Physical Principles Underlying the FQHE

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Report on work spread over 25 years (1989-2000, 2008-2011)

Credits:

Morf

Bieri, Boyarsky, Cheianov, Graf, Kerler, Levkivskyi, Pedrini, Ruchayskiy, Schweigert, Studer, Sukhorukov, Thiran, Walcher, Zee

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Contents:

General Goal (90's):

Classify states of bulk matter and their surface modes, using ideas and concepts from gauge theory and GR, such as Effective Actions (= generating functionals of current Green functions), gauge invariance and anomaly cancellation, "holography", ---

Applications (90's, 2012)

(Topological) Insulators

QHE

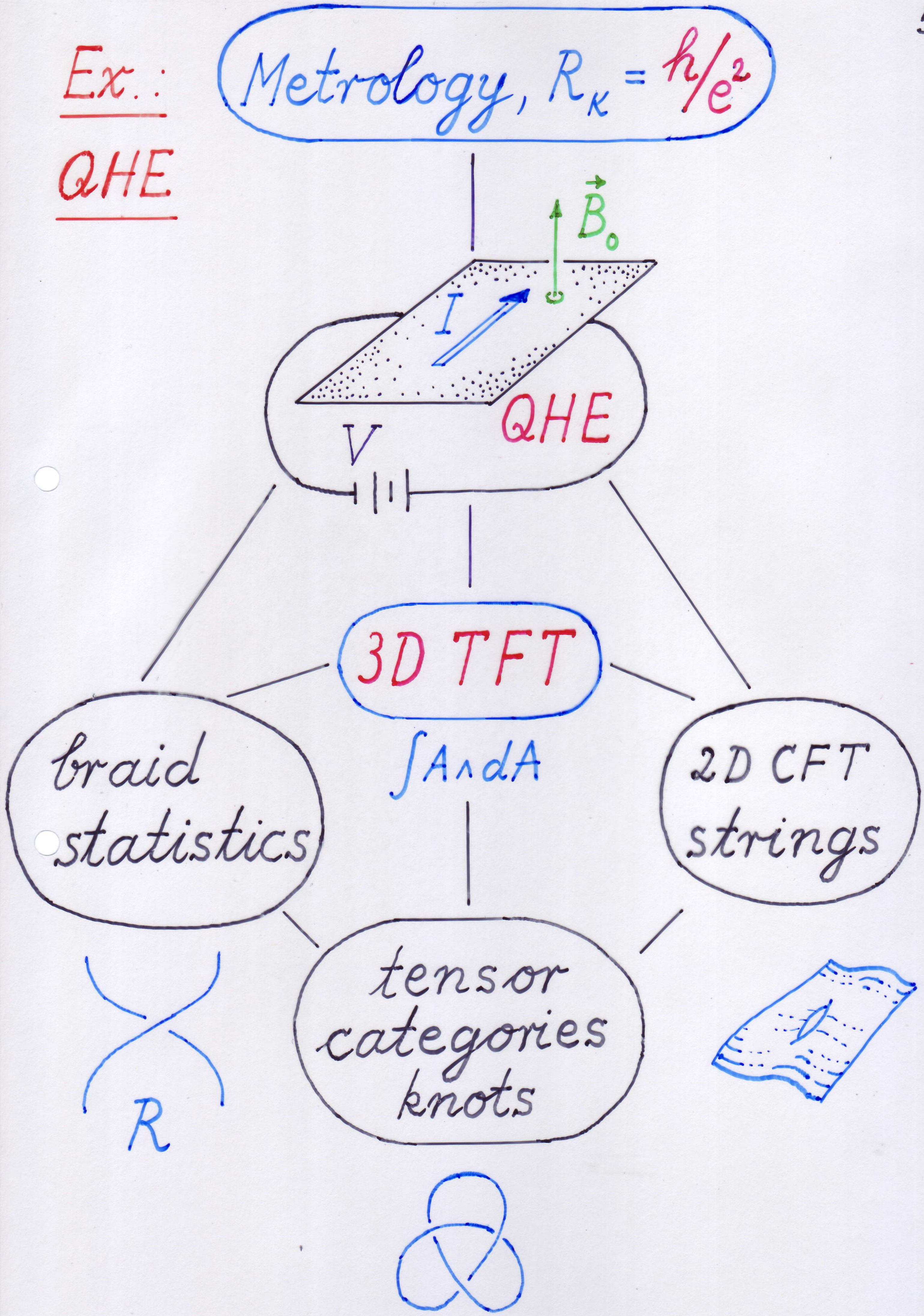
(Topological) Superconductors

Higher-dimensional cousins of QHE ➔ cosmology

Etc.

Ex.:

QHE



Metrology, $R_K = h/e^2$
$\vec{B}_0$
I
QHE
V
3D TFT
$\int A \wedge dA$
braid statistics
2D CFT strings
tensor categories knots
R

1. Remarks on History

1879 : Hall

... holes in semicond.

1966 : Fowler et al. Si MOSFET ... 2DEG

1975 : Kawaji et al. dissip.-less state in Si MOSFET

1978 : Englert & v. Klitzing plateaux

1980 : v. Klitzing $\sigma_H = \frac{e^2}{h} \cdot n$

1982 : Tsui, Störmer, Gossard FQHE in GaAs/GaAlAs; idea of fract. charges

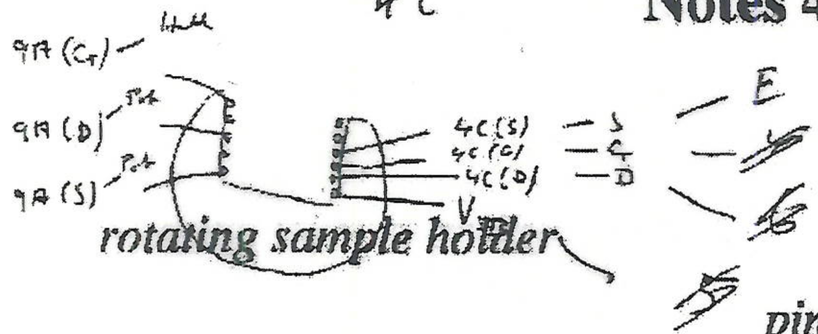
1980–1982 : R. B. Laughlin ... theory; $\geq$ 1982 : et al.

QHE

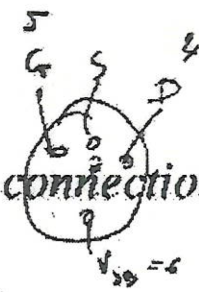
K. von Klitzing

Notes 4./5.2.1980

4 C



rotating sample holder



pin connections

$$E_H = R_H \cdot B \cdot j = \frac{1}{n e} \cdot B \cdot \frac{I}{b}$$

$$U_H = \frac{B}{n \cdot e} \cdot I \qquad N = \frac{eB}{2\pi\hbar} \qquad (g_s \cdot g_v = 1)$$

$$U_H = \frac{2\pi\hbar B}{e \cdot e \cdot B} \cdot I = \boxed{\frac{h}{e^2} \cdot I}$$

~~25,76 kΩ~~
25813

Josephson

$$\frac{\hbar c}{e^2} = \qquad \rho_{xy} = \frac{\alpha}{2} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \Rightarrow 25813\ \Omega$$

notes of the phone call to PTB (5.2.1980)

PTB 531/592-1
2240
Prof. v. Kose

$$\mu_0 = 4\pi \cdot 10^{-7}\ \frac{Vs}{Am}$$

$$\varepsilon_0 = 0{,}8854 \cdot 10^{-13}\ \frac{As}{Vm}$$

10^{-6} 12945

$$\sqrt{\frac{\varepsilon_0}{\mu_0}} = 2{,}65 \cdot 10^{-3}\ \Omega^{-1}$$

$6 \cdot 10^{-?}$ 12907

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$$

25813 Ω : N
1 MΩ parallel

25813 Ω : N	1 MΩ parallel
25813	→ 25163,46
12906.5	12742.04
6453,25	6411.87
3226.63	3216,25
2151.08	2146.47

quantized resistances with and without the input resistance of the x-y recorder

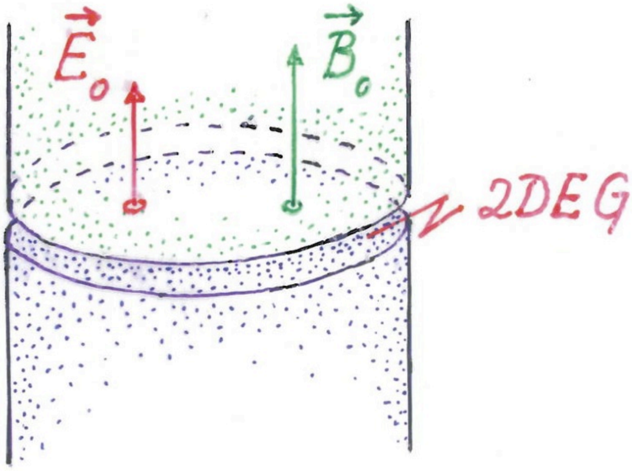
2. What is the FQHE ?

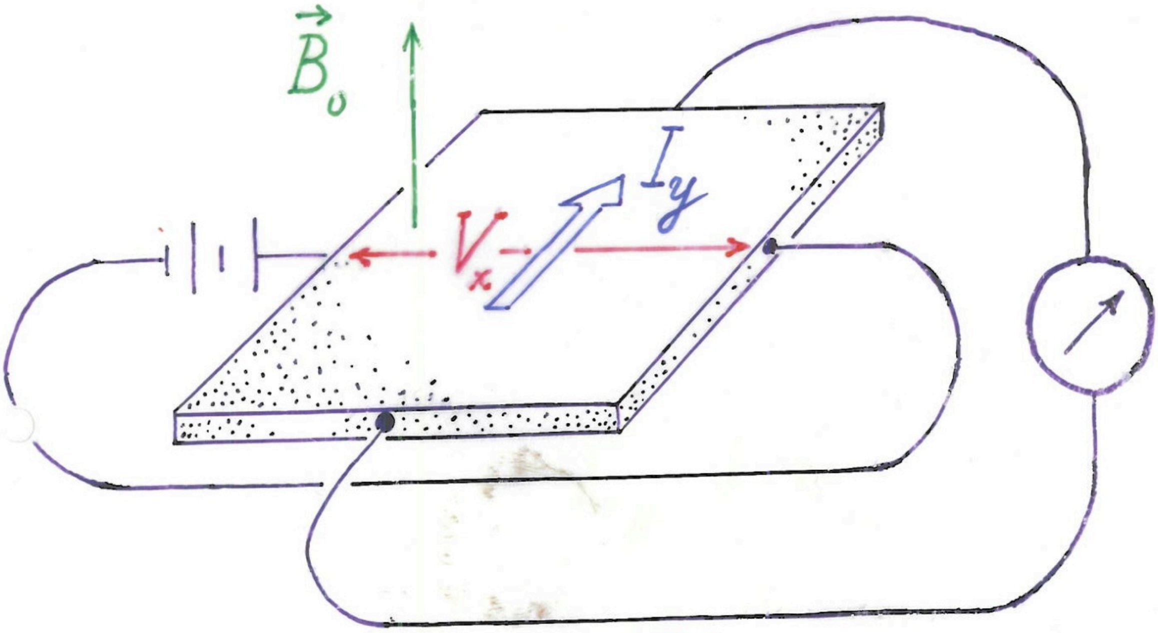
$Ga_x Al_{1-x} As$

Insulator

$Ga As$

Semi-cond





$$\left.\begin{aligned} R_H &= \rho_H = -\frac{V_x}{I_y} \\ R_L &= \frac{V_y}{I_y} \end{aligned}\right\} \text{measured}$$

n : e^- density in 2DEG

$\phi_0 = \frac{hc}{e}$: magn. flux qu.

