equation for the current,

$$\underline{J}(x) \;:=\; \langle \underline{\mathcal{J}}(x) \rangle_{\beta, \underline{\mu}} \;,$$

more realistic than (6.8) may be

$$\underline{J}(x) \;=\; \sigma_L \, \underline{E}\,(x) \;+\; \sigma_T \, V \, \underline{B}\,(x) \;, \tag{6.10}$$

where σ_L is an Ohmic longitudinal conductivity, and

$$\sigma_T \;:=\; - \, \frac{q^2}{4\pi \, h} \tag{6.11}$$

is the analogue of the *"transverse"* or *Hall conductivity*; furthermore,

$$V \;:=\; \mu_\ell - \mu_r \tag{6.12}$$

is the analogue of the *voltage drop* considered in the Hall effect. The quantity σ_T is *"quantized"*, just like the Hall conductivity: If there are $N > 1$ species of charged, massless fermions, with electric charges $q_1, \dots, q_N$, then

$$\sigma_T \;=\; - \, \frac{1}{4\pi \, h} \left(\sum_{j=1}^{N} q_j^2 \right) \;, \tag{6.13}$$

which is the precise analogue of a formula for the quantization of the Hall conductivity derived in [8], and, for $q_j = \pm\, e, \; j = 1, \dots, N$, of eq. (5.6).

Let us temporarily assume that $\sigma_L = 0$, (i.e., we neglect dissipative processes). Then Maxwell's equations, together with eq. (6.10) (for $\sigma_L = 0$) and the assumption that the charge density vanishes, yield the following system of linear equations:

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} \;&=\; 0 \;, \quad \underline{\nabla} \wedge \underline{E} \;+\; \partial_t \, \underline{B} \;=\; 0 \;, \\ \underline{\nabla} \cdot \underline{E} \;&=\; 0 \;, \quad \underline{\nabla} \wedge \underline{B} \;-\; \partial_t \, \underline{E} \;=\; \sigma_T \, V \, \underline{B} \;. \end{aligned} \tag{6.14}$$

Because all coefficients are constant, these equations can be solved by Fourier transformation, and it is enough to construct propagating wave solutions corresponding to an arbitrary, but fixed wave vector $\underline{k}$. The equations $\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot \underline{E} = 0$ imply that

$$\underline{k} \cdot \widehat{\underline{B}} \;=\; \underline{k} \cdot \widehat{\underline{E}} \;=\; 0 \;, \tag{6.15}$$

i.e., that only the components of the Fourier transforms $\widehat{\underline{B}}$ and $\widehat{\underline{E}}$ of $\underline{B}$ and $\underline{E}$ (evaluated at the wave vector $\underline{k}$) perpendicular to $\underline{k}$ can be non-zero. Denoting the components of $\widehat{\underline{B}}$ and $\widehat{\underline{E}}$ perpendicular to $\underline{k}$ by $\widehat{\underline{B}}^T$, $\widehat{\underline{E}}^T$, respectively, the remaining equations in (6.14) yield

$$\partial_t \begin{pmatrix} \widehat{\underline{E}}^T \\ \widehat{\underline{B}}^T \end{pmatrix} = K(\underline{k}) \begin{pmatrix} \widehat{\underline{E}}^T \\ \widehat{\underline{B}}^T \end{pmatrix} , \tag{6.16}$$

where (in an orthonormal basis chosen in the plane perpendicular to $\underline{k}$) the matrix $K(\underline{k})$ is given by

$$K(\underline{k}) = \begin{pmatrix} 0 & 0 & -\sigma_T V & -ik \\ 0 & 0 & ik & -\sigma_T V \\ 0 & ik & 0 & 0 \\ -ik & 0 & 0 & 0 \end{pmatrix} \tag{6.17}$$