$\nu = n \cdot \left(|\vec{B}_{0\perp}| / \phi_0 \right)^{-1}$: dim. less

filling factor; # filled Landau levels

Classical theory:

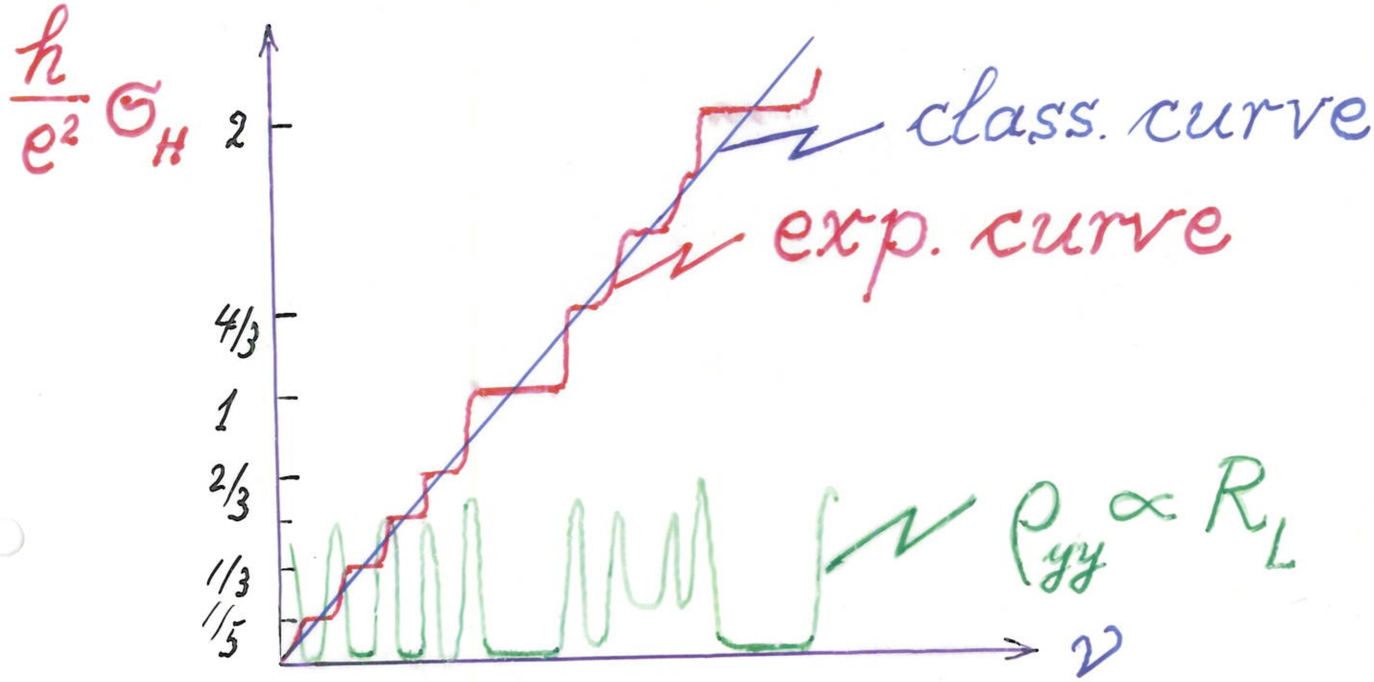
$$\vec{a}_{e^-} = 0 \Leftrightarrow e\vec{E}_{\parallel} = -e \frac{\vec{v}_{e^-}}{c} \wedge \vec{B}_{0\perp}$$

$$\rightarrow \vec{E}_{\parallel} \perp \vec{v}_{e^-}$$

$$\vec{j} = -en\vec{v}_{e^-} =: \sigma_H (\vec{e}_z \wedge \vec{E})$$

$$\Rightarrow \sigma_H = \frac{enc}{|\vec{B}_{0\perp}|} = \frac{e^2}{h} \nu$$

Experiment:



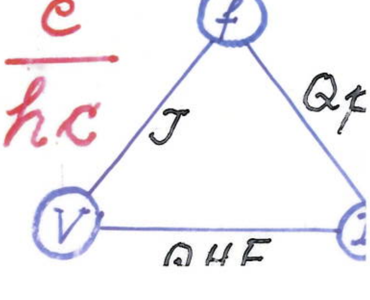
Observations:

I. $R_L = 0 \Leftrightarrow (\nu, \sigma_H) \in$ plateau

II. Plateau heights $\in \frac{e^2}{h}\mathbb{Q}$

Precision - IQHE : $1:10^9$

- <u>QHE</u> $\longrightarrow R_K^{-1} = \frac{e^2}{h}$
- <u>Josephson</u> $\longrightarrow K_J = \frac{e}{hc}$
- <u>Q - pumps</u> $\longrightarrow e$

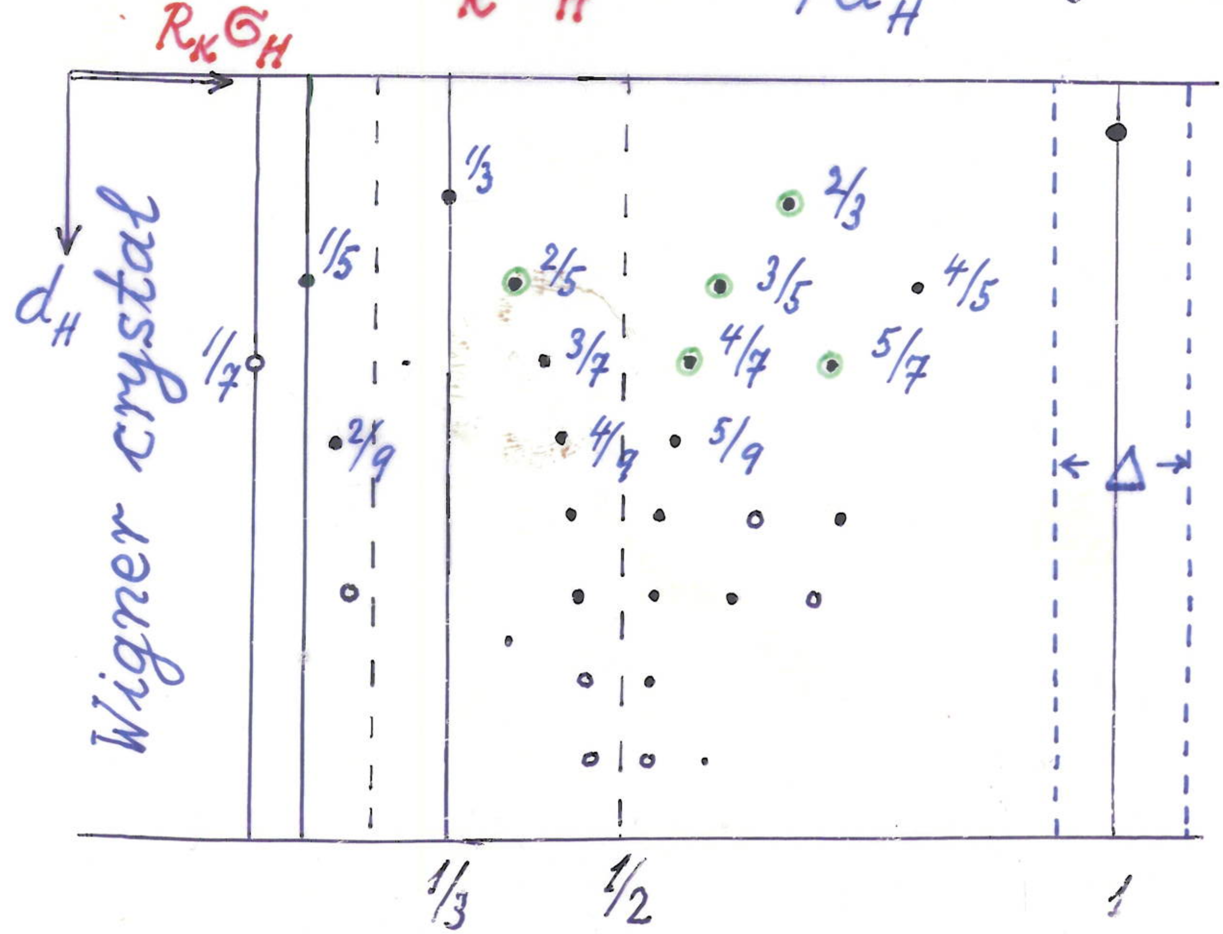


III. The cleaner the sample,

- the more plateaux obs.
- the narrower the plat.

IV. If $R_K \sigma_H \notin \mathbb{Z}$ (FQHE) $\rightarrow$ fract. el. charges (Tsui; Glattli et al.; interferom.)

$$R_K \sigma_H = n_H / d_H \in \mathbb{Q}$$



Tasks for theorists

(1) For what values of ν is $R_L = 0$ (pos. mobility gap)? How do plateau-width, Δ, scale with disorder? …

→ Many-body th., computer

(2) Assuming that $R_L = 0$ (IHF), what can we say about:

(i) possible values of σ_H? ✓

(ii) spectrum & properties of quasi-particles? ✓

(3) Nature of transitions between neighboring IQHF's?

(4) Wigner crystal for $\nu \lesssim 1/7$;

(5) New exp. tests of theor. predictions? (e.g., interferom.)

Applications:

- Metrology & fund. consts.
- Novel computer memories
- q-bits for topological quantum computers (J.F.!) ↔ exploitation of quasi-particles w. braid stat.

before Friedman, Kitaev, ...! →

3. Electrodynamics of IQHF

2DEG confined to planar domain Ω in $\vec{B}_0$; bulk mobility gap $>0 \leftrightarrow R_L = 0$.

Response of 2DEG to small pert. e.m. field, $\vec{E}, \vec{B}$, with

$$\vec{B}^{tot.} = \vec{B}_0 + \vec{B},$$

slowly time-dep.; ("ad. lim.")

Orbital dyn. of e^- dep. only on $B_3^{tot} =: B_0 + B$, $\vec{E}_{\parallel} \equiv \underline{E} = (E_1, E_2)$.

$A := (\phi, A_1, A_2)$ is vector pot. of

$$F := \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix}$$

$\langle (\cdot) \rangle_A$: state of 2DEG

$$j^\mu(x) := \langle J^\mu(x) \rangle_A \,, \quad \mu = 0, 1, 2 \,. \tag{1}$$

(1) Hall's law ($R_L = 0$)

$$j^k(x) = \sigma_H \, \varepsilon^{kl} E_l(x), \tag{2}$$

$k, l = 1, 2, \quad x = (t, \underline{x}) \in \Lambda := \mathbb{R} \times \Omega$

(2) Charge conservation

$$\frac{\partial}{\partial t} \rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0 \tag{3}$$

(3) Faraday's induction law

$$\frac{\partial}{\partial t} B_3^{tot}(x) + \underline{\nabla} \wedge \underline{E}(x) = 0 \tag{4}$$

Laws (1) – (3) imply:

$$\frac{\partial}{\partial t}\rho \overset{(2)}{=} -\underline{\nabla}\cdot\underline{j} \overset{(1)}{=} -\sigma_H \underline{\nabla}\wedge\underline{E}$$

$$\overset{(3)}{=} \sigma_H \frac{\partial}{\partial t} B \qquad (5)$$

Integrate (5) in time t, with

$$j^0(x) := \rho(x) + en$$

$$B(x) = B_3^{tot}(x) - B_0$$

Then (5) $\Rightarrow$

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x) \qquad (6)$$

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(1) & (4) $\Rightarrow$ $\boxed{j^\mu(x) = \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x)}$ (7)

(2) $\Leftrightarrow$ $\partial_\mu j^\mu = 0 \Rightarrow j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$

(3) $\Leftrightarrow$ $\partial_{[\mu} F_{\nu\lambda]} = 0 \Rightarrow F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$

Then (7) $\Rightarrow$

$$j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda = \sigma_H\, \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda,$$

or

$$\boxed{db = \sigma_H\, dA} \qquad (8)$$

But wherever $\sigma_H \neq const.$, e.g. at $\partial\Omega$, Eq.(8) inconsistent:

$$0 = \partial_\mu j^\mu = \varepsilon^{\mu\nu\lambda}(\partial_\mu \sigma_H) F_{\nu\lambda} \neq 0 \quad (8')$$

on $\Sigma :=$ support (grad σ_H)! $\rightarrow$

$$j^\mu = j^\mu_{bulk} \neq j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{edge},$$

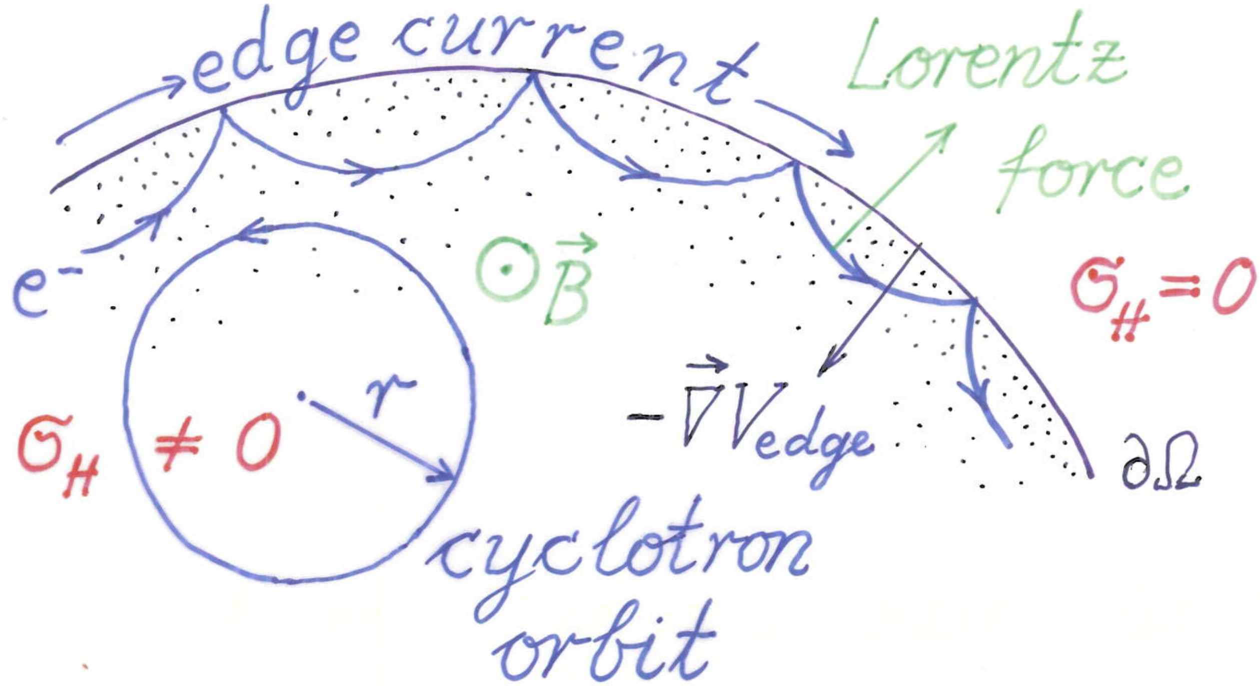
$$\partial_\mu j^\mu_{tot} = 0, \quad supp\, j^\mu_{edge} = \Sigma,$$

$\underline{j}_{edge} \perp \underline{\nabla}\, \sigma_H$. Then (7) $\Rightarrow$

$$\boxed{\partial_\mu j^\mu_{edge} = -\partial_\mu j^\mu_{bulk} = -\sigma_H\, E_{\parallel}\big|_\Sigma} \quad (9)$$

Chiral anomaly in (1+1)D

Edge current, j^μ_{edge}, is anomalos chiral current in (1+1)D:



At edge:

$$e\,\frac{\vec{v}}{c} \wedge \vec{B} = -\vec{\nabla} V_{edge} \rightarrow \vec{v}$$

Analogous phen. in class. physics: Hurricanes!