with $k = |\underline{k}|$. The circular frequency of a propagating wave solution of (6.14) with wave vector $\underline{k}$ is given by $\omega(\underline{k})$, where $i\,\omega(\underline{k})$ is an eigenvalue of $K(\underline{k})$. By (6.17),

$$\omega(\underline{k})^2 = k^2 \pm k\,\sigma_T V , \tag{6.18}$$

as one readily checks. Thus, if

$$|\underline{k}| = k < \sigma_T V \tag{6.19}$$

there are two purely imaginary frequencies, and eqs. (6.14) have solutions $(\underline{B}(\underline{x},t),\ \underline{E}(\underline{x},t))$ growing *exponentially fast in time* and with the property that

$$\underline{B}(\underline{x},t) \cdot \underline{E}(\underline{x},t) \neq 0 . \tag{6.20}$$

It is almost as easy to solve Maxwell's equations (6.9), with $\underline{J}$ given by (6.10), for $\sigma_L \neq 0$. For wave vectors $\underline{k}$ satisfying

$$\sigma_L^2 < |\underline{k}|\,\sigma_T V < (\sigma_T V)^2 , \tag{6.21}$$

one again finds exponentially growing electromagnetic fields; (perturbation theory). Dissipative processes will subsequently damp out electric fields.

In [15], calculations similar to those just presented are used to argue that, in the very early universe, large, cosmic electromagnetic fields may have been generated as a consequence of an asymmetric population of left-handed and right-handed electron modes ($q = -e$). However, these arguments rest on rather shaky hypotheses; (the state of the early universe is assumed to be a thermal equilibrium state, and the charges N_ℓ and N_r, see eq. (6.6), are assumed to be approximately conserved). We propose to reconsider these arguments in the light of the analogy between the (2+1)-dimensional (bulk) description of the Hall effect and the (4+1)-dimensional description of chiral fermions discussed at the beginning of Sect. 3, eqs (3.3) through (3.10). What we have described, so far, in this section are calculations analogous to those reported in eqs. (5.6), (5.8) and (5.9). Next, we generalize our analysis in a way analogous to that followed in eqs. (5.15) through (5.22), starting from the effective action given in (3.10); (see also (3.20)).

We integrate out all degrees of freedom (quarks, gluons, leptons, the weak gauge fields — W, Z — etc.), except for the *electromagnetic* and the *axion field*. We have seen, at the beginning of Sect. 3, eqs. (3.4), (3.10), that the axion could be viewed as the four-component of a five-dimensional electromagnetic vector potential, $\widehat{A}$, which does not depend on the coordinate, x^4, in the direction perpendicular to the four-dimensional branes on which we live; see (3.6). We could pursue a five- (or higher-) dimensional approach to early-universe cosmology (as presently popular), — but let's not! We propose to view the axion as the "model-independent (invisible)" axion first described in [20]. It has a geometrical origin (in superstring theory). It couples to *all* gauge fields present in the system through a term

$$\frac{i\,\ell}{32\,\pi^2}\int \varphi\,(F_W \wedge F_W)\ , \tag{6.22}$$

where F_W is the field strength of a gauge field W appearing in our theoretical description, and to the curvature tensor R; see (3.16). All gauge fields, except for the electromagnetic vector potential A, shall be integrated out. The (Euclidian-region-) functional integrals have the form

$$\int d\mu(W)\ \exp\left[\frac{i\,\ell}{32\,\pi^2}\int \varphi\,(F_W \wedge F_W)\right] \ =:\ e^{-U(\varphi)}\ . \tag{6.23}$$

Since $\frac{i}{32\,\pi^2}$ $(F_W \wedge F_W)$ is the index density, the integrand in $U(\varphi)$ can be shown to be *periodic* in φ, for φ independent of x, with period $\frac{2\pi}{\ell}$. It is known that (somewhat loosely speaking) $d\mu$ is a positive measure and that it is invariant under space reflection, which changes the sign of $\int F_W \wedge F_W$. It follows that $\exp(-U(\varphi))$ is real and has its maxima at $\varphi = \frac{2\pi}{\ell}\,n,\quad n = 0, \pm 1, \pm 2, \dots$. (See e.g. [25] for more details.)