$\vec{B} \rightarrow \vec{\omega}_{earth}$, Lorentz $\rightarrow$ Coriolis

$-\vec{\nabla} V_{edge} \longrightarrow -\vec{\nabla}$pressure

Chiral anomaly in (1+1)D:

$$\partial_\mu j^\mu_{edge} = \frac{e^2}{h}\Big(\sum_{species} Q_i^2\Big) E_{\parallel}\big|_\Sigma ,$$

where $eQ_1, \dots, eQ_n$ are el. charges = c.c.'s to "quasi-part. contributing to j^μ_{edge}. Thus

$$R_K \sigma_H = \sum_{species} Q_i^2 \qquad (10)$$

If $R_K \sigma_H \notin \mathbb{Z} \Rightarrow$ some Q_i's fractional!

<u>IQHF</u>: Each filled "Landau level" contributes one spec. of e^- to j^μ_{edge} $\Rightarrow$

$R_K \sigma_H$ = # filled Landau lev.

4. The bulk of an IHF

$\Sigma = \partial\Omega$, $\Lambda = \mathbb{R} \times \Omega$; $(R_K = 1)$.

$J^\mu(x)$: q.m. current density

$\langle (\cdot) \rangle_A$: state of IHF in ext. field $(\underline{E} = \dot{A}, B = \nabla \wedge A)$

$S_\Lambda(A)$: eff. action of IHF.

$$j^\mu_{bulk}(x) = \langle J^\mu_{bulk}(x) \rangle_A = \delta S_\Lambda(A) / \delta A_\mu(x)$$

$$\stackrel{!}{=} \sigma_H \, \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad (x \notin \partial\Lambda)$$

$$\Rightarrow S_\Lambda(A) = \frac{\sigma_H}{2} \int_\Lambda \varepsilon^{\mu\nu\lambda} A_\mu(x) F_{\nu\lambda}(x) dx + \Gamma_{\partial\Lambda}(a)$$

$$= \underbrace{\frac{\sigma_H}{2} \int_\Lambda A \wedge dA}_{\text{from incompr. + P-breaking}} + \Gamma_{\partial\Lambda}(a) \quad (11)$$

where $a := A|_{\partial\Lambda}$, $\Gamma_{\partial\Lambda}(a)$: gen. fu. of Green fus of edge curr.

$\int_\Lambda A \wedge dA$ not gauge-inv. under $A \to A + d\chi$, $\chi|_{\partial\Lambda} \neq 0$; cured by $\Gamma_{\partial\Lambda}(a) \Rightarrow$

J^μ_{edge} is $U(1)$ Kac-Moody curr.

$$\partial_\mu J^\mu = 0 \Rightarrow J^\mu = \sqrt{\sigma_H}\, \varepsilon^{\mu\nu\lambda} \partial_\nu B_\lambda$$

$S_\Lambda(B, A)$: action of B coupled to vector pot. A.

(11) $\Longleftrightarrow$

$$S_\Lambda(B,A) = \frac{1}{2}\int_\Lambda B \wedge dB + \int_\Lambda J^\mu A_\mu \qquad (12)$$
$$+ \underbrace{\text{bd. term}}_{\text{action for } U(1) \text{ Kac-Moody c.}}$$

$S_\Lambda(B,A)$: action of topol. $U(1)$-Chern-Simons th.

Charge operator:

$$Q_{\mathcal{O}} := \int_{\mathcal{O}} d^2x\, J^0(t,\underline{x})$$

$$= \sqrt{\sigma_H} \int_{\partial\mathcal{O}} B \qquad \text{(Stokes)}$$

$\Rightarrow$ $e^{iQ_{\mathcal{O}}}$ = Wilson loop$[\partial\mathcal{O}]$

Curvature ($\propto J^\mu$) of B-field conc. in <u>loc. static sources</u>, $|(q,\lambda),z\rangle$, with

$$Q_{\mathcal{O}}\,|(q,\lambda),z\rangle = \sqrt{\sigma_H}\, q\, |(q,\lambda),z\rangle, \quad z\in\mathcal{O} \tag{13}$$

q: flux of B-field

λ: "internal" quantum #

Mobility gap in bulk $> 0 \Rightarrow$

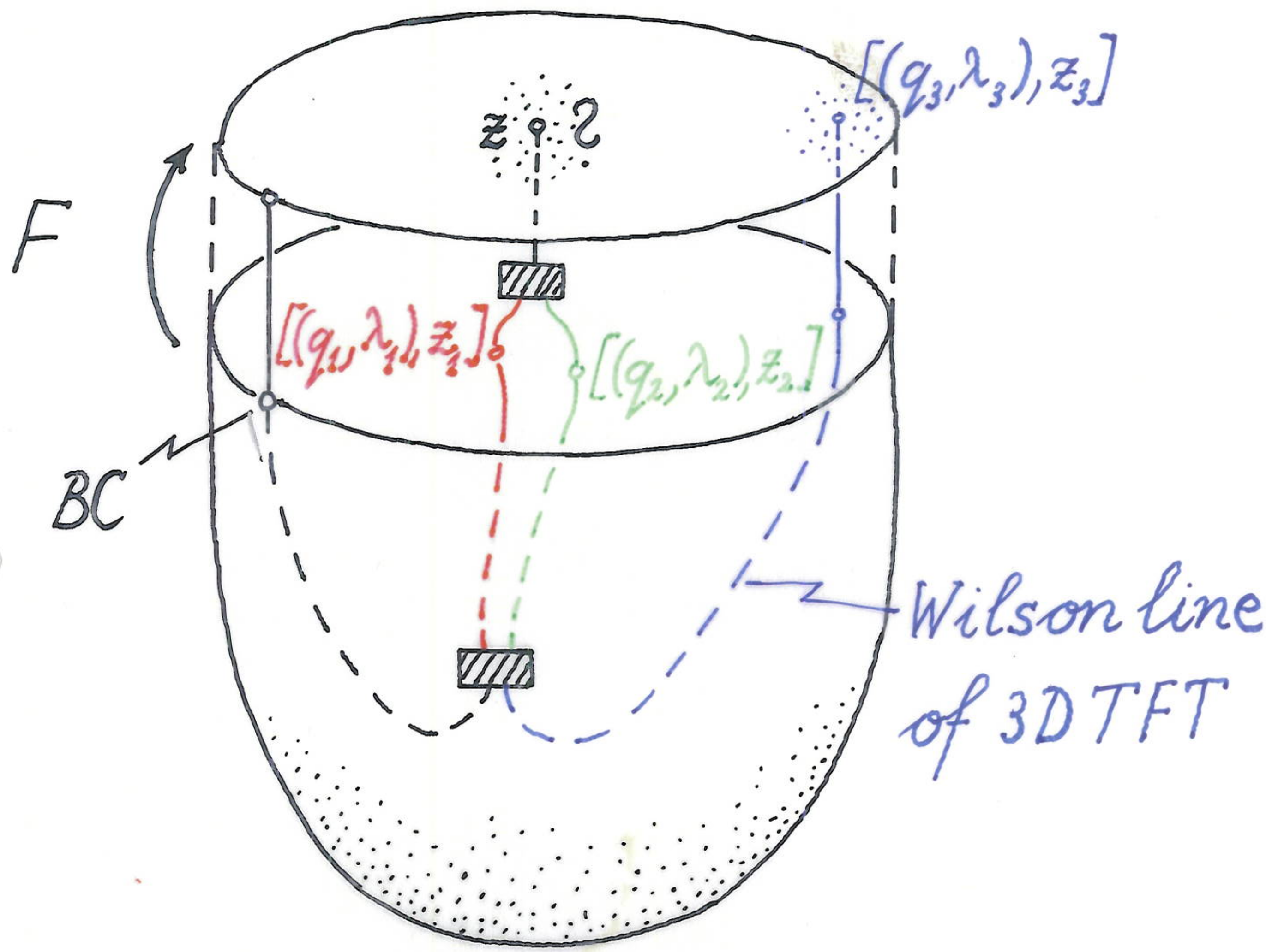
Bulk of IHF described by

3D TFT

$\sim$ {family of "sectors" $[(q,\lambda)]$} $=: \mathcal{C}$

$\mathcal{C}$ equipped w. composition rule, $\otimes$, and quantum statistics given by braiding, ε; $(S, T, \cdots)$.

$[(q,\lambda), z]$: Space of quasi-particle states w. el. charge $\sqrt{\sigma_H}\, q$ & internal quantum # λ localized near $z \in \mathbb{C} \simeq \mathbb{R}^2$



Composition rules (fusion) of quasi-particle sectors:

$$F^{\lambda_3, a}_{\lambda_1, \lambda_2} : [(q_1, \lambda_1), z_1] \otimes [(q_2, \lambda_2), z_2] \longrightarrow [(q_1 + q_2, \lambda_3), z]_a \,, \quad (14)$$

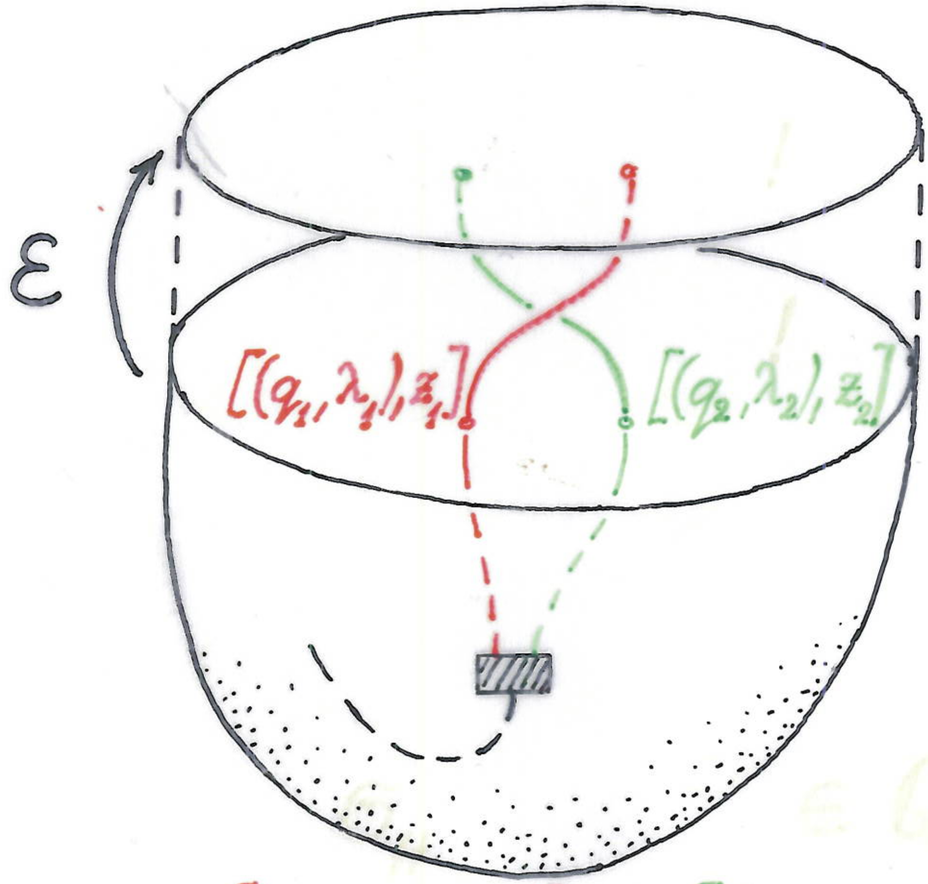
"fusion matrix"
~ Clebsch-Gordan

$$a = 1, \cdots, N^{\lambda_3}_{\lambda_1 \lambda_2} .$$

$N_{\lambda_1 \lambda_2}^{\lambda_3}$: multiplicity of λ_3 in $\lambda_1 \otimes \lambda_2$ – "fusion rules"

Ass.* Given λ_1, λ_2, $\exists$ only $< \infty$ λ_3 with $N_{\lambda_1 \lambda_2}^{\lambda_3} > 0$, & $N_{\lambda_1 \lambda_2}^{\lambda_3} < \infty$.

Braiding ("A-B phases"):



$$\mathcal{E} : [(q_1, \lambda_1), z_1] \otimes [(q_2, \lambda_2), z_2]$$

$$\longrightarrow [(q_2, \lambda_2), z_1] \otimes [(q_1, \lambda_1), z_2] \quad (15)$$

Spin of quasi-particles:

$$U(Rot_{2\pi})\upharpoonright_{[(q,\lambda)]} = e^{2\pi i\, s(q,\lambda)}\, \mathbb{1}\upharpoonright_{[(q,\lambda)]}$$

$$s(q,\lambda) = \frac{q^2}{2} + h_\lambda \quad \Big(\underset{\text{i.g.}}{\notin} \tfrac{1}{2}\mathbb{Z}\,!\Big) \quad (16)$$

$$\text{Ass.}^* \Rightarrow h_\lambda \in \mathbb{Q} \quad (\text{Vafa's thm.})$$

If (q^*,λ^*) = quantum # of e^- then

$$(13) \Rightarrow \sqrt{\sigma_H}\, q^* \overset{!}{=} -1$$

$$(16) \Rightarrow s(q^*,\lambda^*) \overset{!}{=} \ell + \frac{1}{2}, \quad \ell \in \mathbb{Z}.$$

Thus
$$\frac{1}{2\sigma_H} + h_{\lambda^*} = \ell + \frac{1}{2}$$

$$\overset{\text{Vafa}}{\Longrightarrow} \quad \sigma_H =: \frac{n_H}{d_H} \in \mathbb{Q}$$

$$\text{min. el. charge} = e\frac{1}{k\, d_H}, \quad k = 1, 2, 3, \cdots$$

Fractional spin and statistics:

$$U(Rot_{2\pi})\restriction_{[(q_1,\lambda_1)]\otimes[(q_2,\lambda_2)]}$$

$$\overset{(14)}{=} \bigoplus_{\substack{\lambda_3 \\ a=1,\dots,N_{\lambda_1\lambda_2}^{\lambda_3}}} e^{2\pi i\, s(q_1+q_2,\lambda_3)}\, \mathbb{1}\restriction_{[(q_1+q_2,\lambda_3)]_a}$$

$$\Rightarrow \varepsilon^2_{(q_1,\lambda_1)(q_2,\lambda_2)} =$$

$$bl.diag\left(e^{2\pi i(s(q_1+q_2,\lambda_3)-s(q_1,\lambda_1)-s(q_2,\lambda_2))}\right)$$

multiplicity $N_{\lambda_1\lambda_2}^{\lambda_3}$

$\varepsilon^2 \neq \mathbb{1} \iff$ fractional spin & statistics of quasi-p.'s

$$\mathcal{C} = \{(q,\lambda); (N_{\lambda_1\lambda_2}^{\lambda_3}), F, \varepsilon\} \longrightarrow$$

σ_H, fract. el. charges, spins and statistics of quasi-p.'s

Digression on QHE

Classification of incompr. (gapped) bulk theories in scaling limit: 3D TFT's

$\simeq$ quasi-rat., braided, modular tensor cats., $\mathcal{C}$, w. ab. charge, Q_{em}, simple current, J_e, $[Q_{em}, J_e] = -J_e$, describing electrons.

$$\mathcal{C} = \{\mathcal{O}, (N^{\gamma}_{\alpha\beta}), B, F, Q_{em}, J_e\}$$

$$\ell + 1/2 = \frac{1}{2\sigma_H} + \Delta_e, \quad \Delta_e \in \mathbb{Q}$$

$$\Longrightarrow \quad \sigma_H \in \mathbb{Q} \; !$$

"Holography"

$\exists$ QFT of gapless edge degs. of freedom, QFT_{edge}, with:

(i) chiral Kac-Moody current, j_{em}, descr. diam. em edge currents;

(ii) superselection sectors described by $\mathcal{C}$.

In general, QFT_{edge} not conformal, because diff. edge modes have different propagation speeds, $\{v_i\}$.

<u>Example.</u> If $\mathcal{C}$ has abelian braid statistics

$$\mathcal{C} \leftrightarrow \{\Gamma, Q \in \Gamma^* \text{ visible}\},$$

$$\Gamma \ni q_e, \text{ w. } Q \cdot q_e = -1,$$

$$\langle q, q \rangle = Q \cdot q \text{ mod. } 1.$$

$$\Gamma \supset \Gamma_{Kneser} \oplus \underbrace{\Gamma_{Witt}}_{A, D, E_6, E_7 \text{ root lattices}}$$

$$Q \cdot \Gamma_W = 0, \ (\text{for } \sigma_H \leq 1)$$

$$N = \text{rank}\, \Gamma$$

<u>Edge degrees of freedom</u>

$\exists$ N gapless, chiral scalar Bose fields, $\varphi_1, \dots, \varphi_N$, with propagation speeds $v_1, \cdots, v_N$

(possibly all different);

vacuum Ω, with

(i) $j_{em} = \sum_{i=1}^{N} Q_i \cdot \partial\varphi_i \ , \ Q = (Q_1, \cdots, Q_N)$

(ii) Phys. states obt. by appl.

- polyn. in $\{\partial\varphi_1, \cdots, \partial\varphi_N\}$
- vertex operators

$$:\exp i \sum_{j=1}^{N} q_j^{\alpha} \varphi_j :,$$

to Ω,

$$q^{\alpha} = \begin{pmatrix} q_1^{\alpha} \\ \vdots \\ q_N^{\alpha} \end{pmatrix} \in \Gamma$$

(q_j^{α}) analogous to CKM.

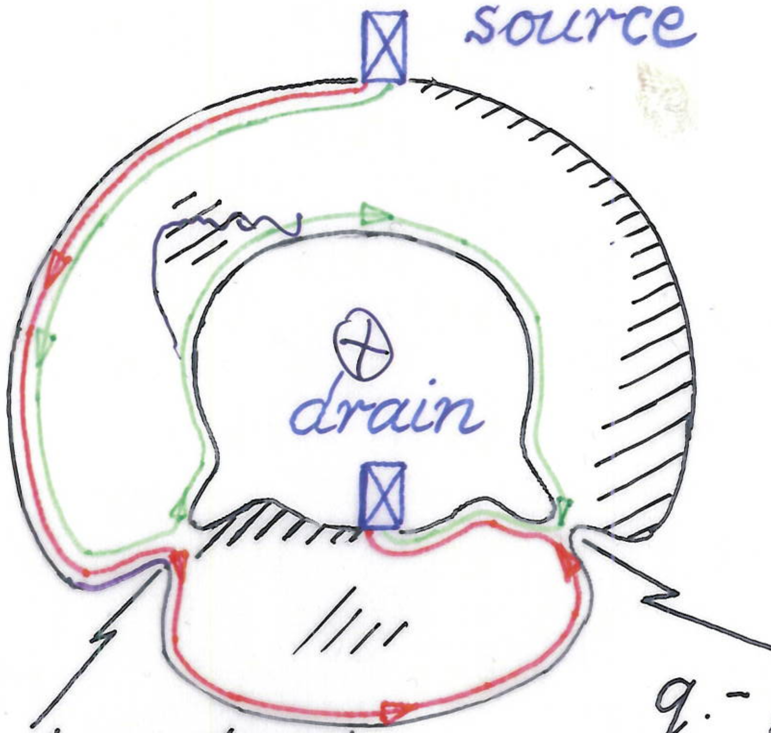
Theory has "approx." Kac–

Moody symm. at level 1 corresp. to Γ_{Witt}. But if $v_1, \dots, v_N$ not all equal symm. not exact, QFT_{edge} not conformal.

Observable consequences.

"Visibility" in interference experiments (M-Z, ↗ L & S) involving tunneling of quasi-particles across quantum point contacts (drawing!)

⟶



Measure $\underline{I}$ as $\underline{\text{fu.}}$ of $\Delta\mu$ & Φ

First proposal of F-P inter-ferometer: F-Pedrini

First proposal to use quasi-particles as Qbits: F.

Topological screening (L-F-S).

Gapless (including conformal) QFT's on graphs, with vertices = q.p.c.'s; mass generation by "inter-edge tunneling".

$\longleftrightarrow$ "Branched world sheets"

Analogous problems for fluids w. non-abelian braid stat. (free field reps., $\nearrow$ A. Wassermann)

?

5. Summary

Physics of IQHF ($\sigma_H, q^{el.}_{min}, \ldots$)

encoded in

3D TFT ($\supset$ CS th. for $\mathcal{B}$)

$\sim$ quasi-rat. braided $\otimes$ cat.

holography $\sim$ chiral 2D QFT

$\uparrow$

descr. edge degs. of freedom

(anomalous chir. edge curr.)

$\rightarrow$ quasi-parts. w. quantum # (q, λ), exh. braid statistics.

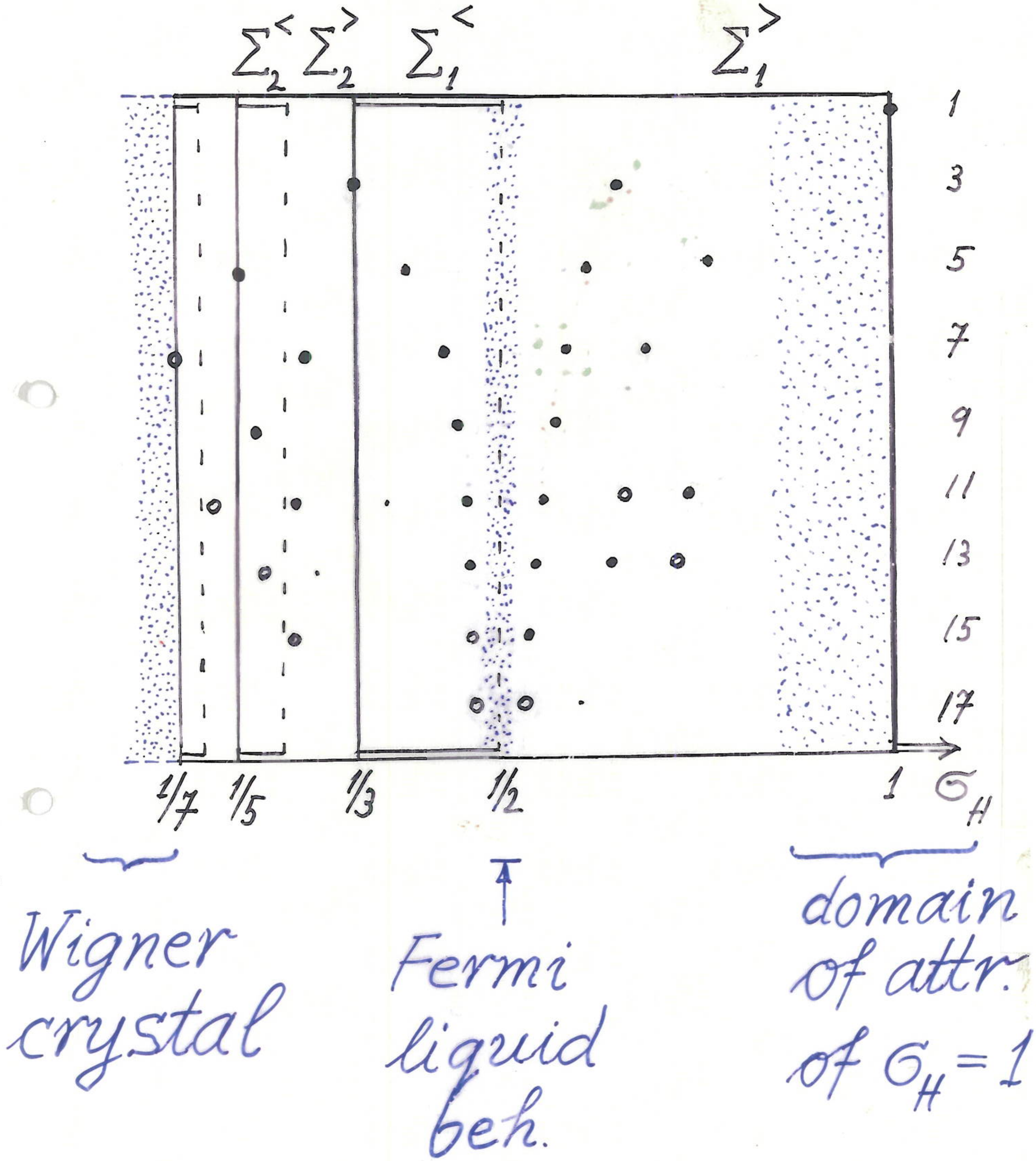
For abelian braid statistics:

$$\text{IQHF} \leftrightarrow (\Gamma, \underline{Q} \in \Gamma^*, \text{"CKM"})$$