A transition amplitude from a configuration $(A_{\text{in}}, \varphi_{\text{in}})$ of the electromagnetic — and the axion field at a very early time, t_1, to a configuration $(A_{\text{out}}, \varphi_{\text{out}})$ at a much later time, t_2, can be computed from the Feynman path integral

$$\int \mathcal{D}A\ \mathcal{D}\varphi\ e^{i\,S_{\text{eff}}(A,\varphi)/\hbar}\ , \tag{6.24}$$

with boundary conditions $(A(t_1), \varphi(t_1)) = (A_{\text{in}}, \varphi_{\text{in}})$ and $(A(t_2), \varphi(t_2)) = (A_{\text{out}}, \varphi_{\text{out}})$. In (6.24), $S_{\text{eff}}(A, \varphi)$ denotes the *total* effective action over Minkowski space. It is obtained from $S^E_{\text{eff}}(A, \varphi)$, the effective action in the Euclidian region, by undoing the Wick rotation described in eq. (1.28). By eqs. (3.10) or (3.20) and (6.23), $S_{\text{eff}}(A, \varphi)$ has the general form

$$\begin{aligned} S_{\text{eff}}(A,\varphi) \ &=\ \frac{1}{4\,e^2}\int d^4x\,\{F_{\mu\nu}(x)\,F^{\mu\nu}(x)\,+\,2\,(\partial_\mu\varphi)\,(x)\,(\partial^\mu\varphi)\,(x)\} \\ &+\ \frac{\ell}{32\,\pi^2}\int \varphi(x)\,(F\wedge F)\,(x)\,-\,U(\varphi)\,+\,W(A)\ , \end{aligned} \tag{6.25}$$

where $W(A)$ is of higher than second order in A and arises from integrating out all charged fields in the theory*; furthermore, e^2 is the effective (one-loop renormalized) feinstructure constant. It is *not* necessary, in this approach, to assume that all the fermions in the theory be massless. They can acquire masses through the Higgs–Kibble mechanism. (The arguments of complex chiral Higgs fields then contain a term proportional to the axion field φ which, however, can

*W depends on the boundary conditions, at times t_1, t_2, imposed on the fields that have been integrated out.

be absorbed in a change of variables.) Furthermore, calculating transition amplitudes with the help of Feynman path integrals does not presuppose that the system is in or close to thermal equilibrium.

We now insert expression (6.25) into the functional integral (6.24) and try to evaluate the latter by using a semi-classical expansion based on the stationary-phase method. The equations for the saddle point are

$$\delta S_{\text{eff}}(A,\varphi)/\delta A_\mu(x) \;=\; 0\;, \quad \delta S_{\text{eff}}(A,\varphi)/\delta\varphi(x) \;=\; 0\;. \tag{6.26}$$

To simplify matters, we consider solutions of these equations describing fairly *small* electromagnetic fields and an axion field that varies only *slowly* in space-time. Then we can neglect the term $W(A)$ in (6.25) and we may omit all contributions to $U(\varphi)$ involving *derivatives*, $\partial_\mu\varphi$, of the axion field φ. The saddle point equations (6.26) then yield the following coupled Maxwell–Dirac-axion equations:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{\ell\, e^2}{8\,\pi^2}\,\partial_\mu\left(\varphi\,\widetilde{F}^{\mu\nu}\right)\;, \\ \Box\,\varphi &= \frac{\ell\, e^2}{32\,\pi^2}\;*\;(F\wedge F) - U'(\varphi)\;, \end{aligned} \tag{6.27}$$