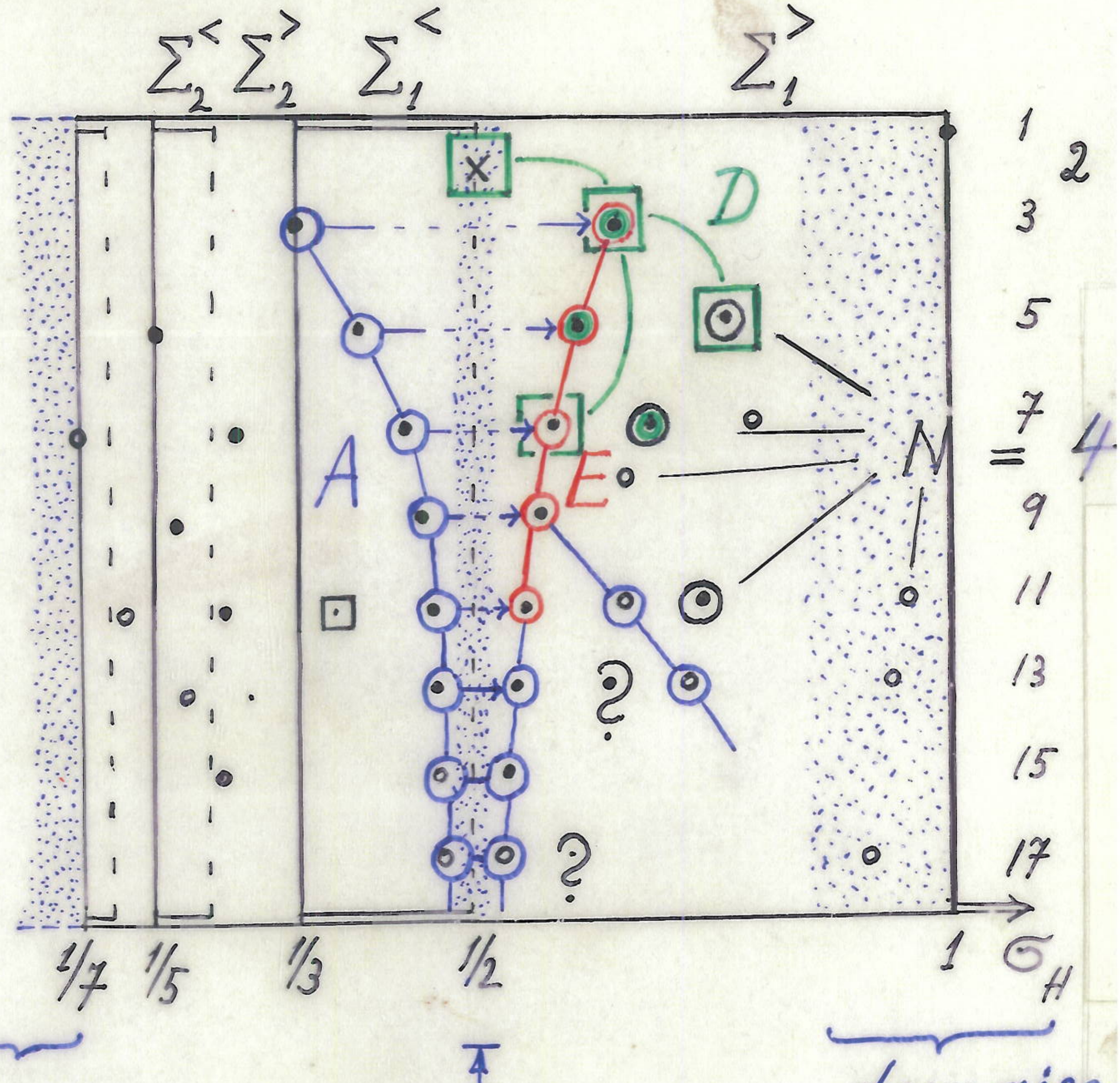
Γ: odd-int. lattice

Q: visible vect. in Γ^*: $\sigma_H = Q \cdot Q \in \mathbb{Q}$

Exp. data.


$\Sigma_2^<$ $\Sigma_2^>$ $\Sigma_1^<$ $\Sigma_1^>$
1
3
5
7
9
11
13
15
17
1/7
1/5
1/3
1/2
1
σ_H
Wigner crystal
Fermi liquid beh.
domain of attr. of $\sigma_H = 1$

Exp. data.



Wigner crystal

Fermi liquid beh. (marginal liquid)

domain of attr. of $\sigma_H = 1$

(iv) Further details on "abelian fluids:

"Minimal models" of incompr. QH fluids: $\exists\, N$ "channels", each carrying cons. vector current, J_j^μ, $j = 1, \cdots, N$, with

$$J_{em}^\mu = \sum_{j=1}^{N} Q_j J_j^\mu \qquad (6)$$

Each J_j^μ derived from vector potential b_j:

$$J_j^\mu = \varepsilon^{\mu\nu\lambda}\, \partial_\nu b_{j,\lambda}$$

In scaling limit, action for J_j^μ, $j = 1, \cdots, N$, given by

$$S^{\Lambda}(b;A) = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{\Lambda} b_j \wedge db_j + \frac{1}{2\pi} \int_{\Lambda} J^{\mu}_{em} A_{\mu} d^3x + \text{b.t.} \quad (7)$$

$\Longrightarrow$ Eqs. of motion for J^{μ}_j :

$$db_j \overset{(6)}{=} Q_j dA \Longleftrightarrow J^{\mu}_j = Q_j \varepsilon^{\mu\nu\lambda} \partial_{\nu} A_{\lambda}$$

$$\Longrightarrow J^{\mu}_{em} = \sum_{j=1}^{N} Q_j J^{\mu}_j$$

$$= \left(\sum_{j=1}^{N} Q_j^2 \right) \varepsilon^{\mu\nu\lambda} \partial_{\nu} A_{\lambda}$$

$$\overset{!}{=} \sigma_H \varepsilon^{\mu\nu\lambda} \partial_{\nu} A_{\lambda}$$

$$\Longrightarrow \sigma_H = \sum_{j=1}^{N} Q_j^2 \quad (8)$$

In scaling limit, excitations described by static charges, $q := (q_1, \dots, q_N)$, corresp. to Wilson lines

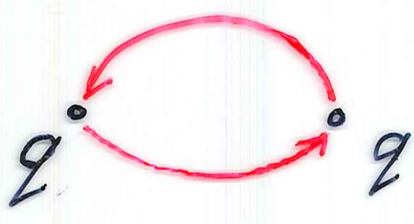
$$\exp\Big(i \sum_{j=1}^{N} q_j \int_{\mathcal{L}} b_j \Big), \qquad (9)$$

in CS - funct. integral.

El. charge of exc. q:

$$Q_{el}(q) = \sum_{j=1}^{N} Q_j \, q_j \equiv \langle Q, q \rangle \qquad (10)$$

Aharonov-Bohm phase for exchange of 2 exc. of "charge" q,



$$\exp i\pi \langle q, q \rangle ,$$

where $\langle q,q \rangle = \sum_{j=1}^{N} q_j^2$. (11)

(Monodromies $= e^{i2\pi \langle q, q' \rangle}, \dots$)

$\Gamma_{phys} := \{ q \mid q$ is charge vector of phys. excit. $\}$

<u>Physical constraints on</u> Γ_{phys}:

(a) Γ_{phys} a <u>lattice</u>

(b) $Q_{el}(q)$ even / odd $\Leftrightarrow$ q corresp. to even / odd # of el., holes.

$\Leftrightarrow \exp(i\pi \langle q,q \rangle) = \pm 1$

i.e.,

$$Q_{el}(q) = \langle Q, q \rangle \equiv \langle q, q \rangle \pmod 2$$

(rel. betw. el charge & stat.)

(c) $\exists\ q_0 \in \Gamma_{phys}$ s.t.

$$Q_{el}(q_0) = \pm 1$$

$$\Gamma := \{ q \mid \langle Q, q \rangle \in \mathbb{Z}, \text{ with (ii) \& (iii)} \} \subseteq \Gamma_{phys} \tag{12}$$

Γ is odd-integral lattice, Q is a primitive vector in Γ^* (= lattice dual to Γ)

$\Longrightarrow$ $\sigma_H = \langle Q, Q \rangle \in \mathbb{Q}$!

$$\Gamma \subseteq \Gamma_{phys} \subseteq \Gamma^*$$

Math. problem: Classify pairs (Γ, Q), $Q \in \Gamma^*$ primitive, Γ odd-int., indecomposable.
(F, Studer, Thiran)

$$\Sigma_p^{>} := \left\{ \frac{1}{2p} \leq \sigma_H \leq \frac{1}{2p-1} \right\}$$

$$\Sigma_p^{<} := \left\{ \frac{1}{2p+1} \leq \sigma_H < \frac{1}{2p} \right\}$$

$$\mathcal{H}_p^{\gtrless} := \left\{ (\Gamma, Q) \,\middle|\, \langle Q, Q \rangle \in \Sigma_p^{\gtrless}, \; \ell_{max} = 2p+1 \right\}$$

$$\mathcal{S}_p : \mathcal{H}_1^{\gtrless} \xrightarrow[1:1]{} \mathcal{H}_{p+1}^{\gtrless}$$

Idea of classification of (Γ, Q)

(1) $\Gamma = \Gamma_e \oplus \Gamma_h$, $Q = Q_e \oplus Q_h$

$\quad\quad >0 \quad <0$

$(\Gamma_{e/h}, Q_{e/h})$: chir. QHL

$\longrightarrow$ Classify the cQHL's.

(2) (Γ, Q) a cQHL.

$$\Gamma = \bigoplus_{k=1}^{K} \Gamma_k, \quad Q = \bigoplus_{k=1}^{K} Q_k$$

$\uparrow$ indecomposable

$\longrightarrow$ Classify indecomp.

cQHL, (Γ, Q).

Use INVARIANTS!

(3) Invariants of Γ:

(a) $c = N = \dim \Gamma$ (rank Γ)

(b) $\Delta := |\Gamma^* / \Gamma|$

(c) $\Delta \langle Q, Q \rangle \bmod 8$

(d) genus of Γ:

$$\left\{ e^{i 2\pi \langle q, q' \rangle} \,\middle|\, q, q' \in \Gamma^* \right\}$$

(e) $\Gamma_W \oplus \mathcal{O} \subseteq \Gamma \subseteq \Gamma_W^* \oplus \mathcal{O}^*$

Γ_W : $\oplus$ A, D, E

Kneser shape

(4) Invariants of (Γ, Q):

(a') $\sigma_H = \langle Q, Q \rangle \equiv \frac{n}{d}$

$$\sigma_H < 2 \Longrightarrow Q^{\perp} \subseteq \Gamma_W^* .$$

(b') $\Delta = \lambda g d$

$(\lambda d)^{-1}$: smallest fract. el. charge of Γ^*

(c') $$l_{max} = \min_{\substack{\{e_i\} \\ \langle Q, e_i \rangle = 1 \\ \forall i}} \left(\max_i \langle e_i, e_i \rangle \right)$$

$$\Delta \leqslant l_{max}^{N}, \quad \sigma_H \geqslant l_{max}^{-1}$$

$$\left\{ {}_N\left(\frac{n}{d}\right)^g_\lambda, \; l_{max} \right\}$$

Results.

$$\#(\Gamma, Q) \text{ w. } \left\{ \begin{array}{l} \sigma_H \leq 1 \\ N \leq N_* \\ \ell_{max} \leq L_* \end{array} \right\} \text{ finite}$$

Stability criterion: Given σ_H ($R_L = 0$), most stable ab. QH-fluid described by cQHL (Γ, Q) with $\langle Q, Q \rangle = \sigma_H$, N and ℓ_{max} as small as possible. – Pheno: $\ell_{max} \leq 7$ (Wigner crystal instab.)

Remarkable fact:

$$\mathcal{H}_1^{<} \xrightarrow[\mathcal{S}_p]{} \mathcal{H}_p^{<} = \left\{ \left(\Gamma_{N,p}, Q_{N,p} \right) \right\}_{N=1,2,3,\cdots}$$

uniqueness!

$$\sigma_H = \langle Q_{N,p}, Q_{N,p} \rangle = \frac{N}{2pN+1}$$

Gramm matrix of $\Gamma_{N,p}$ (in suitable basis):

$$K_{N,p} = \left(\begin{array}{c|cccc} 2p+1 & -1 & 0 & \cdots & \\ \hline -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ \vdots & & & \ddots & \\ & & & -1 & 2 \end{array} \right)$$

Cartan of A_{N-1}; $\widehat{su}(N)_{k=1}$ – Kac–Moody symm.!

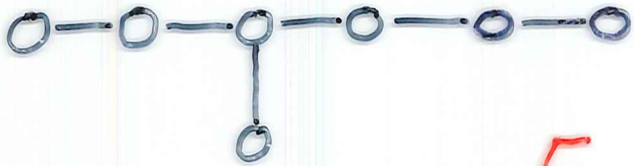
$$Q_{N,p} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

cQHL in $\mathcal{H}^{>}_{1}$: complicated

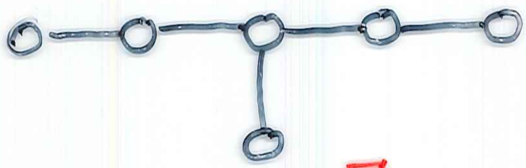
E-series: $\mathcal{H}^{>}_{p}$

p.t.'s

E_7



E_6



$E_5 \equiv D_5$

$E_4 \equiv A_4$

$E_3 \equiv A_2 \oplus A_1$

$$\sigma_H = \frac{n}{2n-1}, \quad n = 2, 3, \cdots, 6.$$

D-series: D_n's, $4 \leq n \leq 7$

$$\sigma_H = \frac{4}{12-n}$$

Secondary A-series:

$$\sigma_H = \frac{N}{N+4}, \quad N \geq 5 \ \underline{\text{odd}}.$$

Classified all cQHL $\in \mathcal{H}_p^{>}$ with

$N=2$ (Gauss), $N=3$ (Dickson)
2 6

$N=4$ (Computer), + E-series,
27 + ...