(and we have set $c=1$ and $\hbar=1$). Let J^μ_M denote the magnetic current that could be present if there were magnetic monopoles moving through the early universe. Then the full set of Maxwell–Dirac-axion equations reads

$$\begin{aligned} \partial_\mu\,\widetilde{F}^{\mu\nu} &= J^\nu_M\;,\quad \partial_\mu\,F^{\mu\nu} \;=\; \frac{\ell\, e^2}{8\,\pi^2}\,\left\{(\partial_\mu\varphi)\;\widetilde{F}^{\mu\nu} + \varphi\,J^\nu_M\right\}\;, \\ \Box\,\varphi &= \frac{\ell\, e^2}{32\,\pi^2}\;*\;(F\wedge F) \;-\; U'(\varphi)\;. \end{aligned} \tag{6.28}$$

The first equation in (6.28) replaces the homogeneous Maxwell equations, $\left(\partial_\mu\widetilde{F}^{\mu\nu}=0,\quad \text{for } J^\nu_M=0\right)$. In vector notation, the system of equations (6.28) reads

$$\begin{aligned} \underline{\nabla}\cdot\underline{B} &= J^0_M\;,\quad \underline{\nabla}\wedge\underline{E} + \underline{\dot{B}} \;=\; \underline{J}_M\;, \\ \underline{\nabla}\cdot\underline{E} &= \frac{\ell\, e^2}{8\,\pi^2}\,\{(\underline{\nabla}\varphi)\cdot\underline{B} + \varphi\,J^0_M\}\;, \\ \nabla\wedge\underline{B} - \underline{\dot{E}} &= -\frac{\ell\, e^2}{8\,\pi^2}\,\{\dot{\varphi}\,\underline{B} + \underline{\nabla}\varphi\wedge\underline{E} + \varphi\,\underline{J}_M\}\;, \\ \Box\,\varphi &= -\frac{\ell\, e^2}{8\,\pi^2}\,\underline{E}\cdot\underline{B} - U'(\varphi)\;. \end{aligned} \tag{6.29}$$

In order to gain some insight into properties of solutions of these highly *non-linear* equations, we study their linearization around various special solutions. Already this part of the analysis, let alone a study of the full, non-linear equations, is quite lengthy; see [26] for a beginning. Here we just sketch results in a few interesting special situations.

We shall first assume that $J^\mu_M\equiv 0$, i.e., that there aren't any magnetic monopoles around.

(i) We set $U(\varphi) = 0$ and consider the following special solution of eqs. (6.29).

$$\begin{aligned} \underline{E} &= \underline{B} \equiv 0 , \\ \varphi(\underline{x}, t) &= \frac{V}{\ell} \cdot t , \end{aligned} \tag{6.30}$$

where V is a constant. Linearizing (6.29) around (6.30), we obtain the equations

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 , \quad \underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = 0 , \\ \underline{\nabla} \cdot \underline{E} &= 0 , \quad \underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = -\frac{e^2}{8\pi^2} V \underline{B} , \\ \Box \varphi &= 0 . \end{aligned} \tag{6.31}$$

With the exception of the wave equation for the axion field φ, these equations are *identical* to eqs. (6.14), with $\sigma_T = \frac{e^2}{8\pi^2}$. Had we not set $\hbar = 1$, the equation for σ_T would read

$$\sigma_T = -\frac{e^2}{4\pi h} ,$$

which is precisely eq. (6.11), with $q = e$! Recall that, in the analysis presented at the beginning of this section,

$$V = \mu_\ell - \mu_r .$$

This equation and (6.30) tell us that, apparently, the field $\ell\dot{\varphi}$ has the interpretation of the *difference of chemical potentials* of *left-* and *right-handed fermions*! This interpretation magically fits with the *five-dimensional* interpretation of the axion field φ as the four-component, $\widehat{A}_4$, of an electromagnetic vector potential $\widehat{A}$ defined over a slab of height ℓ in five-dimensional Minkowski space; see eqs. (3.4), (3.6) and (3.10). Then

$$\dot{\varphi} = \dot{\widehat{A}}_4 = E_4$$

is the four-component of the electric field. Integrating E along an oriented curve, γ, joining a point on the lower face of the slab to a point on the upper face, at fixed time, we obtain

$$\int_\gamma \sum_{K=1}^{4} E_K \cdot ds^K = \int_0^\ell dx^4 \, E_4(\xi) = \int_0^\ell dx^4 \, \dot{\varphi}(\xi) , \tag{6.32}$$

where $\xi = (x, x^4) = (t, \underline{x}, x^4)$, and we have assumed in the first equality that E does not depend on x^4 (see assumption (3.6)) and E_4 does not depend on $\underline{x}$. Since, for solution (6.30),

$$E_4(\xi) = \dot{\varphi}(\xi) = \frac{V}{\ell} ,$$

eq. (6.32) yields

$$\int_\gamma \sum_{K=1}^{4} E_K \cdot ds^K = V . \tag{6.33}$$

This shows that, in the five-dimensional interpretation of the axion, V is the "voltage drop" between the two four-dimensional branes corresponding to the lower and upper face of the five-dimensional slab. This observation makes the analogy between the effects studied here and the Hall effect yet a little more precise.

Solutions of eqs. (6.31) have been studied earlier in this section; see (6.16) through (6.20). They have unstable modes growing exponentially in time, with $\underline{B}(\underline{x},t)\cdot\underline{E}(\underline{x},t)\neq 0$.

(ii) Now $U(\varphi)\neq 0$; $U'(\varphi)(x) := \delta U(\varphi)/\delta\varphi(x)$ is a periodic function with minima at $\frac{2\pi}{\ell}\,n$, $n=0,\pm1,\pm2,\dots$. We linearize equations (6.29) around the solution $\underline{E}=\underline{B}\equiv 0$, $\varphi=\varphi_c(t)$, where $\varphi_c(t)$ solves the equation

$$\ddot{\varphi}(t) \;=\; -\,U'\left(\varphi\left(t\right)\right)\;. \tag{6.34}$$

This is the equation of motion of a planar pendulum in a force field with potential U. We have learnt in our courses on elementary mechanics how to solve (6.34), using energy conservation. For "small energy", a solution, $\varphi_c(t)$, of (6.34) is a periodic function of t; for "large energy", $\varphi_c(t)$ grows linearly in t, with periodic modulations superimposed; and $\dot{\varphi}_c(t)$ is periodic in t.

Eqs. (6.29), with $J^{\mu}_{M}\equiv 0$, linearized around $\underline{E}=\underline{B}=0$, $\varphi=\varphi_c(t)$, yield the equations

$$\begin{aligned}
\underline{\nabla}\cdot\underline{B} \;&=\; 0\,, \quad \underline{\nabla}\wedge\underline{E}\,+\,\dot{\underline{B}} \;=\; 0\,,\\
\underline{\nabla}\cdot\underline{E} \;&=\; 0\,, \quad \underline{\nabla}\wedge\underline{B}\,-\,\dot{\underline{E}} \;=\; -\,\frac{\ell\,e^2}{8\,\pi^2}\,\dot{\varphi}_c\,\underline{B}\,,
\end{aligned} \tag{6.35}$$

which can be solved by Fourier transformation in the space variables. The equations for the components, $\widehat{\underline{B}}^{T}$ and $\widehat{\underline{E}}^{T}$, of the Fourier components of $\underline{B}$ and $\underline{E}$ perpendicular to the wave vector $\underline{k}$ are two Mathieu equations of the form

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} \;=\; \begin{pmatrix}0 & 1\\ h_k(t) & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

where $k=|\underline{k}|$ and $h_k(t)$ depends on k and is linear in $\dot{\varphi}_c(t)$; see [26]. These equations yield

$$\ddot{\xi}(t) \;=\; h_k(t)\,\xi(t)\;. \tag{6.36}$$

In solving this equation one encounters the phenomenon of the *parametric resonance*, i.e., for k in a family of intervals, eq. (6.36) has a solution growing *exponentially* in time. Hence the electromagnetic field has *unstable modes* growing exponentially in time and with $\underline{B}\cdot\underline{E}\neq 0$.