Composite QHL, $\ell_{max}=3$.

$$\sigma_H = \sigma_e - \sigma_h = 1 - \frac{N}{2N+1} = \frac{n}{2n-1}$$

$$\sigma_H = \frac{N}{2N+1} + \frac{M}{2M+1}.$$

Now compare to exp. data:

$$\sigma_H = \frac{1}{2}, \left(\frac{5}{2}\right)$$

$$K = \left(\begin{array}{c|cc} 3 & 1 & 1 \\ \hline 1 & 2 & 0 \\ 1 & 0 & 2 \end{array}\right)$$

$$\Gamma_W = A_1 \oplus A_1$$

$\uparrow$ spin $\quad$ $\uparrow$ 2 layers

$$\Rightarrow \quad \mathfrak{g} = su(2) \oplus su(2)$$

Coset construction:

$$\widehat{su}(2)_k \oplus \widehat{su}(2)_1 \Big/ \widehat{su}(2)_{k+1}$$

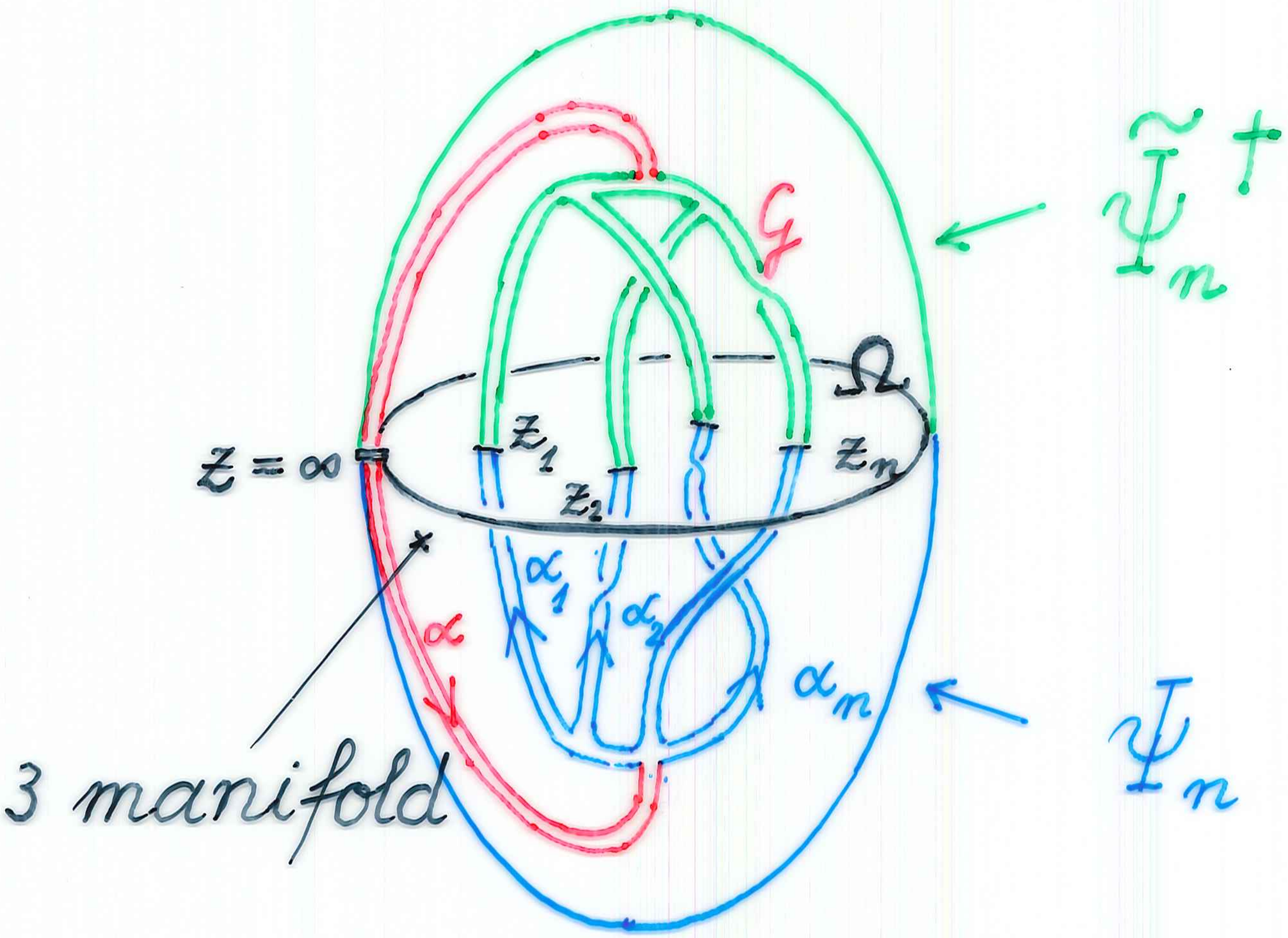
Virasoro min. models

<u>$k = 1$</u> $\longleftrightarrow$ chiral Ising

$\mathcal{E} \longleftrightarrow$ chir. Is. $\rtimes$ $U(1)_{em}$

Pfaffian wave fu. (R-M)

(v) State space of 3D TFT



Ψ_n : conformal block of some 2D CCFT with chiral algebra

$\hat{u}(1) \times \mathcal{A}$

with $\mathrm{Rep}\,\mathcal{A} = \mathcal{T}$

(See F-K, F-P-S-W, F-F-F-S)

Ψ_n depends on "half-ribbon" graph with lines decorated by reps. $\alpha \in \mathcal{C}$ s.t., at each vertex, fusion rule $\neq 0$, & on end points $z_1, \cdots z_n$.

$\langle \tilde{\Psi}_n, \Psi_n \rangle$ = Invariant of decorated Ribbon Graph, $\mathcal{G}$, det. by TFT $\mathcal{C}$; indep. of $z_1, \dots, z_n$!

May represent Ψ_n as horizontal section of vector bundle, E_n, with flat connection, ω_n, over base space $\Omega^{\times n} \setminus$ Diag.; with fibre $\simeq$ {inv. linear functs. $I_{\bar{\alpha}, \alpha_1, \dots, \alpha_n} \overset{\text{III}}{}$ on $[\bar{\alpha}] \otimes [\alpha_1] \otimes \cdots \otimes [\alpha_n]$}

($\simeq$ holomorphic functions on $(\Omega^{\times n} \setminus \text{Diag.})^{\sim}$ w. values in $I_{\bar{\alpha}, \alpha_1, \dots, \alpha_n}$).

If TFT is a CS-theory

$$d\Psi_n = \omega_n \Psi_n ,$$

$$\omega_n = \kappa \sum_{1 \leq i < j \leq n} d \ln (z_i - z_j)\, \Omega_{ij} \qquad \text{(K-Z)}$$

$\kappa = (k + c_G)^{-1}$, Ω_{ij} solus. of
k ← level

class. Yang-Baxter Eqs.

For abelian IHF's considered in Sect. (iv),

$\alpha_i = q_i \in \Gamma_{phys}$, $i = 1, \cdots, n$,

$\Omega_{ij} = \langle q_i, q_j \rangle$, $\bar{q} = -\sum_{i=1}^{n} q_i$

$$\Psi_n(q_1, z_1, \cdots, q_n, z_n, \bar{q})$$

$$\propto \prod_{1 \le i < j \le n} (z_i - z_j)^{\langle q_i, q_j \rangle} \times$$

$$\exp\left[\sum_{i=1}^{n} \langle q_i, \bar{q} \rangle h(z_i)\right],$$

Tantalizing resemblance w. Laughlin wave functions. Reproduces correct stat. phases under braiding, ...

5. Edge degrees of freedom – QH interferometry

(i) General ideas.

If bulk of IHF descr., in scaling lim, by 3D TFT based on braided tensor cat. $\mathcal{C}$,

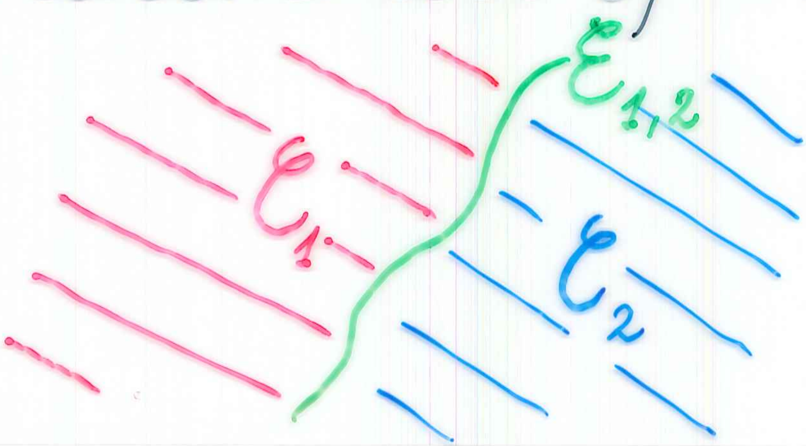
$$\mathcal{C} \subset Rep(\hat{u}(1) \times \mathcal{A}),$$

then one possible descr. of edge degs. of freed. is in terms of CCFT w. chiral algebra $\hat{u}(1) \times \mathcal{A}^{(\prime)}$ and same repr. cat. $\mathcal{C}$ such that all bulk & edge anomalies cancel each other.

An instance of "holography".

However: Edge could consist of several layers of distinct IHF's before vac. is reached. → Describe all possible edge ths. of degs. of freedom on interface betw. two distinct IHF's canceling all anomalies of 2 bulk ths.

?



(ii) Edge theories for abelian IHF's, assuming "holography"

Start from "min. models" of IHF's with N channels, currents J_i^μ, $i = 1, \cdots, N$

$$J_j^\mu = \varepsilon^{\mu\nu\lambda} \partial_\nu b_{j,\lambda}, \tag{1}$$

CS action

$$S^\Lambda(b, A) = \frac{1}{4\pi} \sum_{j=1}^{N} \int_\Lambda b_j \wedge db_j + \frac{1}{2\pi} \int_\Lambda J_{em}^\mu A_\mu \, d^3x + b.t., \tag{2}$$

$$J_{em}^\mu = \sum_{j=1}^{N} Q_j J_j^\mu \tag{3}$$

$$\Rightarrow \quad J_j^\mu = Q_j \, \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda,$$

$$\Rightarrow \partial_\mu \mathcal{J}_j^\mu = \underbrace{\varepsilon^{\mu\nu\lambda}(\partial_\mu Q_j)\partial_\nu A_\lambda}_{\text{anomaly}}, \quad (4)$$

$$\Rightarrow \quad \mathcal{J}_{em}^\mu = \sigma_H \, \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \text{ w.}$$

$$\sigma_H = \sum_{j=1}^N Q_j^2 \qquad (5)$$

<u>Idea</u> ("holography"):
With each $\mathcal{J}_j^\mu$, associate an anomalous chiral edge current j_j^μ such that anomaly (4) is cancelled; i.e., $\partial_\mu j_j^\mu = -Q_j \, F\big|_{\partial\Lambda}$ (6)

General th. of (1+1)-dim. chiral anomaly:

$j^\mu = j^\mu_{L/R}$: anomalous chiral current on $\partial\Lambda$.

$$j^\mu_{L/R} = j^\mu_v \pm j^\mu_a \quad \text{with} \qquad (7)$$

$$\partial_\mu j^\mu_v = 0, \ \partial_\mu j^\mu_a = -QF\big|_{\partial\Lambda} \qquad (8)$$

$$\Downarrow$$

$j^\mu_v = \varepsilon^{\mu\nu}\partial_\nu \varphi$, φ a scalar

$$\partial_0 = \frac{1}{u}\frac{\partial}{\partial t}, \ \partial_1 = \frac{\partial}{\partial x}$$

u: propagation speed of $j^\mu_{L/R}$

In 2 space-time dim.,

$$j^\mu_a = \varepsilon^{\mu\nu} j_{v,\nu}$$

$$\Rightarrow \quad j^\mu_a = \partial^\mu \varphi, \text{ with} \qquad (9)$$

$$\partial_\mu \partial^\mu \varphi = -QF\big|_{\partial\Lambda}, \qquad (10)$$

by (8). Eqs. of motion (10)

of φ can be derived from action functional

$$S(\varphi;a) = \frac{u}{4\pi}\int_{\partial\Lambda} dx\,dt\,\Big\{\partial_\mu\varphi\,\partial^\mu\varphi + 2Q\,\underbrace{\varepsilon^{\mu\nu}(\partial_\nu\varphi)}_{j_v^\mu}a_\mu\Big\} \qquad (11)$$

w. $a = A\big|_{\partial\Lambda}$, $\varepsilon^{\mu\nu}\partial_\mu a_\nu = F\big|_{\partial\Lambda}$.

$$\frac{\delta S(\varphi;a)}{\delta\dot\varphi(\xi)} = \frac{1}{u}\,\dot\varphi(\xi) =: \Pi(\xi),$$

$$[\Pi(x,t),\varphi(y,t)] = -2\pi i\,\delta(x-y) \qquad (12)$$

(CCR)

$$\Rightarrow\quad [j_v^0(x,t), j_v^1(y,s)] = 2\pi i\,\delta'(x-y), \qquad (13)$$

j_L^μ and j_R^ν commute;

but

$$[j_L^0(x,t), j_L^0(y,t)] = -[j_L^0(x,t), j_L^1(y,t)]$$
$$= 4\pi i \delta'(x-y); \quad (14)$$

etc.

Chiral vertex operators

$$V_{\substack{L\\R}}(q;x,t) =$$
$$:\exp\left[\mp iq\left(\varphi(x,t) + \int_S^x \Pi(y,t)\,dy\right)\right]:$$

w. b.c. $\varphi\big|_S = cst.$ (S: "source")

If $\partial\Omega \simeq S^1$ these ops. are multi-valued (shift of S)!