The parametric resonance has appeared in cosmology in other contexts. In our analysis it plays an entirely natural and essentially *model-independent* rôle and may help to explain where large, cosmic (electro) magnetic fields might come from.

Of course, eqs. (6.29) are Lagrangian equations of motion. They are derived from the action functional (6.25), (with $W=0$ and U independent of derivatives of φ). The Lagrangian density does not depend on time explicitly. Therefore, there is a *conserved energy functional*, $\mathcal{E}(A,\varphi)$. The special solutions considered in (6.30) and (6.34) have *infinite* (axionic) energy. The instabilities in the time evolution of the electromagnetic field are due to a reshuffling of energy from axionic to electromagnetic degrees of freedom.

Clearly, it would be interesting to construct finite-energy solutions of eqs. (6.29), with an initial axion field depending not only on time but also on space. Of particular significance is situation (ii), with $U \neq 0$. Interpreting $\ell\dot{\varphi}$ as a difference of chemical potentials for left- and right-handed fermions, we are thus considering states of the universe with spatially varying, time-dependent chemical potentials triggering an asymmetric population of left-handed and right-handed fermionic modes. This asymmetry gradually disappears, due to chirality-changing processes, and the field energy stored in axionic degrees of freedom is reshuffled into certain electromagnetic field modes triggering the growth of cosmic electromagnetic fields. Large electric fields rapidly die out because of dissipative processes; (the energy loss from the electric field into matter degrees of freedom is described by $\underline{E} \cdot \underline{J} \propto \sigma_L |\underline{E}|^2 + \dots$.) But large magnetic fields may survive for a comparatively long time.

Describing these phenomena within the approximation of linearizing eqs. (6.29) (possibly supplemented by a dissipative Ohmic term) around special solutions, including space-dependent ones, of infinite or finite energy, is feasible; [26]. But our understanding of the effects of the *non-linearities* in eqs. (6.29) remains, not surprisingly, very rudimentary.

Some speculations on the rôle played by magnetic monopoles in the effects described here are contained in the last section; see also [26].

7 Conclusions and outlook

In this review we have shown how the chiral, abelian anomaly helps to explain important features of the (quantum) *Hall effect*, such as the existence of edge currents and aspects of the quantization of the Hall conductivity, and of its *four-dimensional cousin*, which may play a significant rôle in explaining the origin of large, cosmic magnetic fields. Our analysis is essentially *model-independent*, a fact that makes it quite trustworthy. How significant the four-dimensional variant of the Hall effect is in early-universe cosmology remains to be understood in more detail. This will require a better understanding of orders of magnitude of various physical quantities and of the properties of solutions of the non-linear Maxwell–Dirac-axion equations (6.29). A beginning has been made in [15, 26]. — There is no doubt that the following equations

$$J_{\text{bulk}} \;=\; \sigma_T \, * \, F \;, \quad \delta \, J_{\text{edge}} \;=\; - \, \sigma_T \, E \;, \tag{7.1}$$

with $\sigma_T = \sigma_H$, for bulk- and edge-currents of an incompressible Hall fluid (see eqs. (2.10) and (2.14)), and

$$J^{\nu} \;=\; \sigma_T \, \ell \, \Big\{ (\partial_\mu \, \varphi) \;\; \widetilde{F}^{\mu\nu} \;+\; \varphi \, J^{\nu}_M \Big\} \;, \tag{7.2}$$

where $\sigma_T = -\frac{1}{4\,\pi\,h} \, \left(\sum_{j=1}^{N} q_j^2 \right)$, with N the number of species of charged fermions with electric charges $q_1, \dots, q_N$, (see eqs. (6.13) and (6.28)) are significant laws of nature connected with the chiral anomaly.

For the future, it would be important to gain a better understanding of the contents of equations (6.29), (possibly corrected by dissipative terms and/or ones coming from $\delta W(A)/\delta A_\mu(x)$, which have been neglected), including the rôle played by magnetic monopoles and dyons ($J^\mu_M \neq 0$). (Eqs. (6.29) and their fully quantized counterparts appear to offer some clue for understanding (axion-driven) monopole–anti-monopole annihilation, triggering the growth of certain modes of the electromagnetic field.) Some understanding of these issues has been gained in [26]; but much work remains to be done. We have also studied the influence of gravitational fields on the processes described in Sect. 6 [26] (in analogy to the "geometric" (or gravitational) Hall effect in 2+1 dimensions described in the third paper quoted under [10] and to the phenomenon of "quantized" heat currents in quantum wires mentioned in Sects. 3 and 5). But there is no room here to describe our results in detail. Our findings will have to be combined with cosmic evolution equations.

In this review, we have only quoted literature that we used in carrying out the calculations described here. Many further references may be found in [7, 8, 10, 15, 20, 26].

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More about the primordial plasma

We start by summarizing some of the laws describing a primordial plasma of charged, light particles in the universe.
The chiral anomaly says that

$$\partial_\mu J_5^\mu = \frac{\alpha}{\pi} \vec{E} \cdot \vec{B}, \tag{1}$$

where α is the fine structure constant.
Let T denote the temperature of the primordial plasma, and μ_5 the "axial chemical potential" conjugate to the conserved charge

$$Q_5 := \int_{\{t=const.\}} d^3x \, \left(J_5^0(\vec{x}, t) - \frac{\alpha}{2\pi} \varepsilon^{0\nu\rho\sigma} A_\nu F_{\rho\sigma}\right), \tag{2}$$

counting the difference of charge of left-handed and of right-handed light leptons. Pseudo-scalar charge density:

$$\rho_5(\vec{x}, t) := \langle J_5^0(\vec{x}, t) \rangle_{T,\mu_5},$$

Implications of chiral anomaly

$$q_5(t) \equiv q_5(t;\, T, \mu_5) := \text{ spatial average of } \rho_5(\vec{x}, t). \qquad (3)$$

Response of q_5 to the chemical potential μ_5:

$$q_5(t;\, T, \mu_5) \approx \mu_5 \frac{\partial q_5}{\partial \mu_5}(t;\, T, \mu_5 = 0) \approx \frac{\mu_5}{6} T^2, \qquad (4)$$

with $q_5(t;\, T, \mu_5 \equiv 0) = 0$. The second equation in Eq. (4) is valid for relativistic particles in a weak magnetic field, ($T^2 \gg e|\vec{B}|$, $T \gg \mu_5$). The chiral anomaly implies that

$$\dot{\mu}_5 = \frac{6}{T^2} \dot{q}_5(T, \mu_5) = \frac{6\alpha}{\pi T^2} \overline{\vec{E} \cdot \vec{B}}. \qquad (5)$$

and, as shown by ACF, that

$$\vec{j}(x) = \langle \vec{J}(x) \rangle_{T,\mu_5} = \underbrace{\frac{\alpha}{\pi} \mu_5 \vec{B}(x)}_{\text{contr. from anomaly}} + \underbrace{\sigma \vec{E}(x)}_{\text{Ohm-Hall law}} . \qquad (6)$$

An instability derived from the anomaly

The first term on the right side of Eq. (6) can be derived from the anomalous equal-time commutator

$$[J_5^0(\vec{y},t), J^0(\vec{x},t)] = \frac{\alpha}{\pi}\vec{B}(\vec{y},t)\cdot\vec{\nabla}\delta(\vec{y}-\vec{x})$$

and from expression (8), below, for the equilibrium state of the plasma. Inserting Eq. (6) into Ampère's law and applying Faraday's induction law, we find that

$$\frac{\partial\vec{B}}{\partial t} = \frac{1}{\sigma}\Box\vec{B} + \frac{\alpha}{\pi}\frac{\mu_5}{\sigma}\vec{\nabla}\wedge\vec{B}. \tag{7}$$

Solutions to this equation exhibit an instability – exp. growth of $\vec{B}$, (as will be shown shortly).