Commutation relations

$$V_L(q';x,t)\,V_L(q;y,t) \quad (15)$$
$$= e^{-i\pi q'\cdot q\, sig(x-y)}\, V_L(q;y,t)\,V_L(q';x,t)$$

$sig(x-y)$ dep. on position of source S!

Charge operators

Currents $\hat{j}^{\mu}_{\substack{L\\R}} := j^{\mu}_{\substack{L\\R}} \mp Q\varepsilon^{\mu\nu}A_{\nu}$ are (not gauge-inv., but) conserved $\rightarrow$ define

$$Q_{\substack{L\\R}} := \frac{Q}{2\pi}\int_{\partial\Omega} dx\ \hat{j}^{0}_{\substack{L\\R}}(x,t) \qquad (16)$$

Then, from (12), (14),

$$[Q_L, V_L(q;x,t)] = Q\cdot q\ V_L(q;x,t) \qquad (17)$$

Generalization to N channels

N chiral edge currents, $j^{\mu}_{L,i}$, $i=1,\cdots,N$, w. propagation speeds $u_1, \dots, u_N$,

$$j^{\mu}_{L,i} = \varepsilon^{\mu\nu}\partial_\nu \varphi_i + \partial^{\mu}\varphi_i, \quad (18)$$

where $\varphi_1, \dots, \varphi_N$ are N scalar Bose fields diag. "velocity matrix";

charges $Q_1, \dots, Q_N$.

$$j^{\mu}_{edge} = \sum_{i=1}^{N} Q_i \, j^{\mu}_{L,i} \quad (19)$$

Then

$$\partial_\mu j^{\mu}_{edge} = -\sigma_H \, F\big|_{\partial\Lambda},$$

$$\sigma_H = \sum_{i=1}^{N} Q_i^2 \quad (20)$$

Vertex operators

$$V_L(q; x, t) = \; :\exp\Big[-i\sum_{i=1}^{N} q_i\Big(\varphi_i(x,t) + \int_S^x \Pi_i(y,t)\,dy\Big)\Big]:;$$

$q = (q_1, \cdots, q_N)$; $(V_R = \cdots)$.

They deposit an el. charge

$$\langle Q, q \rangle = \sum_{i=1}^{N} Q_i q_i$$

and give rise to stat. phases

$$\exp[i\pi \langle q', q \rangle \operatorname{sig}(x-y)],$$

under exchange, where

$$\langle q', q \rangle = \sum_{i=1}^{N} q_i' \cdot q_i$$

Vertex ops. creating single electrons: $V_L(q^\alpha; x, t)$, $\alpha = 1, \cdots, N$

$\langle Q, q^\alpha \rangle = -1$, $\langle q^\alpha, q^\alpha \rangle = 2n_\alpha + 1$,

$n_\alpha \in \mathbb{Z}$, $\forall \alpha$.

$\{q^1, \cdots, q^N\} \to$ odd, int lattice Γ

(q_i^α): analogue of CKM matrix

$$K^{\alpha\beta} := \langle q^\alpha ; q^\beta \rangle \rightarrow$$

K matrices.

There may also be N_1 left-moving & N_2 right-moving currents $\cdots \rightarrow (\Gamma_1, Q^1) \oplus (\Gamma_2, Q^2)$,

$$\sigma_H = \langle Q^1 ; Q^1 \rangle - \langle Q^2 ; Q^2 \rangle$$

(back scattering by impurities becomes possible → possible opening of gaps!)

<u>Physical parameters to be determined:</u>

(1) N_1, N_2

(2) Propagation speeds

$u_1, \dots, u_{N_1}; v_1, \dots, v_{N_2}$

(3) CKM matrices

$(q_{1,i}^{\alpha}), (q_{2,i}^{\alpha});$

in particular $(K_1^{\alpha\beta}), (K_2^{\alpha\beta});$

conf. dimensions

$\langle q_1^{\alpha}, q_1^{\alpha} \rangle, \langle q_2^{\beta}, q_2^{\beta} \rangle$

K-matrices universal, prop. speeds & CKM matrices generally depend on sample preparation & leads.

$\longrightarrow$ Interferometry

(F, P; Levkivskyi, Sukhorukov)

The 4D/5D QHE and Cosmic Magnetic Fields

Jürg Fröhlich
ETH Zurich & IAS, Princeton

110th Conference on Statistical Mechanics, Dec. 2013

Credits:

Work carried out with

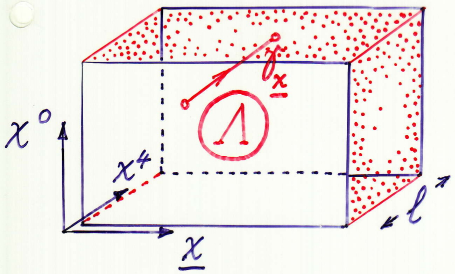
Bill Pedrini (1999)

and continued with

Alexey Boyarsky and Oleg Ruchayskiy (2009-present)

A Higher-diml. Cousin of the QHE

"Space-time," Λ = slab of width $\ell \subset M^5$; $\partial\Lambda$: two $\|$ 3-branes ("visible world")



5D vector pot.

$\hat{A} = (A, A_4)$

$A := \hat{A}_{\|}\big|_{\partial\Lambda}$

Bulk degrees of freedom: Very heavy particles (4-comp. Dirac fermions) coupled to $\hat{A}$, w. breaking of P & T.

$\rightarrow$ ~ massless, chiral surface

(left-handed waves *(L)* at $x^4 = 0$, right-handed ones *(R)* at $x^4 = l$)

waves (masses from tunneling between branes): observed light particles ~ edge degrees of freedom in Hall fluid.

$\Downarrow$

<u>5D analogue of Hall's Law</u>

$$j^\mu = \frac{1}{4}\sigma_H \varepsilon^{\mu\nu\lambda\rho\sigma} \hat{F}_{\nu\lambda} \hat{F}_{\rho\sigma} \quad (13)$$

or

$$\boxed{J = \sigma_H \hat{F} \wedge \hat{F}}$$

valid except where σ_H jumps

$j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{brane}$: conserved

$$j^\mu_{brane} = j^\mu_L \delta_{\partial_-\Lambda} + j^\mu_R \delta_{\partial_+\Lambda}$$

$$(13) \Rightarrow \boxed{\partial_\mu j^\mu_{L/R} = \sigma_H \, \underline{E} \cdot \underline{B}} \tag{14}$$

Chiral anomaly in 3+1 D!

$$\sigma_H = \sum_{\substack{\text{fermion} \\ \text{species}}} \frac{Q_i^3}{4\pi^2} \tag{15}$$

Axion

$$\varphi(\underline{x},t) := \int_{\gamma_{\underline{x}}} \hat{A}_4(\underline{x},x^4,t)\, dx^4$$

$\hat{A}_\mu = A_\mu$, $\mu = 0,1,2,3$, indep. of x^4.

$$\dot{\varphi}(\underline{x},t) = \int_{\gamma_{\underline{x}}} \hat{E}_4(\underline{x},x^4,t)\, dx^4$$

$$= V(\underline{x},t) = \mu_R - \mu_L \tag{16}$$

Then (13) becomes:

$$j^0 = \frac{\sigma_H}{6} \, \underline{\nabla} \cdot (\varphi \underline{B})$$

$$\underline{j} = \frac{\sigma_H}{6} \left\{ (\varphi \underline{B})^{\cdot} + \underline{\nabla} \wedge (\varphi \underline{E}) \right\} \tag{17}$$

For $\varphi = (\mu_R - \mu_L)\, t$,

$$\boxed{\underline{j} = \frac{\sigma_H}{6} (\mu_R - \mu_L) \underline{B}} \tag{18}$$

Plug (17) into Maxwell's Eqs.

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = 0 \tag{19}$$

$$\underline{\nabla} \cdot \underline{E} = \frac{\sigma_H}{6} \underline{\nabla} \varphi \cdot \underline{B}$$

$$\underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = \frac{\sigma_H}{6} \left\{ \dot{\varphi} \underline{B} + \underline{\nabla} \varphi \wedge \underline{E} \right\} \tag{20}$$

5D Maxwell v (13) $\cdots \Longrightarrow$

$$\Box \varphi = -\frac{\sigma_H}{6} \underline{E} \cdot \underline{B} - U'(\varphi) \tag{21}$$

U': periodic fu. of φ.

System of NL hyperbolic PDE's $\rightarrow$ Toy for mathematician

Special solution:

$\underline{E} = \underline{B} = 0; \quad \varphi = \varphi(t)$, ind. of $\underline{x}$,

solution of

$$\ddot{\varphi} = -U'(\varphi) \quad \text{(pendulum)}$$

Linearization of (19)–(21) around this solution:

Parametric resonance $\rightarrow$ unstable Fourier modes, $\underline{\tilde{E}}_{\underline{k}}, \underline{\tilde{B}}_{\underline{k}}$ with $\underline{\tilde{E}}_{\underline{k}} \cdot \underline{\tilde{B}}_{\underline{k}} \neq 0$.

A simple special case:

$$\underline{j} = \sigma_T \underline{B} + \sigma_\Omega \underline{E} \leftarrow \text{Ohm}$$

$$\sigma_T = \sum_i \frac{q_i^2}{4\pi h} (\mu_r^i - \mu_\ell^i)$$

$\sigma_\Omega \gg \sigma_T > 0$: primordial plasma

Maxwell's eqs.

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \cdot \underline{E} = 0.$$

$\Rightarrow$ $\underline{E}, \underline{B}$ transv. pol.

$$\underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = 0, \quad \underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = \sigma_T \underline{B} + \sigma_\Omega \underline{E}$$

Fix wave vector $\underline{k} = k \underline{e}_3$, $k > 0$; $\underline{X}^T$: comp. of $\underline{X} \perp \underline{k}$

Then

$$\begin{pmatrix} \dot{\underline{E}}^T \\ \dot{\underline{B}}^T \end{pmatrix} = K(k) \begin{pmatrix} \underline{E}^T \\ \underline{B}^T \end{pmatrix}, \quad \text{with}$$

$$K(k) = \left(\begin{array}{cc:cc} -\sigma_\Omega & 0 & -\sigma_T & -ik \\ 0 & -\sigma_\Omega & ik & -\sigma_T \\ \hdashline 0 & ik & 0 & 0 \\ -ik & 0 & 0 & 0 \end{array}\right)$$

Eigenvalues of $K(k)$: $i\omega_\alpha(k)$, $\alpha = 1, \dots, 4$ (circ. frequ. of normal modes).

If $i\omega_{\alpha_0}(k) > 0 \Rightarrow$ expon. growing normal mode!

$$i\omega(k) = \frac{-\sigma_\Omega \pm \sqrt{\sigma_\Omega^2 - 4k(k \pm \sigma_T)}}{2}$$

(hand-made calculation)

$\Rightarrow$ For $0 < k < \sigma_T$, $\exists!$ one positive solution,

$$i\omega_{\alpha_0}(k) \approx \frac{k(\sigma_T - k)}{\sigma_\Omega}$$

$\hookrightarrow$ Origin of seed magn. fields in the universe?

But need "large", time-dependent initial axion configuration.

Remarks

(1) The (time derivative of the) axion field really might be a space-time dependent "chiral chemical potential" (rather than a dynamical degree of freedom). Thus, presumably, its equation of motion is a diffusion equation.

(2) 4D Electrodynamics in the presence of an axion field also appears in the theory of 3D topological insulators (TI). Similar instabilities might then be observed when an external electric field is applied: If such a field exceeds a certain critical strength then, in the bulk of a TI, it is screened and converted into a magnetic field; (Ooguri & Oshikawa, Fröhlich & Werner)

Everything else next time!

Thank you!

FROM THE QHE TO "TOPOLOGICAL INSULATORS"

A UNIFIED PERSPECTIVE

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1. Introduction

Purpose of analysis (90's)

Classify

- states of NR bulk (cond.) matter at $T \geq 0$; &
- its surface states —

using ideas & concepts from gauge theory & GR :
(scaling lim of) effective actions, gauge invariance, anomaly cancellation, "holography".

Illustrate program on 2D & 3D electron gases; ($\cdots\to$ atom gases, BEC, ..., primordial plasma, stellar matter, ...: N.t.)

General ideas of approach:

(1) Find all (fund. & accid.) global (int.) symmetries & corresp. conserved currents.

(2) Promote global to local symmetries – "gauging".

(3) Study response of syst. to turning on "tiny" external gauge fields & varying g_{ij}.

→ determine (form of) effective action (free energy) = gener. funct. of <u>current</u> Green fns.

(1) + "order param." → Landau theory

(1) ÷ (3) → "<u>Gauge Theory of States of Matter</u>" (early 90's)

"topological <u>order</u>".