Local thermal equilibrium

On large scales, temperature, $T = \beta^{-1}$, and axial chem. pot. μ_5 depend on location in space-time. Thus, state of local thermal equilibrium (LTE) of primordial plasma described by density operator

$$\rho_{LTE} := \mathcal{Z}_{\Sigma}^{-1} exp\Big(- \int_{\Sigma} [u^{\mu}(x) T_{\mu\nu}(x) - \mu_5(x) J_{5\nu}(x)] \beta(x) d\sigma^{\nu}(x) \Big), \tag{8}$$

where $u^{\mu}(x) =$ 4-velocity field of plasma. Gauge invariance leads to the constraint

$$d(\beta\mu_5) \wedge F|_{\Sigma} \equiv 0.$$

Next, we propose to find a generally covariant form of Eq. (6) linear in the electromagnetic field tensor $F_{\mu\nu}$:

$$j^{\mu}(x) = \frac{\alpha}{2\pi} \varepsilon^{\mu\nu\rho\sigma} (\partial_{\nu}\theta_5)(x) F_{\rho\sigma}(x) - \sigma F^{\mu\nu}(x) u_{\nu}(x), \tag{9}$$

Axial chemical potential and axion

where $\theta_5 \equiv \phi$ is a pseudo-scalar field. The spatial components of this equation in co-moving coordinates are

$$\vec{j}(x) = \frac{\alpha}{\pi}\Big([\dot{\theta}_5(x) + \vec{v}(x)\cdot\vec{\nabla}\theta_5(x)]\vec{B}(x) + \text{term linear in } \vec{E} \propto \vec{\nabla}\theta_5, \vec{v}\Big).$$

Comparing this equation with Eq. (6), we find that

$$\mu_5 = u^\mu \partial_\mu \theta_5. \tag{10}$$

Important Remark: The first term on the right side of Eq. (9) can be derived from an action principle. Upon variation with respect to the electromagnetic vector potential A_μ ($F = dA$), the term

$$\frac{\alpha}{4\pi}\int d^4x\ \theta_5(x)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x) \tag{11}$$

yields precisely the first term on the R.S. in (9).

Equations of motion of axion

Expression (11) suggests that the field θ_5 describes an *axion* coupled to the electromagnetic field.
How do field equations for θ_5 look like? Inspiration from Eqs. (5) and(10): Let's assume that we work in coordinates where the plasma is at rest. Then $\mu_5 = \dot{\theta}_5$, and hence Eq. (5) yields

$$\overline{\frac{\partial^2 \theta_5}{\partial t^2}} = \frac{6\alpha}{\pi T^2} \overline{\vec{E} \cdot \vec{B}} \tag{5'}$$

Eq. (5') suggests that the field equation for θ_5 is one of the following equations:

$$\Lambda^2 \Box \theta_5 + U'(\theta_5) = \frac{\alpha}{4\pi} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

where $\Lambda^2 = \frac{T^2}{6}$, which is an inhomogeneous wave equation; or

$$\Lambda^2 \Big(\frac{\partial^2 \theta_5}{\partial t^2} - D \Delta \dot{\theta}_5 \Big) + U'(\theta_5) = \frac{\alpha}{4\pi} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{12}$$

which is an inhomogeneous diffusion equation for $\dot{\theta}_5 \equiv \dot{\phi} = \mu_5$. ETC.