Today, will apply this to el. systems w. bulk mobility gap: QHE, (2+1)D & (3+1)D "topol. insulators", axion QED

2. Electron Gases in 2&3 D

QM of single el. governed by **Pauli Equation**:

$$i\hbar \frac{\partial}{\partial t} \Psi_t = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 \Psi_t \,, \quad \text{(PE)}$$

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

(1) Symmetries of (PE)

- Global phase rot. : $U(1)$

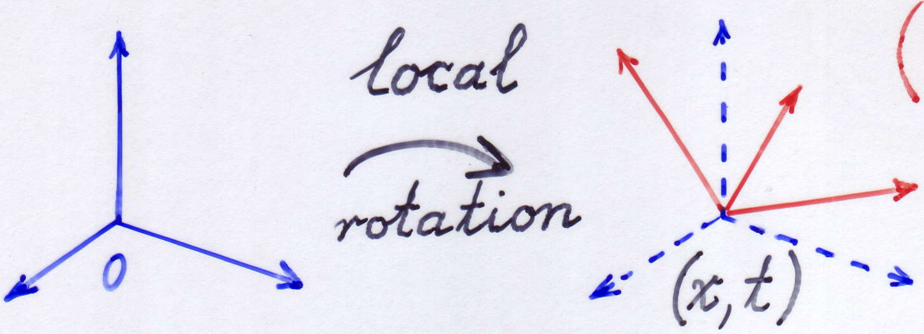
Conserved current: $j^0 = \Psi^* \cdot \Psi$,

$$\vec{j} = \frac{i\hbar}{2m}\left\{ \Psi^* \cdot \vec{\nabla}\Psi - (\vec{\nabla}\Psi^*) \cdot \Psi \right\}$$

- Rotations in spin space: $SU(2)$

$\longrightarrow$ spin current: s^0_κ, $\vec{s}_\kappa \in su(2)$.

(2) Gauge these symmetries:



$\left(\hat{e}_\kappa\right)_{\kappa=1}^{3}$

space-time dep. frames

$$\frac{\hbar}{i}\frac{\partial}{\partial x^j} \longmapsto \frac{\hbar}{i} D_j := \frac{\hbar}{i}\partial_j + a_j + w_j$$

$$i\hbar\frac{\partial}{\partial t} \longmapsto i\hbar D_0 := i\hbar\partial_t + a_0 + w_0$$

$$w_\mu = w_\mu^{\kappa}\sigma_\kappa \qquad\qquad U(1) \quad SU(2)$$

covariant derivatives

$$\left(\delta^{ij}\right) \longmapsto \left(g^{ij}(x)\right)$$

Covariant Pauli Eq.:

$$i\hbar D_0 \Psi_t = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}} D_k\left(\sqrt{g}\, g^{kl}\right) D_l \Psi_t$$

(CPE)

Action:

$$S(\Psi^*, \Psi; g, a, w) :=$$

$$\int dt \int \sqrt{g}\, d^D x \Big[\Psi_t^* \cdot i\hbar D_0 \Psi_t - \frac{\hbar^2}{2m} (D_k \Psi_t^*) \cdot g^{kl} D_l \Psi_t - \lambda |\Psi_t|^2 * \Phi\, |\Psi_t|^2 \Big]$$

For $\lambda = 0$,

$$\frac{\delta S}{\delta \Psi^*} = 0 \iff (\mathrm{CPE})$$

$\lambda \neq 0$: 2-body interactions with potential $\lambda \Phi$

Many-body th.:

$$(\Psi^*, \Psi) \longmapsto (\bar{\Psi}, \Psi) \quad \text{Grassm. V.}$$

(3) Berezin integral:

$$Z(g,a,w) := \int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\, e^{\frac{i}{\hbar} S(\overline{\Psi},\Psi;g,a,w)}$$

$$\boxed{S_{eff}(g,a,w) := \frac{\hbar}{i} \ln Z(g,a,w)}$$

"effective action"

At $T = (k_B \beta)^{-1} > 0$, for g, a, w time-independent, may study thermal equilibr.:

$$S_{eff} \longmapsto F_{eff}(\beta; g, a, w)$$

"eff. free energy"

$$\lim_{\beta\to\infty} F_{eff}(\beta; g,a,w) = \underbrace{\Delta E_0(g,a,w)}_{\text{g.s. energy}}$$

Properties of S_{eff}/F_{eff}

(i) $$\frac{\delta S_{eff}}{\delta g^{ij}(x)} = \Big\langle \underbrace{T_{ij}(x)}_{\text{stress tensor}} \Big\rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta a_\mu(x)} = \Big\langle \underbrace{j^\mu(x)}_{\text{em current density}} \Big\rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta w_\mu^{\kappa}(x)} = \Big\langle \underbrace{S^\mu_\kappa(x)}_{\text{spin current -"-}} \Big\rangle_{g,a,w}$$

Higher derivatives: Connected current Green functions of $n \geq 2$ current densities.

(ii) Gauge invariance

$$S_{eff}(g, a_\mu + \partial_\mu \chi, U w_\mu U^{-1} + U \partial_\mu U^{-1})$$

$$= S_{eff}(g, a_\mu, w_\mu)$$

χ real-valued, U $SU(2)$-valued

$\Leftrightarrow$ j^μ cons., s^μ_κ covar. cons.

(ii′) General covariance

(iii) Bulk mobility gap

$\Downarrow$

Al. props. of current Green fns.

$\Downarrow$

In scaling lim,

$$S_{eff} = \sum_n \underbrace{\int \text{"gauge-inv." local poly.}}_{\text{scaling dim } n\,(=-1,0,1,2,\dots)} + \text{bd. terms}$$

Retain only "most relevant" terms $(n = -1, 0, 1)$. Ex.: $D = 2, w = 0$

$$S_{eff}(a) = \frac{\sigma_H}{2} \int_\Lambda \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \, d^3x$$

QHE

+ bd. term

3. Phys. Interpretation of a & w

$$a_0 = e\varphi - \rho^{-1} p \quad \cdots$$

$$a_k^{tot} = eA_k^{(0)} + \underbrace{eA_k + mV_k}_{\equiv a_k}$$

curl $A^{(0)}$: time-indep., per. or hom.

V: velocity field of moving background - gauge inv., usually div $V = 0$.

p: pressure ...

$$w_0^{tot} = \left(\frac{g\mu_B}{2} \vec{B} + \frac{\hbar}{4} \, curl \vec{V} \right) \cdot \vec{\sigma}$$

Zeeman

$$+ \ \vec{W}_0 \cdot \vec{\sigma}$$

Weiss exch. field

$$+ \ \vec{\omega}_0 \cdot \vec{\sigma}$$

spin conn.

$$w_k^{tot} = \left(\frac{g\mu_B}{2} + \frac{e}{4mc} \right) \left(\vec{\sigma} \wedge \left(\vec{E} + \dot{\vec{V}} \right) \right)_k$$

spin-orbit

$$+ \ \vec{W}_k \cdot \vec{\sigma} \quad + \ \vec{\omega}_k \cdot \vec{\sigma}$$

Weiss   spin conn.

Only spin connection transf. inhomogeneously under SU(2)-gauge trsf.; other contribs. to w_μ^{tot} transf. homogeneously.

4 Effective Actions/Free Energies – Examples

Electron gas in ext. gauge fields $(a_\mu^{tot}, w_\mu^{tot})$ in 2+1 or 3+1 D.

U(1) $a_\mu^{tot} = a_\mu^{(0)} + a_\mu = eA_\mu^{(0)} + eA_\mu + mV_\mu$

gauge-inv

SU(2) $w_\mu^{tot} = w_\mu^{(0)} + w_\mu$

trsf. homog.

$a^{(0)}, w^{(0)}$: diff. geom. of sample backgd., static em field

$\hookrightarrow$ explicit breaking of P, T.

For simplicity, g^{ij} fixed $(g^{ij} = \delta^{ij})$. Apply (i)÷(iii):

- $\underline{2+1\,D}$, $w_\mu = 0$, $a_\mu \neq 0$.

$$S_{eff}(a) = (2\lambda^2)^{-1}\int_\Lambda (a^T)^2 d^3x \qquad \mathrm{I}$$

$$+ \frac{\sigma_H}{2}\left[\int_\Lambda a \wedge da + \Gamma(a|_{\partial\Lambda})\right] \qquad \underline{\mathrm{II}}$$

$$+ \frac{1}{2}\int_\Lambda (\varepsilon\, \underline{\mathcal{E}}^2 + \mu^{-1}\mathcal{B}^2)\, d^3x + \cdots, \qquad \underline{\mathrm{III}}$$

$$\underline{\mathcal{E}} = -e\underline{\nabla}\varphi + e\underline{\dot{A}} + \underline{\nabla}(\rho^{-1}p) + m\underline{\dot{V}}$$

$$\mathcal{B} = \mathrm{curl}\,(e\underline{A} + m\underline{V})$$

I) Supercond., London eq. ("relevant")

$\underline{\mathrm{II}}$) Hall effect ("marginal")

$\underline{\mathrm{III}}$) Maxwell term → diel. cst. ε, magn. perm. μ ("irrelevant" → neglect)

Set $\lambda^{-2} = 0$, neglect $\underline{\underline{III}}$.

Under a gauge trsf., $a \mapsto a + d\chi$,

$$\int_{\Lambda} a \wedge da \mapsto \int_{\Lambda} a \wedge da + \int_{\partial\Lambda} \chi \, da$$

$\Rightarrow \Gamma(a|_{\partial\Lambda})$ is anomalous chir. action = gen. fu. of Green fu. of <u>chiral edge currents</u>.

<u>Response Eqs.</u> – see (i):

$$j^0 = \sigma_H \left(B + \frac{m}{e} \, \mathrm{curl} \, \underline{V} + \varkappa K + \cdots\right)$$

$$j^k = \sigma_H \varepsilon^{kl} \underbrace{\left(E_l + \frac{m}{e} \dot{V}_l\right)}_{\mathcal{E}_l} + \cdots \quad \text{(HE)}$$

$$\frac{\delta \Gamma(a|_{\partial\Lambda})}{\delta a|_{\partial\Lambda}} = j_{edge}$$

$$\partial_\mu j^\mu_{edge} = -\sigma_H \mathcal{E}|_{\partial\Lambda} \quad \text{(Chir. An.)}$$

• $\underline{2+1\,D}$, $a_\mu = 0$, $w_\mu \neq 0$

$$S_{eff}(w) = \chi \int_\Lambda tr(w_0^2)\, d^3x$$
$$+ \tilde{\chi} \int_\Lambda tr(\underline{w}^2)\, d^3x$$
$$+ \frac{k}{4\pi}\Big[\int_\Lambda tr\big(w^{tot} \wedge dw^{tot} + \tfrac{2}{3}(w^{tot})^{\wedge 3}\big)$$
$$+ \Gamma_{WZW}\big(w^{tot}\big|_{\partial\Lambda}\big)\Big]$$
$$+ \cdots \quad , \; k \in \mathbb{Z}$$

$$\boxed{k = 0, \pm 1} \qquad \text{"}\mathbb{Z}_2\text{"}$$

spin (-current) Hall effect

Γ_{WZW} : gen. fu. of Green fus. of chiral edge spin currents at level $k = 2 \times \underline{\text{spin of quasi-pt.}}$

2+1 D "topological insulator"

(J.F. et al. 1993)

- $\underline{3+1\ D}$, $w_\mu = 0$, $a_\mu \neq 0$

$$S_{eff}(a) = S_{Maxwell}(a) + S_\theta(a)$$

$$S_{Maxwell}(a) = \frac{1}{2\alpha}\int [\varepsilon \vec{\mathcal{E}}^2 + \mu^{-1}\vec{\mathcal{B}}^2]\, d^4x$$

$$S_\theta(a) = \theta \frac{1}{4\pi^2}\int \vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\, d^4x$$

∞ ext. sample, $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ vanish at $\infty \Rightarrow$

$$\frac{1}{4\pi^2}\int \vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\, d^4x = n \in \mathbb{Z}$$

$\Rightarrow$ Bulk of system inv. under P & T iff

$$\boxed{\theta = 0, \pi}$$

"topological insulator" in 3 dim.

Consider finite sample confined to Λ w. bd. $\partial\Lambda$.

$$S_\theta^\Lambda(a) \overset{\uparrow}{=} \frac{\theta}{4\pi^2}\int_{\partial\Lambda} a \wedge da$$

Stokes

For $\theta = \pi$, $S_{\theta=\pi}^\Lambda$ is effective action of (2+1)D 2-component relativistic Dirac fermions (F-M-S '76; D-J-T, Redl. '80s)

= surface modes of (3+1)D "topological insulator"

Promote θ to dyn. field φ

= "axion field"

$$S(\varphi, a) = \frac{1}{4\pi^2} \int \varphi \, F_a \wedge F_a \quad \text{(AxTI)}$$

diml. reduction of (4+1)D Hall effect (F-P[-W] '99)

(AxTI) may arise in insulators with 2 filled bands (bonding, anti-bd.). If els. couple to P-, T- breaking background (e.g., anti-ferro order, chiral structure)

$\rightarrow$ (AxTI), φ : backgd. flucts.

Add axion action

$$\frac{J}{2}\int[\dot{\varphi}^2-(v\vec{\nabla}\varphi)^2-U(\varphi)]d^4x$$

→ Possibility of axion domain walls → surface modes ~ 2-comp. Dirac fs.

- Instabilities (F-P '99, ...)

Simplest example:

$\vec{\mathcal{E}}=0$, $\vec{B}$ time-indep.

$\dot{\varphi}=:\Delta\mu$ — " —

$$F_{eff}(\vec{a})=cst\int\vec{B}^2d^3x+\frac{\Delta\mu}{4\pi^2}\int\vec{a}\cdot\vec{B}d^3x$$

Minimize F_{eff} → magnetic instability for $|\vec{k}|\leq cst.\Delta\mu$.